Finite-dimensional Lie superalgebras of Cartan type over fields of prime characteristic

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IN this note the infinite-dimensional Lie superalgebras of Cartan type $X(m, n)$ $(X = W, S, H$ or $K)$ over field F of prime characteristic are constructed. Then the second class of finite-dimensional Lie superalgebras of Cartan type over F is defined. Their simplicity and restrictability are discussed. Finally a conjecture about classification of the finite-dimensional simple Lie superalgebras over F is given.

Let F be a field and $\mathrm{char}F = p > 2$. Let n be a positive integer and $n > 1$. $\Lambda(n)$ denotes the exterior algebra over F with generators $\xi_1, \cdots, \xi_n$. If $u = (i_1, i_2, \cdots, i_r)$, where $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n$, then we set $\xi^u = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_r}$, and $|\xi^u| = r$. Let $\bar{D}_i = \frac{\partial}{\partial \xi_i}$, $i = 1, \cdots, n$.

Imitating the situation that the characteristic number of basic field is zero, we can get the finite-dimensional simple Lie superalgebra of Cartan type over $F^{[1]}$:

$W(n)=\left|\sum_{i=1}^{n}a_i\bar{D}_i \mid a_i\in \Lambda(n)\right|$; $S(n)=\langle D_{ij}(a)\mid a\in\Lambda(n),\ i,j=1,\cdots n\rangle$, where $D_{ij}\cdot(a)=\bar{D}_i(a)\bar{D}_j+\bar{D}_j(a)\bar{D}_i$, ; $H(n)=\langle D_H(\xi^u)\mid |\xi^u|\leqslant n-1\rangle$, where $D_H(\xi^u)=\sum_{i=1}^{n}\bar{D}_i(\xi^u)\cdot\bar{D}_i$; $\tilde{S}(n)=\langle(1-\xi_1\xi_2\cdots\xi_n)D_{ij}(a)\mid a\in\Lambda(n),\ i,j=1,\cdots,n\rangle$, where n is a positive even number.

Let $L=L_{\bar{0}}\oplus L_{\bar{1}}$ be a Lie superalgebra over F. If $L_{\bar{0}}$ is a restricted Lie algebra and $L_{\bar{0}}$-module $L_{\bar{1}}$ is restricted, then L is a restricted Lie superalgebra[2].

Theorem 1. *$W(n)$, $S(n)$, $H(n)$ and $\tilde{S}(n)$ are restricted Lie superalgebras.*

Proof. Because $W(n)=\operatorname{der}\Lambda(n)$, $W(n)$ is restricted. Let $D_{ij}(\xi^u)\in S(n)_{\bar{0}}$. If $|\xi^u|>2$, then we get $(D_{ij}(\xi^u))^2=0$, by direct inspection. So $(D_{ij}(\xi^u))^p=0$. If $|\xi^u|=2$, we can suppose $\xi^u=\xi_i\xi_j$. Then $(D_{ij}(\xi^u))^p=(\xi_i\bar{D}_j-\xi_j\bar{D}_i)^p=\xi_i\bar{D}_j-\xi_j\bar{D}_i=D_{ij}(\xi^u)\in S(n)_{\bar{0}}$. If $\{k,r\}\neq\{i,j\}$, then $(D_{kr}(\xi^u))^2=0$. So $(D_{kr}(\xi^u))^p=0$.

Let c be any element of $S_{\bar{0}}$. Because $\deg c=\bar{0}$, $(\operatorname{ad}c)^p=\operatorname{ad}c^p$. By the above proof we know that $c^p\in S(n)_{\bar{0}}$. So $S(n)_{\bar{0}}$ is a restricted Lie algebra by ref. [3]. Let $c\in S(n)_{\bar{0}}$. Because $(\operatorname{ad}c)^p=\operatorname{ad}c^p$, $S(n)_{\bar{0}}$-module $S(n)_{\bar{1}}$ is restricted. Then $S(n)$ is a restricted Lie superalgebra. Similarly we can prove that $H(n)$ and $\tilde{S}(n)$ are restricted.

We call $W(n)$, $S(n)$, $H(n)$ and $\tilde{S}(n)$ Lie superalgebras of rigid Cartan type.

Let $\mathbb{N}$ be the set of natural numbers and Let N_0 be the set of nongenative integers. Let $s=m+n$, where $m,\ n\in\mathbb{N}$ and $m,\ n>1$. If $\delta=(\delta_1,\cdots,\delta_m)\in N_0^m$, $\eta=(\eta_1,\cdots,\eta_m)\in N_0^m$, then we can let $\binom{\delta+\eta}{\delta}=\prod_{i=1}^{m}\binom{\delta_i+\eta_i}{\delta_i}$. Let $\Lambda(m,n)$ be the F-algebra with generators $\{x^\delta\mid\delta\in N_0^m\}\cup\{\xi_i\mid i=m+1,\cdots,s\}$ defining relation

$$x^\delta x^\eta=\binom{\delta+\eta}{\delta}x^{\delta+\eta},\quad x^\delta\xi_i=\xi_i x^\delta,\quad \xi_i\xi_j=-\xi_j\xi_i,$$

where $\delta,\ \eta\in N_0^m$, $i,\ j=m+1,\cdots,s$. Let $\deg x^\delta=\bar{0}$, $\forall\,\delta\in N_0^m$, $\deg\xi_i=\bar{1}$, $i=m+1,\cdots,s$. Then $\Lambda(m,n)$ is an F-superalgebra. We write $x^{\varepsilon_i}=x_i$, $i=1,\cdots,m$, where $\varepsilon_i=(\delta_{i_1},\ \delta_{i_2},\cdots,\delta_{i_m})$.

Let $B(n)=\bigcup_{r=0}^{n}B_r$, where $B_0=\{0\}$, $B_r=\{(i_1,\cdots,i_r)\mid m+1\leqslant i_1<\cdots<i_r\leqslant s\}$. If $u=(i_1,\cdots,i_r)\in B_r$, then ξ^u denotes the elements $\xi_{i_1},\cdots,\xi_{i_r}$. Assume that $\xi^0=1$. Then $\{x^\delta\cdot\xi^u\mid\delta\in N_0^m,\ u\in B(n)\}$ consists of an F-basis of $\Lambda(m,n)$.

Let $\tau:\{1,2,\cdots,s\}\to Z_2$ be a mapping such that $\tau(i)=\bar{0}$ $(1\leqslant i\leqslant m)$, $\tau(i)=\bar{1}$ $(m<i\leqslant s)$. Suppose that D_i are the linear mappings, $i=1,\cdots,s$ such that

$$D_i(x^\delta\xi^u)=\begin{cases}x^{\delta-\varepsilon_i}\xi^u, & i=1,\cdots,m,\\ x^\delta\dfrac{\partial\xi^u}{\partial\xi_i}, & i=m+1,\cdots,s.\end{cases}$$

Then D_i is a superderivation of $\Lambda(m,n)$ and $\deg D_i=\tau(i)$, $i=1,\cdots,s$. Furthermore

$$D_iD_j-(-1)^{\tau(i)\tau(j)}D_jD_i=[D_i,D_j]=0,\ 1\leqslant i,j\leqslant s. \tag{1}$$

By direct inspection we know that $\overline{W}(m,n)\stackrel{d}{=}\left|\sum_{i=1}^{s}a_iD_i\mid a_i\in\Lambda(m,n)\right|$ is an infinite-dimensional subalgebra of Lie superalgebra der($\Lambda(m,n)$).

If $\deg x$ occurs in some expression, then it is assumed that x is a homogeneous element about Z_2-grading. Using equality (1) we obtain the following equality:

$$[aD_i, bD_j] = aD_i(b)D_j - (-1)^{\deg(aD_i)\deg(bD_j)} bD_j(a)D_i. \quad (2)$$

Let $1 \leqslant i,\ j \leqslant s,\ a \in \Lambda(m,n)$. Suppose that $D_{ij}:\ \Lambda(m,n) \to \overline{W}(m,n)$ is a linear mapping such that $D_{ij}(a) = a_iD_i + a_jD_j$, where $a_i = -(-1)^{\deg a(\tau(i)+\tau(j))} D_j(a), a_j = (-1)^{\tau(i)\tau(j)} D_i(a)$. Let

$$\overline{S}(m,n) = \langle D_{ij}(a) \mid a \in \Lambda(m,n),\ i,j = 1,\cdots,S \rangle.$$

Using equality (2) we can prove the following lemmas.

Lemma 1. *Let* $D_{ij}(a) = a_iD_i + a_jD_j \in \overline{S}(m,\ n)_\alpha,\ D_{kt}(b) = b_kD_k + b_tD_t \in \overline{S}(m,\ n)_\beta$, *where* $\alpha, \beta \in Z_2$. *Let* $\lambda_{rh} = (-1)^{\tau(r)(\alpha + \tau(r) + \tau(h))}$, *where* $r \in \{i,\ j\},\ h \in \{k,\ t\}$. *Then*

$$[D_{ij}(a),\ D_{kt}(b)] = \lambda_{ik}D_{ik}(a_ib_k) + \lambda_{it}D_{it}(a_ib_t) + \lambda_{jk}D_{jk}(a_jb_k) + \lambda_{jt}D_{jt}(a_jb_t).$$

Let $m = 2k$ be an even number. And let

$$i' = \begin{cases} i+k, & \text{if } 1 \leqslant i \leqslant k, \\ i-k, & \text{if } k < i \leqslant 2k, \\ i, & \text{if } 2k < i \leqslant s, \end{cases} \quad \sigma(i) = \begin{cases} 1, & \text{if } 1 \leqslant i \leqslant k, \\ -1, & \text{if } k < i \leqslant 2k, \\ 1, & \text{if } 2k < i \leqslant s. \end{cases} \quad (3)$$

Suppose that $a \in \Lambda(m,\ n)_\alpha, \alpha \in Z_2$. Let $D_H:\ \Lambda(m,n) \to \overline{W}(m,\ n)$ be a linear mapping such that $D_H(a) = \sum_{i=1}^{s} a_iD_i$, where $a_i = \sigma(i')(-1)^{\tau(i')\alpha} D_{i'}(a),\ i = 1,\cdots,s$. Then $\deg a_i = \alpha + \tau(i)$. By direct inspection we have the equality:

$$D_i(a_{j'}) = (-1)^{\tau(i)\tau(j) + (\tau(i)+\tau(j))\alpha} \sigma(i)\sigma(j) D_j(a_{i'}), \quad (4)$$

where $i,\ j = 1,\ \cdots s$, Let $\overline{H}(m,n) = \langle D_H(a) \mid a \in \Lambda(m,n) \rangle$. Using equalities (2)—(4) and by calculation we can prove

Lemma 2. *Let* $D_H(a) = \sum_{i=1}^{s} a_iD_i \in \overline{H}(m,n)_\alpha,\ D_H(b) = \sum_{i=1}^{s} b_iD_i \in \overline{H}(m,n)_\beta$, *where* $\alpha,\ \beta \in Z_2$. *Then*

$$[D_H(a), D_H(b)] = D_H \sum_{i=1}^{s} \sigma(i)(-1)^{\tau(i)\beta} a_ib_{i'}.$$

Let $m = 2k+1$ be an odd number. The definitions of i' and $\sigma(i)$ are the same as equality (3), where $1 \leqslant i \leqslant s$. Let $a \in \Lambda(m,n)_\alpha, \alpha \in Z_2$. Let $\widetilde{D}_k:\ \Lambda(m,n) \to \overline{W}(m,n)$ be a linear mapping such that

$$\widetilde{D}_k(a) = \sum_{i=1}^{2k} (x_iD_m(a) + \sigma(i')D_{i'}(a))D_i + \sum_{i=m+1}^{s} (-1)^\alpha (\xi_iD_m(a) + D_i(a))D_i + \left(2a - \sum_{i=m+1}^{s} \xi_iD_i - \sum_{i=1}^{2k} x_iD_i(a)\right)D_m.$$

Lemma 3. *Let* $a \in \Lambda(m,n)_\alpha,\ b \in \Lambda(m,n)_\beta, \alpha, \beta \in Z_2$. *Then* $[\widetilde{D}_k(a),\ \widetilde{D}_k(b)] = \widetilde{D}_k(\widetilde{D}_k(a)(b) - 2D_m(a)b)$.

Let $\overline{K}(m,n) = \langle \widetilde{D}_k(a) \mid a \in \Lambda(m,n) \rangle$. By lemmas 1—3 we have

Theorem 2. $\overline{S}(m,n)$, $\overline{H}(m,n)$ *and* $\overline{K}(m,\ n)$ *are infinite-dimensional subalgebras of* $\overline{W}(m,\ n)$.

Equalities $|x_i| = |\xi_j| = 1\ (i = 1,\ \cdots,\ m,\ j = m+1,\ \cdots,\ s)$, $|D_i| = -1\ (i = 1,\ \cdots,\ s)$ define a Z-grading of $\overline{X}(m,\ n)$: $\overline{X}(m,\ n) = \bigoplus_{i \geqslant -1} \overline{X}(m,n)_{[i]}$, where $X = W$, S or H. It induces a filtration $\{\overline{X}(m,\ n)_i\}_{i \geqslant -1}$ of $\overline{X}(m,\ n)$. Equalities $|x_i| = |\xi_j| = 1$, $|D_i| =$

$-1\ (i \neq m)$, $|x_m| = 2$, $|D_m| = -2$ define a Z-grading of $\overline{K}(m, n)$. It induces a filtration $\{\overline{K}(m, n)_i\}_{i \geqslant -2}$ of $K(m, n)$.

Suppose that $t = (t_1, \cdots, t_m) \in N^m$. Let $\boldsymbol{q} = (q_1, \cdots, q_m)$, where $q_i = p^{t_i} - 1$. Then subspace $\langle x^{\delta}\xi^u \mid 0 \leqslant \delta^i \leqslant q_i, u \in B(n)\rangle$ is a subalgebra of $\Lambda(m, n)$. This subalgebra is denoted by $\Lambda(m, n, t)$. Let

$$\overline{W}(m, n, t) = \left\{\sum_{i=1}^{s} a_i D_i \mid a_i \in \Lambda(m, n, t),\ i = 1, \cdots, s\right\}.$$

Then $\overline{W}(m, n, t)$ is a finite-dimensional subalgebra of $\overline{W}(m, n)$.

Suppose that L is a Lie superalgebra, $i \in N_0$. Then $L^{(i)}$ denotes the derived algebra of i degree of L. Let $L^{(\infty)} = \bigcap_{i \geqslant 0} L^{(i)}$.

Suppose that φ is an automorphism of $\overline{W}(m, n)$. Let

$$\overline{X}(m, n, t, \varphi) = \varphi(\overline{X}(m, n)) \cap \overline{W}(m, n, t),$$

where X denote W, S, H or K. Obviously the filtration of $\overline{X}(m, n)$ induces a filtration $\{\overline{X}(m, n, t, \varphi)_i\}_{i \geqslant -(1+\delta_{XK})}$.

Definition 1. If $\overline{X}(m, n, t, \varphi)_{2+\delta_{XK}} \neq 0$, and

$$\overline{X}(m, n, t, \varphi) + \varphi(\overline{X}(m, n)_{1+\delta_{XK}}) = \varphi(\overline{x}(m, n)),$$

then we call $\overline{X}(m, n, t, \varphi)^{(\infty)}$ the Lie superalgebras of generalized Cartan type.

Because the proof and results in sec. 2 of ref. [4] hold for Lie superalgebras, we can prove the following.

Theorem 3. *Suppose that $X = W$, S, H or K. Then the finite-demensional Lie superalgebras $\overline{X}(m, n, t, \varphi)^{(\infty)}$ are simple.*

If φ is the identical automorphism, then we write $\overline{X}(m, n, t)$ instead of $\overline{X}(m, n, t, \varphi)$. Then $\overline{X}(m, n, t) = \overline{X}(m, n) \cap \overline{W}(m, n, t)$.

Let $\lambda = \sum_{i=1}^{m} q_i + n$, Then $\Lambda(m, n, t) = \bigoplus_{i=1}^{\lambda} \Lambda(m, n, t)_{[i]}$. Let $W(m, n, t) = \overline{W}(m, n, t)$; $S(m, n, t) = \langle D_{ij}(a) \mid a \in \Lambda(m, n, t),\ i, j = 1, \cdots, s\rangle$; $H(m, n, t) = \langle D_H(a) \mid a \in \bigoplus_{i=1}^{\lambda-1} \Lambda(m, n, t)_{[i]}\rangle$. If $n - m - 3 \not\equiv 0 \pmod{p}$, then let $K(m, n, t) = \langle \widetilde{D}_k(a) \mid a \in \Lambda(m, n, t)\rangle$. If $n - m - 3 \equiv 0 \pmod{p}$, then let $K(m, n, t) = \langle \widetilde{D}_k(a) \mid a \in \bigoplus_{i=0}^{\lambda-1} \Lambda(m, n, t)_{[i]}\rangle$.

Theorem 4. *Let $X = W$, S, H or K. Then $\overline{X}(m, n, t)^{(\infty)} = X(m, n, t)$. Thus $X(m, n, t)$ is the finite-dimensional Lie superalgebra.*

We call $X(m, n, t)$ the simple Lie superalgebras of Cartan type, where $X = W$, S, H or K.

Theorem 5. *Let $X = W$, S, H or K. Then $X(m, n, t)$ is a restricted Lie superalgebra if and only if $t = \mathbf{1}$, where $\mathbf{1} = (1, 1, \cdots, 1)$.*

Imitating the situation that the characteristic number of basic field is zero, we can construct Lie superalgebras over F: $A(m, n)$, $A(n, n)$, $B(m, n)$, $D(m, n)$, $C(n)$, $P(n)$, $Q(n)$, $F(4)$, $G(3)$ and $D(2, 1 : \alpha)$[1]. Because char$F > 2$, some of them have a center which is not zero (for example, $A(m, n)$). By constructing quatient algebra with the center, we can obtain the corresponding simple Lie superalgebra. We call them classical Lie superalgebras. Finally we give a conjecture about classification of the finite-dimensional simple Lie superalgebras over F.

Conjecture. *Let* char$F > 7$. *Besides Lie algebras (which are Lie superalgebra with triv-*

ial Z_2-grading), any finite-dimensional simple Lie superalgebra must be isomorphic to a classical Lie superalgebra, or a Lie superalgebra of rigid Cartan type, or a Lie superalgebra of generalized Cartan type.

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