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A product of two generalized derivations on polynomials in prime rings

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Abstract

Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R, F and G non-zero generalized derivations of R and $f(x_1, \dots, x_n)$ a polynomial over C. Denote by $f(R)$ the set $\{f(r_1, \dots, r_n) : r_1, \dots, r_n \in R\}$ of all the evaluations of $f(x_1, \dots, x_n)$ in R. Suppose that $f(x_1, \dots, x_n)$ is not central valued on R. If R does not embed in $M_2(K)$, the algebra of 2×2 matrices over a field K, and the composition (FG) acts as a generalized derivation on the elements of $f(R)$, then (FG) is a generalized derivation of R and one of the following holds:

1. there exists $\alpha \in C$ such that $F(x) = \alpha x$, for all $x \in R$;
2. there exists $\alpha \in C$ such that $G(x) = \alpha x$, for all $x \in R$;
3. there exist $a, b \in U$ such that $F(x) = ax$, $G(x) = bx$, for all $x \in R$;
4. there exist $a, b \in U$ such that $F(x) = xa$, $G(x) = xb$, for all $x \in R$;
5. there exist $a, b \in U$, $\alpha, \beta \in C$ such that $F(x) = ax + xb$, $G(x) = \alpha x + \beta(ax - xb)$, for all $x \in R$.

Throughout this paper, R always denotes a prime ring with center $Z(R)$, U the Utumi quotient ring of R and $C = Z(U)$ the center of U. We refer the reader to [3] for the definitions and the related properties of these objects.

Let $F : R \longrightarrow R$ be an additive mapping of R into itself. It is said to be a derivation of R if $F(xy) = F(x)y + xF(y)$, for all $x, y \in R$. If $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$ and d a derivation of R, then the mapping F is called a generalized derivation on R. Obviously any derivation of R is a generalized derivation of R.

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A typical example of a generalized derivation is a map of the form $x \mapsto ax + xb$, where a, b are fixed elements in R; such generalized derivations are called inner. The well known Posner's first theorem states that if δ and d are two non-zero derivations of R, then the composition $(d\delta)$ cannot be a non-zero derivation of R ([11, Theorem 1]). An analogue of Posner's result for Lie derivations was proved by Lanski in [8]. More precisely Lanski showed that if δ and d are two non-zero derivations of R and L is a Lie ideal of R, then $(d\delta)$ cannot be a Lie derivation of L into R unless $char(R) = 2$ and either R satisfies $s_4(x_1, ..., x_4)$, the standard identity of degree 4, or $d = \alpha\delta$, for $\alpha \in C$.

In [6] Hvala initiated the algebraic study of generalized derivations. In particular, generalized derivations whose product is again a generalized derivation was characterized. More precisely Hvala in ([6, Theorem 1]) proved that:

Theorem

Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R, F and G non-zero generalized derivations of R. If the composition FG acts as a generalized derivation on R, then one of the following holds:

1. *there exists $\alpha \in C$ such that $F(x) = \alpha x$, for all $x \in R$;*
2. *there exists $\alpha \in C$ such that $G(x) = \alpha x$, for all $x \in R$;*
3. *there exist $a, b \in U$ such that $F(x) = ax$, $G(x) = bx$, for all $x \in R$;*
4. *there exist $a, b \in U$ such that $F(x) = xa$, $G(x) = xb$, for all $x \in R$;*
5. *there exist $a, b \in U$, $\alpha, \beta \in C$ such that $F(x) = ax + xb$, $G(x) = \alpha x + \beta(ax - xb)$, for all $x \in R$.*

One might wonder if it is possible that the composition of two generalized derivations with special forms may act like a generalized derivation on some subset of prime rings. Following this line of investigation, our main theorem gives a description of the forms of two generalized derivations F and G of a prime ring R, in the case when (FG) acts as a generalized derivation on the elements of the subset $f(R)$, where $f(R)$ is a the set of all evaluations in R of a polynomial $f(x_1, \ldots, x_n)$ over C in n non-commuting variables. More precisely we assume that this means $(FG)(st) = (FG)(s)t + sh(t)$, for all $s, t \in f(R)$ and for a derivationn h of R. The statement of our result is the following:

Theorem 1

Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R, F and G non-zero generalized derivations of R and $f(x_1, \ldots, x_n)$ a polynomial over C. Denote by $f(R)$ the set $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in R\}$ of all the evaluations of $f(x_1, \ldots, x_n)$ in R. Suppose that $f(x_1, \ldots, x_n)$ is not central valued on R. If R does not embed in $M_2(K)$, the algebra of 2×2 matrices over a field K, and the composition (FG) acts as a generalized derivation on the elements of $f(R)$, then (FG) is a generalized derivation of R and one of the following holds:

1. *there exists $\alpha \in C$ such that $F(x) = \alpha x$, for all $x \in R$;*

2. *there exists* $\alpha \in C$ *such that* $G(x) = \alpha x$, *for all* $x \in R$;
3. *there exist* $a, b \in U$ *such that* $F(x) = ax$, $G(x) = bx$, *for all* $x \in R$;
4. *there exist* $a, b \in U$ *such that* $F(x) = xa$, $G(x) = xb$, *for all* $x \in R$;
5. *there exist* $a, b \in U$, $\alpha, \beta \in C$ *such that* $F(x) = ax + xb$, $G(x) = \alpha x + \beta(ax - xb)$, *for all* $x \in R$.

The assumption that R does not embed in $M_2(K)$, for K a field, is essential to the main result. For example let e_{ij} be the usual matrix unit in $R = M_2(K)$ and consider $F(x) = e_{22}x - xe_{22}$, $G(x) = (e_{12}+e_{21})x + x(e_{12}+e_{21})$. Then $FG([R,R]) = (0)$, but FG does not act on R like a generalized derivation as described by the main theorem.

1. The matrix case and inner generalized derivations

In this section we will study the case when $R = M_m(K)$ is the algebra of $m \times m$ matrices over an infinite field K. Here we will assume that there exist a, b, c, q, v, w elements of R such that $a(cx + xq) + (cx + xq)b = vx + xw$ for all $x \in [R, R]$. Notice that the set $[R, R] = \{[r_1, r_2] : r_1, r_2 \in R\}$ is invariant under the action of all inner automorphisms of R. Let us denote as usual by e_{ij} the matrix unit with 1 in (i, j)-entry and zero elsewhere, moreover let I be the identity matrix in R. In this section we will prove that, in case $m \geq 3$, one of the following holds:

- c and q are central matrices;
- a and b are central matrices;
- b, q and w are central matrices;
- a, c and v are central matrices;
- there exists $\eta \in K$ such that $a + \eta c$, $b - \eta q$ are central matrices.

In order to prove this result we will make implicit use of the following easy remarks:

Remark 1.1 For any inner automorphism φ of $M_m(K)$, we have that

$$0 = \varphi\big(a(cs + sq) + (cs + sq)b - vs - sw\big) = \varphi(a)\big(\varphi(c)s + s\varphi(q)\big) + \big(\varphi(c)s + s\varphi(q)\big)\varphi(b) - \varphi(v)s - s\varphi(w)$$

for all $s \in [R, R]$, since $[R, R]$ is invariant under the action of all inner automorphisms of R. Clearly

- c and q are central matrices if and only if $\varphi(c)$ and $\varphi(q)$ are central matrices;
- a and b are central matrices if and only if $\varphi(a)$ and $\varphi(b)$ are central matrices;
- b, q and w are central matrices if and only if $\varphi(b)$, $\varphi(q)$ and $\varphi(w)$ are central matrices;
- a, c and v are central matrices if and only if $\varphi(a)$, $\varphi(c)$ and $\varphi(v)$ are central matrices;
- $a + \alpha b$, $c - \alpha q$ and $b - \eta q$ are central matrices if and only if $\varphi(a) + \alpha\varphi(b)$, $\varphi(c) - \alpha\varphi(q)$ and $\varphi(b) - \eta\varphi(q)$ are central matrices.

Hence, to prove our result, we may replace a, b, c, q, v, w respectively with $\varphi(a)$, $\varphi(b)$, $\varphi(c)$, $\varphi(q)$, $\varphi(v)$, $\varphi(w)$.

Remark 1.2 The matrix unit e_{kl} is an element of $[R,R]$ for all $k \neq l$.

We need the following:

Remark 1.3 Let R be a prime ring and $a, c \in R$ such that $ax + xc = 0$ for all $x \in R$. Then $a = -c \in Z(R)$.

Proof. Consider the assumed identity

$$ax + xc = 0\,. \tag{1}$$

Left multiplying (1) by any $t \in R$, we have $tax + txc = 0$ $(1')$. On the other hand, by replacing x with tx in (1), we also have $atx + txc = 0$ $(1'')$. Comparing $(1')$ with $(1'')$ it follows $[a,t]x = 0$ and, by the primeness of R, a must be central. So $x(a+c) = 0$, that is $a = -c$. □

Remark 1.4 Let R be a prime ring and $a, b, c \in R$ such that $axb + xc = 0$ for all $x \in R$. Then either $a \in Z(R)$ and $c + ab = 0$, or a, b, c are central elements of R.

Proof. Consider the assumed identity

$$axb + xc = 0. \tag{2}$$

Right multiplying (2) by any $t \in R$, we have $axbt + xct = 0$ $(2')$. On the other hand, by replacing x with xt in (2), we also have $axtb + xtc = 0$ $(2'')$. Comparing $(2')$ with $(2'')$ it follows

$$ax[b,t] + x[c,t] = 0. \tag{3}$$

As above, left multiplying (3) by any $z \in R$, we have $zax[b,t] + zx[c,t] = 0$ $(3')$. Moreover, by replacing x with zx in (3), we also have $azx[b,t] + zx[c,t] = 0$ $(3'')$. Comparing $(3')$ with $(3'')$ it follows $[a,z]x[b,t] = 0$ and, by the primeness of R, either $a \in Z(R)$ or $b \in Z(R)$. In the first case $x(ab+c) = 0$, which implies $ab + c = 0$. In the second case $abx + xc = 0$, and the conclusion follows from Remark 1.3. □

We also need the following lemma:

Lemma 1.5

Let F be a infinite field and $n \geq 2$. If $A_1, \ldots, A_k$ are not scalar matrices in $M_n(F)$ then there exists some invertible matrix $Q \in M_n(F)$ such that each matrix $QA_1Q^{-1}, \ldots, QA_kQ^{-1}$ has all non-zero entries.

Proof. First we show that if $A \in M_n(F)$ is not scalar then there exists a conjugate QAQ^{-1} having a non-zero entry in any particular position.

Assume that A is not diagonal, hence for some $i \neq j$ the (i,j)-entry A_{ij} of A is non-zero. Clearly if $p \neq q$ then there exists a permutation $\sigma \in S_n$ such that $\sigma(i) = p$ and $\sigma(j) = q$. We consider the automorphism φ_σ on $M_n(F)$ defined by $\varphi_\sigma(e_{rs}) = e_{\sigma(r)\sigma(s)}$, for any matrix unit e_{rs}. Let $Q \in M_n(F)$ be the permutation matrix which induces in $M_n(F)$ this automorphism φ_σ, hence the (p,q)-entry of QAQ^{-1} is A_{ij}. Assume now that $p = q$. By the previous argument, for $s \neq p$, some conjugate A' of A has non-zero (p,s)-entry. Let $\lambda \in F$, and put $A'_\lambda = (I + \lambda e_{sp})A'(I - \lambda e_{sp})$. Then the (p,p)-entry of

A'_λ is $A'_{pp} - \lambda A'_{ps}$. Of course we can choose λ in F such that $A'_{pp} - \lambda A'_{ps}$ is not zero. This proves our claim in the case when A is not diagonal. If A is a diagonal matrix which is not a scalar one, there exist $i \neq j$ such that $A_{ii} \neq A_{jj}$. The (i,j)-entry of the conjugate $A'' = (I + e_{ij})A(I - e_{ij})$ is $A_{jj} - A_{ii}$ which is not zero. Hence A'' is not diagonal and by the previous case we are done.

Consider the set $\{x_{ij} : 1 \leq i, j \leq n\}$ of n^2 commutative indeterminates and let $M_n(F[x_{ij}])$ be the algebra of $n \times n$ matrices over the polynomial ring $F[x_{ij}]$. Let $P = \sum_{ij} x_{ij} e_{ij}$ be the generic matrix and consider, for $l = 1, \ldots, k$, $P_l = P \cdot A_l \cdot adj(P)$. Any substitution of the indeterminates x_{ij} with elements $c_{ij} \in F$ induces a homomorphism $\varphi : M_n(F[x_{ij}]) \longrightarrow M_n(F)$. If $\varphi(P)$ is an invertible matrix Q then $\varphi(P_l)$ is a non-zero scalar multiple of QA_lQ^{-1}. Clearly any matrix $Q \in M_n(F)$ is the image of P under the action of some of such homomorphisms. Now each entry of $adj(P)$ is a homogeneous polynomial in $\{x_{ij}\}$ so the entries of P_l are homogeneous polynomials in $\{x_{ij}\}$ without constant terms. None of these entries is zero by our observation above: in any particular position some conjugate of A_l has a non-zero entry. Also $\det(P)$ is a non-zero polynomial of $F[x_{ij}]$. Let $G(x_{ij})$ be the product of $\det(P)$ and all entries of P_l, for $l = 1, \ldots, k$. Clearly $G(x_{ij})$ is a non-zero polynomial and, since the field F is infinite, some evaluation of $G(x_{ij})$ is not zero in F. As above let φ be the homomorphism induced by this evaluation, then $Q = \varphi(P)$ is invertible and $QA_lQ^{-1} = \frac{1}{\det(Q)}\varphi(P_l)$ is a matrix with all non-zero entries, for $l = 1, \ldots, k$. □

We start the proof of the main theorem of this section by studying the following case:

Lemma 1.6

Let K be an infinite field, let $R = M_m(K)$ be the algebra of $m \times m$ matrices over K, $Z(R)$ the center of R and $S = [R, R]$. Assume that there exist $a, b, c, q, v, w \in R$ such that $a(cs + sq) + (cs + sq)b = vs + sw$ for all $s \in S$. If $q \in Z(R)$ then one of the following holds:

1. *c is a central matrix;*
2. *b and w are central matrices.*

Proof. Since $q \in Z(R)$, by the assumption we have that $a(c+q)s + (c+q)sb = vs + sw$ for all $s \in S$. Clearly if $c + q \in Z(R)$ we are done. Suppose that $b \in Z(R)$. Then $(a+b)(c+q)s = vs+sw$ for all $s \in S$, in other words for all $i \neq j$, $X = (a+b)(c+q)e_{ij} - ve_{ij} - e_{ij}w = 0$. In particular the (i,i)-entry of X is $-e_{ij}we_{ii} = 0$, that is w is a diagonal matrix, say $w = \sum_{k=1}^{m} w_k e_{kk}$, for $w_k \in K$. Let χ be any inner automorphism of R; of course $\chi(q)$ and $\chi(b)$ are central matrices and $\chi((a + b)(c + q)s - vs - sw) = 0$ for all $s \in S$. Thus $\chi(w)$ must be a diagonal matrix, say $\chi(w) = \sum_{k=1}^{m} w'_k e_{kk}$, for some $w'_k \in K$. In particular for $r \neq s$ and $\chi(x) = (1 + e_{rs})x(1 - e_{rs})$, we have $\chi(w) = w + e_{rs}w - we_{rs}$. Since the (r,s)-entry of $\chi(w)$ is zero, it follows $w_r = w_s$, for all $r \neq s$. This means that w is a central matrix in R and we are done.

In light of this, we consider $c + q$ and b both non-scalar matrices. We will prove that in this case we get a contradiction.

By Remark 1.1 and Lemma 1.5, we can assume that $c + q$ and b have all non-zero entries, say $c + q = \sum_{kl} t_{kl} e_{kl}$ and $b = \sum_{kl} b_{kl} e_{kl}$, for $0 \neq t_{kl}, 0 \neq b_{kl} \in K$.

Since $e_{ji} \in S$ for all $i \neq j$, then for any $i \neq j$

$$X = a(c+q)e_{ji} + (c+q)e_{ji}b - ve_{ji} - e_{ji}w = 0$$

in particular the (i,j)-entry of X is $t_{ij}b_{ij} = 0$, a contradiction. □

Analogously one may prove the following (we omit the proof for brevity):

Lemma 1.7

Let K be an infinite field, let $R = M_m(K)$ be the algebra of $m \times m$ matrices over K, $Z(R)$ the center of R and $S = [R,R]$. Assume that there exist $a,b,c,q,v,w \in R$ such that $a(cs+sq)+(cs+sq)b = vs+sw$ for all $s \in S = [R,R]$. If $c \in Z(R)$ then one of the following holds:

1. *q is a central matrix;*
2. *a and v are central matrices.*

Lemma 1.8

Let K be an infinite field, let $R = M_m(K)$ be the algebra of $m \times m$ matrices over K, $Z(R)$ the center of R and $S = [R,R]$. Assume that there exist $a,b,c,q,v,w \in R$ such that $a(cs+sq)+(cs+sq)b = vs+sw$ for all $s \in S$. If $b \in Z(R)$ then one of the following holds:

1. *a is a central matrix;*
2. *q and w are central matrices.*

Proof. We assume both a and q non-scalar matrices and prove that a contradiction follows. Denote $q = \sum_{kl} q_{kl}e_{kl}$ and $a = \sum_{kl} a_{kl}e_{kl}$, $w = \sum_{kl} w_{kl}e_{kl}$, for $w_{kl}, q_{kl}, a_{kl} \in K$.

By Remark 1.1 and Lemma 1.5, we may assume that q and a have all non-zero entries. Since $b \in Z(R)$, we have that $(a+b)(cs+sq) = vs+sw$ for all $s \in S$, that is $((a+b)c-v)s + (a+b)sq - sw = 0$ for all $s \in S$, in other words for all $i \neq j$, $X = ((a+b)c-v)e_{ij} + (a+b)e_{ij}q - e_{ij}w = 0$. In particular the (j,i)-entry of X is $a_{ji}q_{ji} = 0$, which contradicts our assumption.

In particular, in case $q \in Z(R)$, by Lemma 1.6, either c is central or w is central. If $c \in Z(R)$, one has $(a+b)(c+q)s = vs+sw$ for all $s \in S$. For any $i \neq j$ and $s = e_{ij}$: $0 = Y = (a+b)(c+q)e_{ij} = ve_{ij} + e_{ij}w$. In particular the (i,i)-entry of Y is $w_{ji} = 0$, that is w is a diagonal matrix. Let χ be any inner automorphism of R; of course $\chi(q)$, $\chi(b)$ and $\chi(c)$ are central matrices and $\chi\left((a+b)(c+q)s - vs - sw\right) = 0$ for all $s \in S$. Thus $\chi(w)$ must be a diagonal matrix, say $\chi(w) = \sum_{k=1}^{m} w'_k e_{kk}$, for some $w'_k \in K$. In particular for $r \neq s$ and $\chi(x) = (1+e_{rs})x(1-e_{rs})$, we have $\chi(w) = w + e_{rs}w - we_{rs}$. Since the (r,s)-entry of $\chi(w)$ is zero, it follows $w_r = w_s$, for all $r \neq s$. This means that w is a central matrix in R and we are done. □

Lemma 1.9

Let K be an infinite field, let $R = M_m(K)$ be the algebra of $m \times m$ matrices over K with $m \geq 3$, $Z(R)$ the center of R. Assume that there exist $a,b,c,q,v,w \in R$ such that $a(cs+sq)+(cs+sq)b = vs+sw$ for all $s \in S = [R,R]$. If $q \notin Z(R)$ and $b - \alpha q \in Z(R)$, for a suitable $\alpha \in K$, then $a + \alpha c$ is a central matrix.

Proof. Assume that $a+\alpha c$ is not a scalar matrix. By Remark 1.1 and Lemma 1.5, we can assume that $a+\alpha c$ and q have all non-zero entries, say $a+\alpha c=\sum_{kl} t_{kl}e_{kl}$ and $q=\sum_{kl} q_{kl}e_{kl}$, for $0\neq t_{kl}, 0\neq q_{kl}\in K$.

Since $b=\beta I+\alpha q$, for a suitable $\beta\in K$, by our assumption we have that

$$a(cx+xq)+(cx+xq)(\beta+\alpha q)-vx-xw=0$$

that is

$$(ac+\beta c)x+(a+\alpha c)xq+x(\alpha q^2+\beta q)-vx-xw=0$$

for all $x\in S$, and for $x=e_{ij}$, with $i\neq j$

$$0=X=(ac+\beta c)e_{ij}+(a+\alpha c)e_{ij}q+e_{ij}(\alpha q^2+\beta q)-ve_{ij}-e_{ij}w=0.$$

By calculations one has that the (j,i)-entry of X is $0=t_{ji}q_{ji}$, a contradiction. Therefore $a+\alpha c$ must be a central matrix in R and we are done. □

Lemma 1.10

Let K be an infinite field, let $R=M_m(K)$ be the algebra of $m\times m$ matrices over K with $m\geq 3$ and $S=[R,R]$. Suppose there exist $a,b,c,q,u,p,v,w\in R$ such that $ux+axq+cxb+xp=vx+xw$ for all $x\in S$. Denote

$$a=\sum_{kl}a_{kl}e_{kl},\ b=\sum_{kl}b_{kl}e_{kl},\ c=\sum_{kl}c_{kl}e_{kl},\ q=\sum_{kl}q_{kl}e_{kl},$$

for suitable a_{kl}, b_{kl}, c_{kl} and q_{kl} elements of K. If there are $i\neq j$ such that $q_{ji}\neq 0$, $c_{ji}\neq 0$ and $b_{ji}=0$, then $a_{ri}=0$ and $b_{rk}=0$ for all $r\neq i$ and $k\neq r$ (that is the only non-zero off-diagonal elements of b fall in the i-th row).

Proof. Consider the assumption

$$ux+axq+cxb+xp-vx-xw=0\quad \forall x\in[R,R].$$

In particular, for $x=e_{ij}$ we have:

$$X=ue_{ij}+ae_{ij}q+ce_{ij}b+e_{ij}p-ve_{ij}-e_{ij}w=0$$

and for $x=e_{it}$, with $t\neq i,j$, we also have

$$Y=ue_{it}+ae_{it}q+ce_{it}b+e_{it}p-ve_{it}-e_{it}w=0.$$

From the previous equalities it follows that:

1. for all $r\neq i$, the (r,i)-entry of the matrix X is $0=a_{ri}q_{ji}+c_{ri}b_{ji}=a_{ri}q_{ji}$;
2. for all $s\neq j$, the (j,s)-entry of the matrix X is $a_{ji}q_{js}+c_{ji}b_{js}=0$;
3. the (j,i)-entry of the matrix Y is $a_{ji}q_{ti}+c_{ji}b_{ti}=0$;
4. for all $k\neq i,t$, the (j,k)-entry of the matrix Y is $a_{ji}q_{tk}+c_{ji}b_{tk}=0$ (note that this holds also in case $k=j$);

From (1) and since $q_{ji}\neq 0$, one has $a_{ri}=0$ for all $r\neq i$, in particular $a_{ji}=0$. Thus by (2) and since $c_{ji}\neq 0$, we have $b_{js}=0$ for all $s\neq j$. So by (3) $b_{ti}=0$ for all $t\neq i$. Finally by (4), $b_{tk}=0$ for all $t\neq i,j$ and $k\neq t$. □

Lemma 1.11

Let K be an infinite field, let $R = M_m(K)$ be the algebra of $m \times m$ matrices over K with $m \geq 3$ and $S = [R, R]$. Suppose there exist $a, b, c, q, u, p, v, w \in R$ such that $ux + axq + cxb + xp = vx + xw$ for all $x \in S$. Denote

$$b = \sum_{kl} b_{kl} e_{kl}, \; c = \sum_{kl} c_{kl} e_{kl}, \; q = \sum_{kl} q_{kl} e_{kl},$$

for suitable b_{kl}, c_{kl} and q_{kl} elements of K. Assume there are $i \neq j$ such that $b_{ji} = 0$. If $q_{rs} \neq 0$, $c_{rs} \neq 0$ for all $r \neq s$, then b is central in R.

Proof. The first step is to apply twice Lemma 1.10: this forces b to be a diagonal matrix. In fact $b_{ji} = 0$, $q_{ji} \neq 0$ and $c_{ji} \neq 0$ imply that $b_{rk} = 0$ for all $r \neq i$ and $k \neq r$, in particular, since $m \geq 3$, ther exists $t \neq i$ such that $b_{lt} = 0$, for all $l \neq t$. Since $q_{lt} \neq 0$, $c_{lt} \neq 0$ we have $b_{rk} = 0$ for all $r \neq t$ and $k \neq r$, so $b_{ik} = 0$ for all $k \neq i$, as required. Say $b = \sum_k b_{kk} e_{kk}$.

Consider now the inner automorphism of R induced by the invertible matrix $P = I + e_{rj}$, for $r \neq i, j$: $\varphi(x) = P^{-1} x P$. Of course

$$\varphi(u)x + \varphi(a)x\varphi(q) + \varphi(c)x\varphi(b) + x\varphi(p) = \varphi(v)x + x\varphi(w),$$

for all $x \in R$. Moreover the (j, i)-entries of $\varphi(q)$, $\varphi(c)$, $\varphi(b)$ are respectively $q_{ji} \neq 0$, $c_{ji} \neq 0$ and $b_{ji} = 0$. Therefore, again by Lemma 1.10, any (r, j)-entry of $\varphi(b)$ is zero, for all $r \neq i$. By calculations $0 = (\varphi(b))_{rj} = b_{jj} - b_{rr}$, that is $b_{jj} = b_{rr}$.

On the other hand, if χ is the inner automorphisms induced by the invertible matrix $Q = I + e_{ri}$, as above $\chi(u)x + \chi(a)x\chi(q) + \chi(c)x\chi(b) + x\chi(p) = \chi(v)x + x\chi(w)$, for all $x \in R$. Since the (i, j)-entries of $\chi(q)$, $\chi(c)$ and $\chi(b)$ are respectively $q_{ij} \neq 0$, $c_{ij} \neq 0$ and $b_{ij} = 0$, again any (r, i)-entry of $\chi(b)$ is zero, for all $r \neq j$, that is $0 = (\varphi(b))_{ri} = b_{ii} - b_{rr}$ and $b_{ii} = b_{rr} = b_{jj} = \beta$, for all $r \neq i, j$. Thus $b = \beta I$ is a central matrix in R. □

Now we are ready to prove the main result of this section:

Proposition 1.12

Let K be an infinite field, let $R = M_m(K)$ be the algebra of $m \times m$ matrices over K with $m \geq 3$ and $S = [R, R]$. Suppose there exist $a, b, c, q, v, w \in R$ such that $a(cx + xq) + (cx + xq)b = vx + xw$ for all $x \in S$. Then one of the following holds:

1. *c, q are central matrices;*
2. *a, b are central matrices;*
3. *b, q and w are central matrices;*
4. *a, c and v are central matrices;*
5. *there exists $\eta \in K$ such that $a + \eta c$ and $b - \eta q$ are central matrices.*

Proof. Let

$$a = \sum_{kl} a_{kl} e_{kl}, \; b = \sum_{kl} b_{kl} e_{kl}, \; c = \sum_{kl} c_{kl} e_{kl}, \; q = \sum_{kl} q_{kl} e_{kl},$$

for suitable a_{kl}, b_{kl}, c_{kl} and q_{kl} elements of K.

Clearly if one of q or c is a scalar matrix we are done by Lemmas 1.6 and 1.7. In order to prove the proposition, we may assume that q and c are non-central matrices.

By Remark 1.1 and Lemma 1.5, there exists some invertible matrix $Q \in M_m(K)$ such that $QqQ^{-1} = q'$ and $QcQ^{-1} = c'$ have all non-zero entries. By this conjugation we denote

$$a' = \sum_{kl} a'_{kl} e_{kl}, \ b' = \sum_{kl} b'_{kl} e_{kl}, \ c' = \sum_{kl} c'_{kl} e_{kl}, \ q' = \sum_{kl} q'_{kl} e_{kl},$$

for suitable a'_{kl}, b'_{kl}, c'_{kl} and q'_{kl} elements of K, the conjugates of elements a, b, c, q. Moreover let u' and v' be the conjugates of elements u and v. Of course

$$a'c'x + a'xq' + c'xb' + xq'b' = v'x + xw' \quad \text{for all} \quad x \in S.$$

Since $q'_{rs} \neq 0$ and $c'_{rs} \neq 0$ for all $r \neq s$, then the following holds: if for some $i \neq j$ there is some $b'_{ji} = 0$ then by Lemma 1.11 b' is a central matrix, that is also b is a central matrix and we are finished by Lemma 1.8.

Hence assume that $b'_{rs} \neq 0$ for all $r \neq s$. Let $\eta = \frac{b'_{ji}}{q'_{ji}} \neq 0$ and $a'' = a' + \eta c'$. By replacing a' with $a'' - \eta c'$ in the main equation we get

$$(a'' - \eta c')c'x + (a'' - \eta c')xq' + c'xb' + xq'b' = v'x + x'w \quad \text{for all} \quad x \in S.$$

By calculations it follows that

$$(a'' - \eta c')c'x + a''xq' + c'x(b' - \eta q') + xq'b' = v'x + x'w \quad \text{for all} \quad x \in S.$$

Note that the (j, i)-entry of the matrix $(b' - \eta q')$ is zero; since $q'_{rs} \neq 0$ and $c'_{rs} \neq 0$ for all $r \neq s$, then by Lemma 1.11 $b' - \eta q'$ must be a central matrix, that is $b - \eta q$ is central in R. Let $b = \eta q + \beta$ for a suitable $\beta \in Z(R)$. Thus by the main assumption we get

$$acx + axq + \eta cxq + \eta xq^2 + \beta cx + \beta xq = vx + xw \quad \text{for all} \quad x \in S.$$

Assume finally that $a + \eta c$ is not a scalar matrix. Since q is not a scalar matrix, then there exists some invertible matrix $P \in M_m(K)$ such that $PqP^{-1} = q'''$ and $P(a + \eta c)P^{-1} = c'''$ have all non-zero entries. As above, by this conjugation we denote

$$a''' = \sum_{kl} a'''_{kl} e_{kl}, \quad c''' = \sum_{kl} c'''_{kl} e_{kl}, \quad q''' = \sum_{kl} q'''_{kl} e_{kl},$$

for suitable a'''_{kl}, c'''_{kl} and q'''_{kl} elements of K, the conjugates of elements a, c, q, and v''', w''' the conjugates of elements u and v. Then

$$a'''c'''x + a'''xq''' + \eta c'''xq''' + \eta x(q''')^2 + \beta c'''x + \beta xq''' = v'''x + xw''' \quad \text{for all} \quad x \in S.$$

Choose $x = e_{ji}$ for $i \neq j$. Hence the matrix

$$a'''c'''e_{ji} + a'''e_{ji}q''' + \eta c'''e_{ji}q''' + \eta e_{ji}(q''')^2 + \beta c'''e_{ji} + \beta e_{ji}q''' - v'''e_{ji} + e_{ji}w'''$$

is zero. In particular the (j, i)-entry is $(a'''_{ij} + \eta c'''_{ij})q'''_{ij} = 0$. This contradiction shows that also $a + \eta c$ must be a central matrix and we are done. □

2. The proof of the Theorem 1

We begin this section by studying in detail the case when F, G and H are all inner generalized derivations. More precisely, if $F(x) = ax + xb$ is the inner generalized derivation induced by the elements $a, b \in U$, $G(x) = cx + xq$ the one induced by $c, q \in U$, and $H(x) = vx + xw$ the one induced by $v, w \in U$, is the composition FG on $[R, R]$. Thus

$$\Phi(x_1, x_2) = a\big(c[x_1, x_2] + [x_1, x_2]q\big) + \big(c[x_1, x_2] + [x_1, x_2]q\big)b - v[x_1, x_2] - [x_1, x_2]w$$

is a generalized polynomial identity for R.

We observe the following:

Remark 2.1 If B is a basis of U over C then any element of $T = U *_C C\{x_1, \dots, x_n\}$, the free product over C of the C-algebra U and the free C-algebra $C\{x_1, \dots, x_n\}$, can be written in the form $g = \sum_i \alpha_i m_i$. In this decomposition the coefficients α_i are in C and the elements m_i are B-monomials, that is $m_i = q_0 y_1 q_1 \cdots y_h q_h$, with $q_i \in B$ and $y_i \in \{x_1, \dots, x_n\}$. In [4] it is shown that a generalized polynomial $g = \sum_i \alpha_i m_i$ is the zero element of T if and only if all α_i are zero. Let $a_1, \dots, a_k \in U$ be linearly independent over C and $a_1 g_1(x_1, \dots, x_n) + \dots + a_k g_k(x_1, \dots, x_n) = 0 \in T$, for some $g_1, \dots, g_k \in T$. If, for any i, $g_i(x_1, \dots, x_n) = \sum_{j=1}^n x_j h_j(x_1, \dots, x_n)$ and $h_j(x_1, \dots, x_n) \in T$, then $g_1(x_1, \dots, x_n)$,..., $g_k(x_1, \dots, x_n)$ are the zero element of T. The same conclusion holds if $g_1(x_1, \dots, x_n)a_1 + \dots + g_k(x_1, \dots, x_n)a_k = 0 \in T$, and $g_i(x_1, \dots, x_n) = \sum_{j=1}^n h_j(x_1, \dots, x_n)x_j$ for some $h_j(x_1, \dots, x_n) \in T$. (We refer the reader to [2] and [4] for more details on generalized polynomial identities).

We will make frequent use of the previous remark in our next result:

Lemma 2.2

If $\Phi(x_1, x_2) = 0$ in $T = U *_C C\{x_1, x_2\}$, then one of the following holds:

1. $c, q \in C$;
2. $a, b \in C$;
3. $b, q, w \in C$;
4. $a, c, v \in C$;
5. *there exists $\eta \in C$ such that $a + \eta c \in C$, and $b - \eta q \in C$.*

Proof. By our hypothesis, $\Phi(r_1, r_2) = 0$ for all $r_1, r_2 \in R$, that is R satisfies the generalized polynomial identity $\Phi(x_1, x_2)$.

If $a \in C$, then

$$\Phi(x_1, x_2) = \big(c[x_1, x_2] + [x_1, x_2]q\big)(a + b) - v[x_1, x_2] - [x_1, x_2]w.$$

Notice that in case $c \in C$, it follows

$$\Phi(x_1, x_2) = [x_1, x_2](c + q)(a + b) - v[x_1, x_2] - [x_1, x_2]w$$

and this implies $v \in C$, since $\Phi(x_1, x_2) = 0$ in T. In this case we are done. On the other hand, if $c \notin C$, since $\{c, v, 1\}$ must be linearly C-dependent, there exist $\lambda, \mu \in C$ such that $v = \lambda c + \mu$ since $\{c, 1\}$ is linearly C-independent. As a consequence R satisfies

$$\Phi(x_1, x_2) = c[x_1, x_2](a + b - \lambda) + [x_1, x_2](q(a + b) - w - \mu),$$

which is a non-trivial generalized polynomial identity for R, unless $a+b=\lambda$, which means $b\in C$. Also in this case we are done.

Analogously is suppose $b\in C$, by using the same argument on the right of the identity $\Phi(x_1,x_2)$, one may prove that either $q,w\in C$ or $a\in C$.

Assume now that $c\in C$ and $a\notin C$, then

$$\Phi(x_1,x_2)=a[x_1,x_2](c+q)+[x_1,x_2](c+q)b-v[x_1,x_2]-[x_1,x_2]w.$$

Thus $\{a,v,1\}$ is linearly C-dependent and since $a\notin C$, we may write $v=\lambda a+\mu$, for suitable $\lambda,\mu\in C$. It follows that

$$\Phi(x_1,x_2)=a[x_1,x_2](c+q-\lambda)+[x_1,x_2]((c+q)b-w-\mu)$$

which is a non-trivial generalized polynomial identity for R, unless $c+q=\lambda$. In this last case, it follows that $q\in C$, and we are finished.

Using a similar argument we may prove that if $q\in C$ and $b\notin C$, then we obtain the conclusion $c\in C$.

Clearly in all that follows we may assume that a,b,c,q are all non-central elements of U.

Remark that, since

$$\Phi(x_1,x_2)=(ac-v)[x_1,x_2]+a[x_1,x_2]q+c[x_1,x_2]b+[x_1,x_2](qb-w)$$

is the zero element in T, then $\{(ac-v),a,c,1\}$ must be C-linearly dependent, and also $\{(qb-w),q,b,1\}$ must be C-linearly dependent.

We divide the rest of the proof into three steps:

- Suppose that $\{a,c,1\}$ is linearly C-independent. Since $\{(ac-v),a,c,1\}$ must be C-linearly dependent, there exist $\alpha,\beta,\gamma\in C$ such that $ac-v=\alpha a+\beta c+\gamma$. Hence R satisfies

  $$\Phi(x_1,x_2)=(\alpha a+\beta c+\gamma)[x_1,x_2]+a[x_1,x_2]q+c[x_1,x_2]b+[x_1,x_2](qb-w)$$

  that is

  $$\Phi(x_1,x_2)=a[x_1,x_2](\alpha+q)+c[x_1,x_2](\beta+b)+[x_1,x_2](qb-w+\gamma).$$

  This implies that $q=-\alpha\in C$, $b=-\beta\in C$, $w=qb+\gamma=\alpha\beta+\gamma\in C$ and we are done.
- Suppose now that $\{b,q,1\}$ is linearly C-independent.

Since $\{(qb-w),q,b,1\}$ must be C-linearly dependent, there exist $\alpha,\beta,\gamma\in C$ such that $qb-w=\alpha b+\beta q+\gamma$. Hence R satisfies

$$\Phi(x_1,x_2)=(ac-v)[x_1,x_2]+a[x_1,x_2]q+c[x_1,x_2]b+[x_1,x_2](\alpha b+\beta q+\gamma)$$

that is

$$\Phi(x_1,x_2)=(ac-v+\gamma)[x_1,x_2]+(a+\beta)[x_1,x_2]q+(c+\alpha)[x_1,x_2]b.$$

This implies that $c=-\alpha\in C$, $a=-\beta\in C$, $v=qb+\gamma=\alpha\beta+\gamma\in C$ and we are done again.

- Finally suppose that there exist $0 \neq \alpha \in C$, $0 \neq \beta \in C$ and $\gamma, \eta \in C$ such that $a = \alpha c + \gamma$, $b = \beta q + \eta$. In order to obtain the last conclusion of the Lemma, our aim is now to prove that $\alpha = -\beta$.

In this case R satsfies the generalized identity

$$(\alpha c^2 - v)[x_1, x_2] + c[x_1, x_2](\alpha q + \beta q + \gamma + \eta) + [x_1, x_2](\gamma q + \beta q^2 + \eta q - w).$$

Since $\Phi(x_1, x_2) = 0$ in T, then $\{\alpha c^2 - v, c, 1\}$ is linearly C-dependent and, since $c \notin C$, there exist $\lambda, \mu \in C$ such that $\alpha a c^2 - v = \lambda c + \mu$. Therefore R satisfies

$$c[x_1, x_2](\lambda + \alpha q + \beta q + \gamma + \eta) + [x_1, x_2](\gamma q + \beta q^2 + \eta q - w + \mu) = 0 \in T.$$

Hence $(\alpha + \beta)q + (\lambda + \gamma + \eta) = 0$. Since $q \notin C$, that is $\{q, 1\}$ is linearly C-independent, it follows $\lambda + \gamma + \eta = 0$ and $\alpha + \beta = 0$, as required. □

When R is a matrix algebra over the field K, then its Utumi quotient ring coincides with R. In this case we have the following consequence of Proposition 1.12.

Proposition 2.3

Let $R = M_m(K)$ be the algebra of $m \times m$ matrices over a field K with $m \geq 3$ and $char(K) \neq 2$. If there exist $a, b, c, q, v, w \in R$ such that $a(cs + sq) + (cs + sq)b = vs + sw$ for all $s \in [R, R]$, then one of the following holds:

1. *c and q are central matrices;*
2. *a and b are central matrices;*
3. *b, q and w are central matrices;*
4. *a, c and v are central matrices;*
5. *there exists $\alpha \in K$ such that $a + \alpha c$ and $b - \alpha q$ are central matrices.*

Proof. Let L be an infinite extension of K and let $\overline{R} = M_m(L) \cong R \otimes_K L$. Recall that any multilinear generalized polynomial is an identity for R if and only if it is an identity also for $\overline{R}$. As in the previous lemma, we consider the generalized polynomial

$$\Phi(x_1, x_2) = a\big(c[x_1, x_2] + [x_1, x_2]q\big) + \big(c[x_1, x_2] + [x_1, x_2]q\big)b - v[x_1, x_2] - [x_1, x_2]w$$

and we remark that $\Phi(x_1, x_2)$ is a generalized multilinear polynomial identity for R. Clearly the multilinear polynomial $\Phi(x_1, x_2)$ is a generalized polynomial identity for $\overline{R}$ too. We obtain $\Phi(r_1, r_2) = 0$, for all $r_1, r_2 \in \overline{R}$, and the conclusion follows from Proposition 1.12. □

In order to prove our final result in the inner case, we observe the following one, which is a reduced version of Hvala's theorem we recalled in the beginning of the paper in ([6, Theorem 1]):

Proposition 2.4

Let R be a prime ring with $char(R) \neq 2$. If there exist $a, b, c, q, v, w \in R$ such that $a(cs + sq) + (cs + sq)b = vs + sw$ for all $s \in R$, then one of the following holds:

1. *c and q are central matrices;*

2. *a and b are central matrices;*
3. *b, q and w are central matrices;*
4. *a, c and v are central matrices;*
5. *there exists $\alpha \in K$ such that $a + \alpha c$ and $b - \alpha q$ are central matrices.*

Proof. Let $F(x) = ax + xb$, $G(x) = cx + xq$ and $H(x) = vx + xw$ be generalized derivations of R. Our assumption is $FG = H$ in R. From [6, Theorem 1], one of the following possibilities holds:

1. there exists $\gamma \in C$ such that either $F(x) = ax + xb = \gamma x$ or $G(x) = cx + xq = \gamma x$. Thus either $(a - \gamma)x + xb = 0$ or $(c - \gamma)x + xq = 0$, for all $x \in R$. By Remark 1.3, either $a, b \in C$ and $a + b = \gamma$, or $c, q \in C$ and $c + q = \gamma$ (conclusions 1 and 2 of proposition).
2. there exist $p, u \in U$ such that $F(x) = xp$ and $G(x) = xu$. Hence $ax + xb = xp$ and $cx + xq = xu$, that is $ax + x(b - p) = 0$ and $cx + x(q - u) = 0$, for all $x \in R$. Also in this case we apply Remark 1.3 and obtain $a = p - b \in C$ and $c = u - q \in C$, moreover $H(x) = vx + xw = FG(x) = xup$ implies $v \in C$ (conclusion 4).
3. there exist $p, u \in U$ such that $F(x) = px$ and $G(x) = ux$, that is $ax + xb = px$ and $cx + xq = ux$. As above, by applying Remark 1.3, we obtain $b = p - a \in C$, $q = u - c \in C$ and $w \in C$ (conclusion 3).
4. there exist $\lambda, \mu \in C$ such that $G(x) = (\lambda + \mu a)x - x(\mu b)$. In this case $cx + xq = (\lambda + \mu a)x - x(\mu b)$, that is $c - \mu a - \lambda = -q - \mu b \in C$. If $\mu \neq 0$ we get the conclusion 5 of the proposition. On the other hand, in case $\mu = 0$ then $c - \lambda = -q \in C$, that is $c, q \in C$ and we obtain the conclusion 1.

□

Proposition 2.5

Let R be a prime ring with $char(R) \neq 2$. Assume that R does not embed in $M_2(L)$, the algebra of 2×2 matrices over a field L. If there exist $a, b, c, q, v, w \in R$ such that $a(cs + sq) + (cs + sq)b = vs + sw$ for all $s \in [R, R]$, then one of the following holds:

1. *c and q are central matrices;*
2. *a and b are central matrices;*
3. *b, q and w are central matrices;*
4. *a, c and v are central matrices;*
5. *there exists $\alpha \in K$ such that $a + \alpha c$ and $b - \alpha q$ are central matrices.*

Proof. We consider the generalized polynomial

$$\Phi(x_1, x_2) = a\big(c[x_1, x_2] + [x_1, x_2]q\big) + \big(c[x_1, x_2] + [x_1, x_2]q\big)b - v[x_1, x_2] - [x_1, x_2]w.$$

By Lemma 2.2 we may assume that $\Phi(x_1, x_2)$ is a non-trivial generalized polynomial identity for R. By a theorem due to Beidar in ([2, Theorem 2]) this generalized polynomial identity is also satisfied by the symmetric Martindale quotient ring Q of R. Let K be an algebraic closure of C. By [8, Theorem 1], either $\Phi(x) = a(cx + xq) + (cx + xq)b - vx - xw$ is a generalized polynomial identity for $Q \bigotimes_C K$, so in R, and we are finished by Proposition 2.3, or $\Phi(x_1, x_2)$ is an identity $Q \bigotimes_C K \cong M_m(K)$. In this last case the conclusion follows by Proposition 2.2. □

Before proving the main theorem of this paper, we need a well known result:

Remark 2.6 We would like to point out that in [9] Lee proves that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U. In particular Lee proves the following result:

In **[9, Theorem 3].** *Every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$, for some $a \in U$ and a derivation d on U.*

Finally we are able to prove our main result:

Theorem 2.7

Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R, F and G non-zero generalized derivations of R and $f(x_1, \ldots, x_n)$ a polynomial over C. Denote by $f(R)$ the set $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in R\}$ of all the evaluations of $f(x_1, \ldots, x_n)$ in R. Suppose that $f(x_1, \ldots, x_n)$ is not central valued on R. If R does not embed in $M_2(K)$, the algebra of 2×2 matrices over a field K, and the composition (FG) acts as a generalized derivation on the elements of $f(R)$, then (FG) is a generalized derivation of R and one of the following holds:

1. *there exists $\alpha \in C$ such that $F(x) = \alpha x$, for all $x \in R$;*
2. *there exists $\alpha \in C$ such that $G(x) = \alpha x$, for all $x \in R$;*
3. *there exist $a, b \in U$ such that $F(x) = ax$, $G(x) = bx$, for all $x \in R$;*
4. *there exist $a, b \in U$ such that $F(x) = xa$, $G(x) = xb$, for all $x \in R$;*
5. *there exist $a, b \in U$, $\alpha, \beta \in C$ such that $F(x) = ax + xb$, $G(x) = \alpha x + \beta(ax - xb)$, for all $x \in R$.*

Proof. Let S be the additive subgroup of R generated by the set $f(R)$. In [5] it is proved that, if characteristic of R is not 2 and $f(x_1, \ldots, x_n)$ is not central-valued on R, then S contains a non-central Lie ideal L of R. Moreover it is well known that, in case of characteristic different from 2, there exists a non-central ideal I of R such that $[I, R] \subseteq L$. Of course it is easy to see that if FG acts as a generalized derivation on $f(R)$, then it acts as a generalized derivation also on L and $[I, R]$. Therefore there exists a generalized derivation H of R such that $FG([r_1, r_2]) = H([r_1, r_2])$ for all $r_1, r_2 \in I$. As we said in Remark 2.6, we can write $F(x) = ax + d(x)$, $G(x) = bx + \delta(x)$ and $H(x) = cx + h(x)$, for suitable $a, b, c \in U$ and d, δ, h derivations of U. Therefore I satisfies the differential identity

$$a\big(b[x_1, x_2] + \delta([x_1, x_2])\big) + d\big(b[x_1, x_2] + \delta([x_1, x_2])\big) - c[x_1, x_2] - h\big([x_1, x_2]\big).$$

Since R and I satisfy the same differential identities (see [10]), then also R satisfies

$$\begin{aligned} a\big(b[x_1, x_2] + [\delta(x_1), x_2] + [x_1, \delta(x_2)]\big) + d(b)[x_1, x_2] \\ + b[d(x_1), x_2] + b[x_1, d(x_2)] + [d\delta(x_1), x_2] \\ + [\delta(x_1), d(x_2)] + [d(x_1), \delta(x_2)] + [x_1, d\delta(x_2)] \\ - c[x_1, x_2] - [h(x_1), x_2] - [x_1, h(x_2)]. \end{aligned} \tag{4}$$

First consider the case when $\{d, \delta, h\}$ is a set of linearly C-independent derivations modulo X-inner derivations (*i.e.* modulo the space of inner derivations of R). In light of Kharchenko's theory (see [7]) and starting from (4), R satisfies:

$$a\big(b[x_1,x_2]+[y_1,x_2]+[x_1,y_2]\big)+d(b)[x_1,x_2] \\ +b[z_1,x_2]+b[x_1,z_2]+[t_1,x_2]+[y_1,z_2]+[z_1,y_2]+[x_1,t_2] \\ -c[x_1,x_2]-[u_1,x_2]-[x_1,u_2]$$

in particular R satisfies the blended component

$$a[y_1,x_2]+a[x_1,y_2]+b[z_1,x_2]+b[x_1,z_2]+[t_1,x_2] \\ +[y_1,z_2]+[z_1,y_2]+[x_1,t_2]-[u_1,x_2]-[x_1,u_2]$$

and for $y_1 = y_2 = z_1 = z_2 = u_1 = u_2 = t_2 = 0$ we have the contradiction that R satisfies $[t_1, x_2]$, that is R should be commutative. This conclusion contradicts the assumption that $f(x_1, \ldots, x_n)$ is not central valued on R.

Hence we assume that $\{d, \delta, h\}$ is linearly C-dependent modulo X-inner derivations. In case d, δ and h are all inner derivations of U, then there exist $p, q, v \in U$ such that $d(x) = [p, x]$, $\delta(x) = [q, x]$, $h(x) = [v, x]$. Hence

$$FG(x) = (a+p)\left((b+q)x + x(-q)\right) + \left((b+q)x + x(-q)\right)(-p)$$

and

$$H(x) = (c+v)x + x(-v).$$

Since $FG([r_1, r_2]) = H([r_1, r_2])$ for all $r_1, r_2 \in R$, by Proposition 2.5 we have that one of the following holds:

$b, q \in C$ and $G(x) = bx$;

$a, p \in C$ and $F(x) = ax$;

$p, q, v \in C$ and $F(x) = ax$, $G(x) = bx$, $H(x) = cx$;

$(a+p), (b+q), (c+v) \in C$ and $F(x) = xa$, $G(x) = xb$, $H(x) = xc$;

there exists $\alpha \in C$ such that $(a+p) + \alpha(b+q) = \eta \in C$ and $(-p) - \alpha(-q) = \lambda \in C$, for suitable $\eta, \lambda \in C$. In this case it follows that: $F(x) = a'x + xb'$, where $a' = a + p$ and $b' = -p$; $G(x) = \mu(a'x - xb') + \nu x$, where $\mu = -\alpha^{-1}$ and $\nu = \alpha^{-1}(\eta - \lambda)$.

In any case we are done.

In light of previous argument, here we may assume that there exist $\alpha, \beta, \gamma \in C$ such that

$$\alpha d + \beta\delta + \gamma h = ad(p)$$

the inner derivation induced by some element $p \in U$, moreover at least one of $\{d, \delta, h\}$ is not an inner derivation.

If $\{\delta, h\}$ is linearly C-independent modulo X-inner derivations, then $\alpha \neq 0$ and d cannot be an inner derivation, and so at least one of β and γ is not zero. Thus we write $d = \beta'\delta + \gamma' h + ad(p)$, for $\beta' = \alpha^{-1}\beta$, $\gamma' = \alpha^{-1}\gamma$. Starting from (4), R satisfies

$$\begin{aligned} &a\big(b[x_1,x_2] + [\delta(x_1),x_2] + [x_1,\delta(x_2)]\big) \\ &\quad + \big(\beta'\delta + \gamma' h + ad(p)\big)(b)[x_1,x_2] + b[(\beta'\delta + \gamma' h + ad(p))(x_1),x_2] \\ &\quad + b[x_1,\big(\beta'\delta + \gamma' h + ad(p)\big)(x_2)] + [\big(\beta'\delta^2 + \gamma' h\delta + ad(p)\delta\big)(x_1),x_2] \\ &\quad + [\delta(x_1),\big(\beta'\delta + \gamma' h + ad(p)\big)(x_2)] + [\big(\beta'\delta + \gamma' h + ad(p)\big)(x_1),\delta(x_2)] \\ &\quad - [x_1,\big(\beta'\delta^2 + \gamma' h\delta + ad(p)\delta\big)(x_2)] - c[x_1,x_2] - [h(x_1),x_2] - [x_1,h(x_2)]. \end{aligned}$$

By Kharchenko's theory R satisfies

$$\begin{aligned} &\big(b[x_1,x_2] + [y_1,x_2] + [x_1,y_2]\big) \\ &\quad + \big(\beta'\delta + \gamma' h + ad(p)\big)(b)[x_1,x_2] + b\big[\beta' y_1 + \gamma' z_1 + [p,x_1],x_2\big] \\ &\quad + b\big[x_1,\beta' y_2 + \gamma' z_2 + [p,x_2]\big] + \big[\beta' t_1 + \gamma' u_1 + [p,y_1],x_2\big] \\ &\quad + \big[y_1,\beta' y_2 + \gamma' z_2 + [p,x_2]\big] + \big[\beta' y_1 + \gamma' z_1 + [p,x_1],y_2\big] \\ &\quad + \big[x_1,\beta' t_2 + \gamma' u_2 + [p,y_2]\big] - c[x_1,x_2] - [z_1,x_2] - [x_1,z_2]. \end{aligned}$$

In particular, for $x_2 = y_2 = z_2 = 0$, R satisfies $[x_1, \beta' t_2 + \gamma' u_2]$, which forces R to be commutative, since either $\beta' \neq 0$ or $\gamma' \neq 0$, a contradiction.

Consider now the case when there exist $\lambda,\mu \in C$, not both zero, such that

$$\lambda\delta + \mu h = ad(q)$$

for some $q \in U$. We will prove that the last assumption implies a number of contradictions. We divide the proof into three cases:

The case $\lambda = 0$.

For $\lambda = 0$, we have $\mu \neq 0$ and $h = ad(\mu^{-1}q)$, the inner derivation induced by $\mu^{-1}q$. It follows that $\alpha d + \beta\delta = ad(p - \gamma\mu^{-1}q)$, with $\alpha \neq 0$ and $\beta \neq 0$, since at least one of δ, d and h must be not inner.

Then $\delta = \alpha' d + \beta' ad(p')$, for $p' = p - \gamma\mu^{-1}q$ and $0 \neq \alpha' = -\beta^{-1}\alpha$, $0 \neq \beta' = \beta^{-1}$. By (4) R satisfies:

$$\begin{aligned} &a\big(b[x_1,x_2] + [\alpha' d(x_1) + \beta'[p',x_1],x_2] + [x_1,\alpha' d(x_2) + \beta'[p',x_2]]\big) \\ &\quad + d(b)[x_1,x_2] + b[d(x_1),x_2] + b[x_1,d(x_2)] \\ &\quad + [d(\alpha')d(x_1) + \alpha' d^2(x_1) + d(\beta')[p,x_1] + \beta'[d(p'),x_1] + \beta'[p',d(x_1)],x_2] \\ &\quad + [\alpha' d(x_1) + \beta'[p',x_1],d(x_2)] + [d(x_1),\alpha' d(x_2) + \beta'[p',x_2]] \\ &\quad + [x_1,d(\alpha')d(x_2) + \alpha' d^2(x_2) + d(\beta')[p,x_2] + \beta'[d(p'),x_2] + \beta'[p',d(x_2)]] \\ &\quad - c[x_1,x_2] - [[q',x_1],x_2] - [x_1,[q',x_2]]. \end{aligned}$$

In this case, Kharchenko's result implies that R satisfies

$$\begin{aligned} &a\big(b[x_1,x_2] + [\alpha' y_1 + \beta'[p',x_1],x_2] + [x_1,\alpha' y_2 + \beta'[p',x_2]]\big) \\ &\quad + d(b)[x_1,x_2] + b[y_1,x_2] + b[x_1,y_2] \\ &\quad + [d(\alpha')y_1 + \alpha' z_1 + d(\beta')[p,x_1] + \beta'[d(p'),x_1] + \beta'[p',y_1],x_2] \\ &\quad + [\alpha' y_1 + \beta'[p',x_1],y_2] + [y_1,\alpha' y_2 + \beta'[p',x_2]] \\ &\quad + [x_1,d(\alpha')y_2 + \alpha' z_2 + d(\beta')[p,x_2] + \beta'[d(p'),x_2] + \beta'[p',y_2]] \\ &\quad - c[x_1,x_2] - [[q',x_1],x_2] - [x_1,[q',x_2]] \end{aligned}$$

in particular R satisfies the blended component $\alpha'[x_1, z_2]$, a contradiction again.

The case $\lambda \neq 0$ and $\mu = 0$.

In this case $\delta = \lambda^{-1}ad(q) = ad(v)$, for $v = \lambda^{-1}q$.

Suppose first that $\{d, h\}$ is linearly C-independent modulo X-inner derivations. By (4) it follows that R satisfies

$$\begin{aligned} a\big(b[x_1,x_2] &+ [[v,x_1],x_2] + [x_1,[v,x_2]]\big) \\ &+ d(b)[x_1,x_2] + b[d(x_1),x_2] + b[x_1,d(x_2)] + [[d(v),x_1] + [v,d(x_1)],x_2] \\ &+ [[v,x_1],d(x_2)] + [d(x_1),[v,x_2]] + [x_1,[d(v),x_2] + [v,d(x_2)]] \\ &- c[x_1,x_2] - [h(x_1),x_2] - [x_1,h(x_2)] \end{aligned}$$

and using Kharchenko's theorem, R satisfies

$$\begin{aligned} a\big(b[x_1,x_2] &+ [[v,x_1],x_2] + [x_1,[v,x_2]]\big) \\ &+ d(b)[x_1,x_2] + b[y_1,x_2] + b[x_1,y_2] + [[d(v),x_1] + [v,y_1],x_2] \\ &+ [[v,x_1],y_2] + [y_1,[v,x_2]] + [x_1,[d(v),x_2] + [v,y_2]] \\ &- c[x_1,x_2] - [z_1,x_2] - [x_1,z_2] \end{aligned}$$

and in particular R satisfies the blended component $[z_1, x_2]$, a contradiction.

In the case $\{d, h\}$ is linearly C-dependent modulo X-inner derivations, there are $\eta_1, \eta_2 \in C$ and $w \in U$ such that $\eta_1 d + \eta_2 h = ad(w)$, the inner derivation induced by w. Of course both d and h are outer derivations, moreover at least one of η_1 and η_2 must be non-zero. Without loss of generality, say $\eta_1 \neq 0$. So we may write $d = \eta_1^{-1}(-\eta_2 h + ad(w)) = \eta h + ad(u)$, for $\eta = -\eta_1^{-1}\eta_2$ and $u = \eta_1^{-1}w$. Hence $d\delta(x) = [\eta h(v), x] + [v, \eta h(x)] + [u, [v, x]]$. So by (4), R satisfies

$$\begin{aligned} a\big(b[x_1,x_2] &+ [[v,x_1],x_2] + [x_1,[v,x_2]]\big) \\ &+ (\eta h(b) + [u,b])[x_1,x_2] + b[\eta h(x_1) + [u,x_1],x_2] + b[x_1,\eta h(x_2) + [u,x_2]] \\ &+ [[\eta h(v),x_1] + [v,\eta h(x_1)] + [u,[v,x_1]],x_2] \\ &+ [[v,x_1],\eta h(x_2) + [u,x_2]] + [\eta h(x_1) + [u,x_1],[v,x_2]] \\ &+ [x_1,[\eta h(v),x_2] + [v,\eta h(x_2)] + [u,[v,x_2]]] \\ &- c[x_1,x_2] - [h(x_1),x_2] - [x_1,h(x_2)] \end{aligned}$$

and again by Kharchenko's result, R satisfies

$$\begin{aligned} a\big(b[x_1,x_2] &+ [[v,x_1],x_2] + [x_1,[v,x_2]]\big) \\ &+ (\eta h(b) + [u,b])[x_1,x_2] + b[\eta y_1 + [u,x_1],x_2] + b[x_1,\eta y_2 + [u,x_2]] \\ &+ [[\eta h(v),x_1] + [v,\eta y_1] + [u,[v,x_1]],x_2] \\ &+ [[v,x_1],\eta y_2 + [u,x_2]] + [\eta y_1 + [u,x_1],[v,x_2]] \\ &+ [x_1,[\eta h(v),x_2] + [v,\eta y_2] + [u,[v,x_2]]] \\ &- c[x_1,x_2] - [y_1,x_2] - [x_1,y_2]. \end{aligned}$$

From this last, R satisfies

$$b[x_1,\eta y_2] + [[v,x_1],\eta y_2] + [x_1,[v,\eta y_2]] - [x_1,y_2]$$

which is

$$b[x_1, \eta y_2] + [\eta v, [x_1, y_2]] - [x_1, y_2].$$

For $\eta = 0$ we have that R satisfies $[x_1, y_2]$ that is R is commutative, a contradiction. Assume $\eta \neq 0$ and denote by H the following generalized derivation of R: $H(x) = (\eta G)(x) = (\eta b)x + [\eta v, x]$ for all $x \in R$. Therefore $[H(u), u] = 0$ for all $u \in [R, R]$. By [1, Theorem 1] either both $v \in C$ and $b \in C$ and we obtain conclusion 2 of the Theorem; or R satisfies the standard identity $s_4(x_1, \dots, x_4)$, that is $U = M_2(C)$, and there exists $\gamma \in C$ such that $b = -2v + \gamma$. In this last case, by calculations it follows that R and U satisfy the identity $\eta v[x_1, x_2] + [x_1, x_2]\eta v + \eta\gamma[x_1, x_2] - [x_1, x_2]$ Now choose $x_1 = e_{ij}, x_2 = e_{jj}$ and multiply on the right by e_{ii}, for $i \neq j$ and $i, j \in \{1, 2\}$. We get $e_{ij}\eta v e_{ii} = 0$, which means that v is a diagonal matrix in $M_2(C)$. As in Lemma 1.11, standard argument shows that v is a central matrix, as well as b. Also in this case we are done (conclusion 2).

The case $\lambda \neq 0$ and $\mu \neq 0$.

In this case we may write $\delta = \mu' h + \lambda' ad(q)$, with $\mu' = -\lambda^{-1}\mu \neq 0$, $\lambda' = \lambda^{-1} \neq 0$. Moreover we may consider h as an outer derivation of R; in fact if h is an inner derivation, then also d and δ should be inner.

Hence $\alpha d + \beta\mu' h + \beta\lambda' ad(q) + \gamma h = ad(p)$, with $\alpha \neq 0$ and $d \neq 0$, since h is not inner. Also here we show that a number of contradictions follows.

Write $d = \beta' h + \beta'' ad(c)$, for $\beta' = -\alpha^{-1}(\beta\mu' + \gamma)$, $\beta'' = \alpha^{-1} \neq 0$ and $c = p - \beta\lambda' q$. By (4), R satisfies

$$\begin{aligned}
ab[x_1, x_2] &+ a[\mu' h(x_1) + \lambda'[q, x_1], x_2] + a[x_1, \mu' h(x_2) + \lambda'[q, x_2]] \\
&+ (\beta' h(b) + \beta''[c, b])[x_1, x_2] + b[\beta' h(x_1) + \beta''[c, x_1], x_2] \\
&+ b[x_1, \beta' h(x_2) + \beta''[c, x_2]] \\
&+ \Big[\beta' h(\mu') h(x_1) + \beta'\mu' h^2(x_1) + \beta' h(\lambda')[q, x_1] + \beta'\lambda'[h(q), x_1] \\
&+ \beta'\lambda'[q, h(x_1)] + [c, \beta''\mu' h(x_1)] + [c, \beta''\lambda'[q, x_1]], x_2\Big] \\
&+ [\mu' h(x_1) + \lambda'[q, x_1], \beta' h(x_2) + \beta''[c, x_2]] \\
&+ [\beta' h(x_1) + \beta''[c, x_1], \mu' h(x_2) + \lambda'[q, x_2]] \\
&+ \Big[x_1, \beta' h(\mu') h(x_2) + \beta'\mu' h^2(x_2) + \beta' h(\lambda')[q, x_2] + \beta'\lambda'[h(q), x_2] \\
&+ \beta'\lambda'[q, h(x_2)] + [c, \beta''\mu' h(x_2)] + [c, \beta''\lambda'[q, x_2]]\Big] \\
&- c[x_1, x_2] - [h(x_1), x_2] - [x_1, h(x_2)]
\end{aligned}$$

and since h is outer, R satisfies

$$
\begin{aligned}
ab[x_1,x_2] &+ a[\mu' y_1 + \lambda'[q,x_1],x_2] + a[x_1,\mu' y_2 + \lambda'[q,x_2]] \\
&+ (\beta' h(b) + \beta''[c,b])[x_1,x_2] + b[\beta' y_1 + \beta''[c,x_1],x_2] \\
&+ b[x_1,\beta' y_2 + \beta''[c,x_2]] \\
&+ \Big[\beta' h(\mu')y_1 + \beta'\mu' z_1 + \beta' h(\lambda')[q,x_1] + \beta'\lambda'[h(q),x_1] \\
&+ \beta'\lambda'[q,y_1] + [c,\beta''\mu' y_1] + [c,\beta''\lambda'[q,x_1]],x_2\Big] \\
&+ [\mu' y_1 + \lambda'[q,x_1],\beta' y_2 + \beta''[c,x_2]] + [\beta' y_1 + \beta''[c,x_1],\mu' y_2 + \lambda'[q,x_2]] \\
&+ \Big[x_1,\beta' h(\mu')y_2 + \beta'\mu' z_2 + \beta'\lambda'[h(q),x_2] \\
&+ \beta'\lambda'[q,y_2] + [c,\beta''\mu' y_2] + \big[c,\beta''\lambda'[q,x_2]\big]\Big] \\
&- c[x_1,x_2] - [y_1,x_2] - [x_1,y_2]
\end{aligned} \tag{5}
$$

in particular R satisfies the component $\beta'\mu'[z_1,x_2]$, which is a contradiction unless when $\beta' = 0$.

In case $\beta' = 0$, we write (5) as follows

$$
\begin{aligned}
ab[x_1,x_2] &+ a\big[\mu' y_1 + \lambda'[q,x_1],x_2\big] + a\big[x_1,\mu' y_2 + \lambda'[q,x_2]\big] \\
&+ \beta''[c,b][x_1,x_2] + b\big[\beta''[c,x_1],x_2\big] + b\big[x_1,\beta''[c,x_2]\big] \\
&+ \big[[c,\beta''\mu' y_1] + [c,\beta''\lambda'[q,x_1]],x_2\big] \\
&+ \big[\mu' y_1 + \lambda'[q,x_1],\beta''[c,x_2]\big] + \big[\beta''[c,x_1],\mu' y_2 + \lambda'[q,x_2]\big] \\
&+ \big[x_1,[c,\beta''\mu' y_2] + [c,\beta''\lambda'[q,x_2]]\big] \\
&- c[x_1,x_2] - [y_1,x_2] - [x_1,y_2]
\end{aligned}
$$

and R satisfies the component

$$
\begin{aligned}
a[\mu' y_1,x_2] + a[x_1,\mu' y_2] &+ \big[[c,\beta''\mu' y_1],x_2\big] + \big[x_1,[c,\beta''\mu' y_2]\big] \\
&- c[x_1,x_2] - [y_1,x_2] - [x_1,y_2].
\end{aligned}
$$

For $y_1 = x_2$ and $y_2 = x_1 = 0$ it follows that R satisfies $\beta''\mu'([c,y_1]_2$, which implies $[c,x]_2 = 0$, for all $x \in R$, since $\mu' \neq 0$and $\beta'' \neq 0$. Denote by $\varphi = ad(c)$ the inner derivation of R induced by c. Hence $[\varphi(x),x] = 0$ for all $x \in R$, thus by Posner's result in [11] it follows $c \in C$. Therefore, since $\beta' = 0$ and c is central, it follows $d = 0$, which is a contradiction again. □

References

1. N. Argac, L. Carini, and V. De Filippis, An Engel condition with generalized derivations on Lie ideals, *Taiwanese J. Math.* **12** (2008), 419–433.
2. K.I. Beidar, Rings with generalized identities, *Moscow Univ. Math. Bull.* **33** (1978), 53–58.
3. K.I. Beidar, W.S. Martindale, and A.V. Mikhalev, *Rings with Generalized Identities*, Monographs and Textbooks in Pure and Applied Mathematics **196**, Marcel Dekker, Inc., New York, 1996.

4. C.L. Chuang, GPIs having coefficients in Utumi quotient rings, *Proc. Amer. Math. Soc.* **103** (1988), 723–728.
5. C.L. Chuang, The additive subgroup generated by a polynomial, *Israel J. Math.* **59** (1987), 98–106.
6. B. Hvala, Generalized derivations in rings, *Comm. Algebra* **26** (1998), 1147–1166.
7. V.K. Kharchenko, Differential identities of prime rings, *Algebra i Logika* **17** (1978), 155–168.
8. C. Lanski, Differential identities, Lie ideals and Posner's theorems, *Pacific J. Math.* **134** (1988), 275–297.
9. T.K. Lee, Generalized derivations of left faithful rings, *Comm. Algebra* **27** (1999), 4057–4073.
10. T.K. Lee, Semiprime rings with differential identities, *Bull. Inst. Math. Acad. Sinica* **20** (1992), 27–38.
11. E.C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.* **8** (1957), 1093–1100.