Data envelopment analysis

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Abstract **This review introduces the history and present status of data envelopment analysis (DEA) research, particularly the evaluation process. And extensions of some DEA models are also described. It is pointed out that mathematics, economics and management science are the main forces in the DEA development, optimization provides the fundamental method for the DEA research, and the wide range of applications enforces the rapid development of DEA.**

Keywords: data envelopment analysis, non-dominated solution, production frontier.

Data envelopment analysis (DEA) is a new research area which synthesizes operations research, management science and econometrics. It was first proposed by Charnes et al. in 1978[1]. DEA is a mathematical programming approach to provide a relative efficiency assessment (called DEA efficient) for a group of decision making units (DMU) with multiple number of inputs and outputs. In addition, the input and output vectors of the DMUs expand the production possibility set. Determining whether a DMU is efficient from the observed data is equivalent to testing whether the DMU is on the "frontier" of the production possibility set. The concept of the production frontier is extended from the production function to the case of multiple outputs. The methods and models of DEA can be used to comprehensively describe the structure of the production frontier. Therefore, DEA is also recognized as a non-parametric statistical estimation method. To evaluate the relative efficiency of a set of DMUs by DEA methods, we can obtain some insightful management information with the economic background. Therefore, the research and applications of DEA attract a great amount of interests from both academic field and industrial practice[2].

This review introduces the history and the current status of DEA research. In particular, we aim to describe the development process of DEA research. Extensions on some DEA models are discussed. It is pointed out that mathematics, economics and management science form the solid foundation of DEA research, optimization theory provids the major methodology, and a wide range of applications is the driving force of the rapid development of DEA. We can see the profound influence on the DEA studies from the effort and research results of Charnes et al. in different fields such as operations research, management science and economics since the 1950s.

A research field or branch is formed from a new "starting point" by the long-term and joint effort of many people. In the field of DEA research, Charnes, Cooper, their research co-investigators, their students, and other scholars working on DEA conducted a series of fundamental works. These works mainly include the following aspects: (i) A large amount of application cases in different industries are implemented. The work comprehensively shows the broad applications of DEA. (ii) Various numerical methods on DEA models and related softwares are developed. The work significantly supported the application of DEA. (iii) Different DEA models are extended and thoroughly discussed. This includes the additive model, Log-type DEA models, DEA models with a cone ratio of decision makers' preference, semi-infinite programming DEA models with infinitely many DMUs, stochastic DEA models, etc. (iv) The economic and management background of DEA models and methods are extensively investigated. The research established the position of DEA in economics and management science. (v) The mathematical theories for DEA research are discussed. The work involves convex analysis, mathematical programming, game theory, etc.

The research of DEA in China was started from 1986. The research of the Chinese scholars on the DEA theory, model, and related softwares have been well recognized[2–4]. In 1988, the first book on DEA research, *DEA Methods for Relative Efficiency Evaluation——A New Area in Operations Research*, was published in Chinese[5]. The book systematically described DEA methods and models. From an initial estimation in our "China Mainland Index of DEA Research", there are more than 300 research articles or books published in either Chinese or English up to the date of January, 2000[6]. We will comprehensively introduce and survey the research results on DEA from Chinese scholars in the next paper.

1 The first DEA model, C²R model

In 1978, Charnes et al. proposed the first DEA model in their work[1]. Assume that there are n departments or organizations (called decision making units, denoted as DMUs), each DMU has m types of inputs and s types of outputs, given as in the following:

1	2	⋯	n
x_1	x_2	⋯	x_n
y_1	y_2	⋯	y_n

In above, $x_j = (x_{1j}, x_{2j}, \cdots, x_{mj})^T > 0$, $y_j = (y_{1j}, y_{2j}, \cdots, y_{sj})^T > 0$, x_{ij} = the amount of the ith input to DMU-j, y_{rj} = the amount of the rth output of DMU-j, for $j = 1, 2, \cdots, n$; $i = 1, 2, \cdots, m$; and $r = 1, 2, \cdots, s$. For convenience, denote the inputs and outputs of DMU-j_0 by $x_0 = x_{j_0}$ and $y_0 = y_{j_0}$, $1 \leqslant j_0 \leqslant n$, respectively. The DEA model (C²R model) for evaluating DMU-j_0 is given by

$$\begin{cases} \max \dfrac{u^{\mathrm{T}} y_0}{v^{\mathrm{T}} x_0}, \\ \dfrac{u^{\mathrm{T}} y_j}{v^{\mathrm{T}} x_j} \leqslant 1, \ j = 1, 2, \cdots, n, \\ v \geqslant 0, \ u \geqslant 0, \ v \neq 0, \ u \neq 0. \end{cases}$$

In this model, $v = (v_1, v_2, \cdots, v_m)^{\mathrm{T}}$, $u = (u_1, u_2, \cdots, u_s)^{\mathrm{T}}$ are the weighting parameters for m inputs and s outputs respectively. Using Charnes-Cooper transformation, the fractional programming is transferred into an equivalent linear programming problem. Let

$$t = \frac{1}{v^{\mathrm{T}} x_0}, \quad \omega = t\,u, \quad \mu = t\,v.$$

The corresponding linear programming problem of C^2R model is

$$(P_{C^2R}) \begin{cases} \max \mu^{\mathrm{T}} y_0 = h^0, \\ \omega^{\mathrm{T}} x_j - \mu^{\mathrm{T}} y_j \geqslant 0, \ j = 1, 2, \cdots, n, \\ \omega^{\mathrm{T}} x_0 = 1, \\ \omega \geqslant 0, \ \mu \geqslant 0. \end{cases}$$

And its dual programming,

$$(D_{C^2R}) \begin{cases} \min \theta, \\ \sum_{j=1}^{n} x_j \lambda_j \leqslant \theta x_0, \\ \sum_{j=1}^{n} y_j \lambda_j \geqslant y_0, \\ \lambda_j \geqslant 0, \ j = 1, 2, \cdots, n, \ \theta \in E^1. \end{cases}$$

Definition 1[5, 7]**.** If (P_{C^2R}) has optimal objective value $h^0 = 1$, then DMU-j_0 is called weak DEA efficient. h^0 is called efficiency index.

Definition 2. If (P_{C^2R}) has an optimal solution (ω^0, μ^0) such that $\omega^0 > 0$, $\mu^0 > 0$ and $\mu^0 y_0 = 1$, then DMU-j_0 is called DEA efficient.

From the dual theorem and complementary theorem of linear programming, we have the following equivalent definition of the DEA efficiency.

Definition 3. If any optimal solution of (D_{C^2R}), θ^0 and λ^0_j, $j = 1, 2, \cdots, n$, satisfies

$$\theta^0 = 1, \quad \sum_{j=1}^{n} x_j \lambda_j^0 = \theta^0 x_0, \quad \sum_{j=1}^{n} y_j \lambda_j^0 = y_0,$$

then DMU-j_0 is called DEA efficient.

The DEA models (P_{C^2R}) and (D_{C^2R}) above extend the method of efficiency evaluation on single input and single output given by economist Farrell in 1957 to the case with multiple inputs and outputs, by mathematical programming models.

However, applying models (P_{C^2R}) and (D_{C^2R}) to determining the efficiency often faces the problem that all weight ω and μ are not guaranteed to be positive, and the corresponding slack variables in dual programming to be 0. In 1952, when Charnes dealt with the issue of degeneration in linear programming, he introduced the concept of non-Archimedean value ε1). For DEA model (D_{C^2R}), Charnes et al. gave a DEA model $(D^{\varepsilon}_{C^2R})$ with non-Archimedean value ε as follows:

$$(D^{\varepsilon}_{C^2R}) \begin{cases} \min[\theta - \varepsilon(\hat{e}^{\mathrm{T}} s^- + e^{\mathrm{T}} s^+)], \\ \sum_{j=1}^{n} x_j \lambda_j + s^- = \theta x_0, \\ \sum_{j=1}^{n} y_j \lambda_j - s^+ = y_0, \\ \lambda_j \geqslant 0, \ j = 1, 2, \cdots, n, \\ s^+ \geqslant 0, s^- \geqslant 0, \theta \in E^1, \end{cases}$$

where

$$\hat{e} = (1, 1, \cdots, 1)^{\mathrm{T}} \in E^m, \quad e = (1, 1, \cdots, 1)^{\mathrm{T}} \in E^s.$$

We then have the following theorem (that is, there exists a positive value ε, such that the following theorem holds (see ref. [5])).

Theorem 1. If $(D^{\varepsilon}_{C^2R})$ has an optimal solution $\theta^0, \lambda^0_j, j = 1, 2, \cdots, n$, such that

$$\theta^0 = 1, \ s^{-0} = 0, \ s^{+0} = 0,$$

then DMU-j_0 is DEA efficient.

2 Postulate system of production possibility set and extension of DEA models

In the research on econometrics, it is often needed to introduce some postulates for investigating the structure of an economic system. Denote the production possibility set by

$T = \{(x, y) \mid y \in E^s_+$ is a possible output while $x \in E^m_+$ is an input$\}$.

For production possibility set T, we have the following postulates[8, 9].

1) The non-Archimedean value is a small amount such that $\forall a > 0$ and $\forall N > 0$, we have $N \cdot \varepsilon < a$. That is, "$\varepsilon > 0$" is a number which is smaller than any positive number.

Postulate 1(Convexity postulate). If $(x, y) \in T$, and $(\hat{x}, \hat{y}) \in T$, then $(\lambda x+(1-\lambda)\hat{x},\ (\lambda y+(1-\lambda)\hat{y}) \in T$, for $\lambda \in [0,1]$.

Postulate 2 (Inefficiency postulate). If $(x, y) \in T$, and $\hat{x} \geqslant x,\ \hat{y} \leqslant y$, then $(\hat{x}, \hat{y}) \in T$.

Postulate 3 (Ordinary postulate). The observed $(x_j, y_j) \in T$, for all $j = 1, 2, \cdots, n$.

Postulate 4.1 (Ray unboundness postulate). If $(x, y) \in T$, then $\alpha(x, y) \in T$, for all $\alpha \geqslant 0$.

Postulate 4.2 (Contraction postulate). If $(x, y) \in T$, then $\alpha(x, y) \in T$, for all $0 \leqslant \alpha \geqslant 1$.

Postulate 4.3 (Expansion postulate). If $(x, y) \in T$, then $\alpha(x, y) \in T$, for all $\alpha \geqslant 1$.

Postulate 5 (Minimum extrapolation postulate). T is the intersection set of all $\widetilde{T}$ satisfying Postulates 1, 2 and 3 or satisfying Postulates 1, 2 and 3 together with one of the postulates from Postulates 4.1, 4.2 and 4.3.

Now, we introduce three 0-1 binary parameters δ_1, δ_2 and δ_3. The production possibility set has the following unique form.

$$T = \left\{ (x, y) \,\middle|\, \sum_{j=1}^{n} x_j \lambda_j \leqslant x,\ \sum_{j=1}^{n} y_j \lambda_j \geqslant y,\ \delta_1 \left(\sum_{j=1}^{n} \lambda_j + \delta_2 (-1)^{\delta_3} \lambda_{n+1} \right) = \delta_1,\quad \lambda_j \geqslant 0,\ j = 1, \cdots, n, n+1 \right\}.$$

In particular,

(i) When T satisfies Postulates 1, 2, 3, 4.1 and 5 (correspondingly $\delta_1 = 0$), then

$$T_{\mathrm{C^2R}} = \left\{ (x, y) \,\middle|\, \sum_{j=1}^{n} x_j \lambda_j \leqslant x,\ \sum_{j=1}^{n} y_j \lambda_j \geqslant y,\ \lambda_j \geqslant 0,\ j = 1, 2, \cdots, n \right\}.$$

(ii) When T satisfies Postulates 1, 2, 3 and 5 (correspondingly $\delta_1 = 1$, $\delta_2 = 0$), then

$$T_{\mathrm{BC^2}} = \left\{ (x, y) \,\middle|\, \sum_{j=1}^{n} x_j \lambda_j \leqslant x,\ \sum_{j=1}^{n} y_j \lambda_j \geqslant y,\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geqslant 0,\ j = 1, 2, \cdots, n \right\}.$$

(iii) When T satisfies Postulates 1, 2, 3, 4.2 and 5 (correspondingly $\delta_1 = \delta_2 = 1,\ \delta_3 = 0$), then

$$T_{\mathrm{FG}} = \left\{ (x, y) \,\middle|\, \sum_{j=1}^{n} x_j \lambda_j \leqslant x,\ \sum_{j=1}^{n} y_j \lambda_j \geqslant y,\ \sum_{j=1}^{n} \lambda_j \leqslant 1,\ \lambda_j \geqslant 0,\ j = 1, 2, \cdots, n \right\}.$$

(iv) When T satisfies Postulates 1, 2, 3, 4.3 and 5 (correspondingly $\delta_1 = \delta_2 = \delta_3 = 1$), then

$$T_{\mathrm{ST}} = \left\{ (x, y) \,\middle|\, \sum_{j=1}^{n} x_j \lambda_j \leqslant x,\ \sum_{j=1}^{n} y_j \lambda_j \geqslant y,\ \sum_{j=1}^{n} \lambda_j \geqslant 1,\ \lambda_j \geqslant 0,\ j = 1, 2, \cdots, n \right\}.$$

These results represent the four most typical DEA models: $\mathrm{C^2R}$ model given by Charnes et al. in 1978[1], $\mathrm{BC^2}$ model given by Banker et al. in 1984[8], FG model given by Färe et al. in 1985[10] and ST model given by Seiford et al. in 1990[11]. These models can be given in a unified form.

$$\begin{cases} \min \theta, \\ (\theta x_0, y_0) \in T, \end{cases}$$

or

$$(P) \quad \begin{cases} \min\ \theta, \\ \sum_{j=1}^{n} x_j \lambda_j \leqslant \theta x_0, \quad \sum_{j=1}^{n} y_j \lambda_j \geqslant y_0, \\ \delta_1 \left(\sum_{j=1}^{n} \lambda_j + \delta_2 (-1)^{\delta_3} \lambda_{n+1} \right) = \delta_1, \\ \lambda_j \geqslant 0,\ j = 1, 2, \cdots, n, n+1, \theta \in E^1. \end{cases}$$

And its dual

$$(D) \quad \begin{cases} \max\ (\mu^{\mathrm{T}} y_0 - \delta_1 \mu_0), \\ \omega^{\mathrm{T}} x_j - \mu^{\mathrm{T}} y_j + \delta_1 \mu_0 \geqslant 0,\ j = 1, 2, \cdots, n, \\ \omega^{\mathrm{T}} x_0 = 1, \\ \omega \geqslant 0,\ \mu \geqslant 0,\ \delta_1 \delta_2 (-1)^{\delta_3} \mu_0 \geqslant 0. \end{cases}$$

Given different values of δ_1, δ_2 and δ_3, the corresponding models $(P_{\mathrm{C^2R}})$, $(D_{\mathrm{C^2R}})$; $(P_{\mathrm{BC^2}})$, $(D_{\mathrm{BC^2}})$; (P_{FG}), (D_{FG}); and (P_{ST}), (D_{ST}) could be obtained. For the generalized DEA models (P) and (D), we can similarly define the weak DEA efficiency and DEA efficiency for a particular DMU, and the DEA model with non-Archimedean value ε.

Theorem 2. The following results hold[12].

(i) Weak DEA efficiency ($\mathrm{C^2R}$) $\Rightarrow$ Weak DEA efficiency (FG) $\Rightarrow$ Weak DEA efficiency ($\mathrm{BC^2}$);

(ii) Weak DEA efficiency ($\mathrm{C^2R}$) $\Rightarrow$ Weak DEA efficiency (ST) $\Rightarrow$ Weak DEA efficiency ($\mathrm{BC^2}$);

where "weak DEA efficiency ($\mathrm{C^2R}$)" means that DMU-j_0

is weak DEA efficiency under the C^2R model, and others have the similar meaning, "$\Rightarrow$" means "implies".

3 Additive model and Log-type model

In 1961, Charnes et al. introduced goal programming when they studied a linear programming problem with an application background, where no feasible solution existed. The goal programming involves both positive and negative error variables. Later on, the error variables are introduced into the objective function, and the model is used to deal with the problem with multiple objectives. Based on the results, Charnes et al.[13] proposed their C^2GS^2 model. In the model, the objective function simply consists of the sum of all error variables. The model is then called an additive model. For the generalized DEA model, we can also give the additive models for determining the DEA efficiency and the weak DEA efficiency in the similar way. Consider the following additive model,

$$(\overline{P})\begin{cases}\max\ (\hat{e}^{\mathrm{T}}s^{-}+e^{\mathrm{T}}s^{+}),\\ \sum_{j=1}^{n}x_j\lambda_j+s^{-}=x_0,\\ \sum_{j=1}^{n}y_j\lambda_j-s^{+}=y_0,\\ \delta_1\left(\sum_{j=1}^{n}\lambda_j+\delta_2(-1)^{\delta_3}\lambda_{n+1}\right)=\delta_1,\\ \lambda_j\geqslant 0,\ j=1,2,\cdots,n,n+1,\\ s^{+}\geqslant 0,\quad s^{-}\geqslant 0;\end{cases}$$

and its dual

$$(\overline{D})\begin{cases}\min\left(\omega^{\mathrm{T}}x_0-\mu^{\mathrm{T}}y_0+\delta_1\mu_0\right),\\ \omega^{\mathrm{T}}x_j-\mu^{\mathrm{T}}y_j+\delta_1\mu_0\geqslant 0,\ j=1,2,\cdots,n,\\ \omega\geqslant 0,\quad \mu\geqslant 0,\\ \delta_1\delta_2(-1)^{\delta_3}\mu_0\geqslant 0.\end{cases}$$

Theorem 3. DMU-j_0 is DEA efficient if and only if $(\overline{P})$ and $(\overline{D})$ have the same optimal objective value 0.

Using the additive model to determine the DEA efficiency of a DMU, the computational inconvenience due to the non-Archimedean value ε can be avoided. Similar to model $(D^{\varepsilon}_{C^2R})$, we can obtain the projection of the DMU on the production frontier of T. The corresponding models for determining the weak DEA efficiency can be constructed in the same way. It can be shown that DMU-j_0 is weak DEA efficiency if and only if the following linear programming problem has optimal objective value 0.

$$(P)\begin{cases}\max t,\\ \sum_{j=1}^{n}x_j\lambda_j+t\,\hat{e}=x_0,\ \sum_{j=1}^{n}y_j\lambda_j-te=y_0,\\ \delta_1\left(\sum_{j=1}^{n}\lambda_j+\delta_2(-1)^{\delta_3}\lambda_{n+1}\right)=\delta_1,\\ \lambda_j\geqslant 0,\ j=1,2,\cdots,n,n+1,\ t\geqslant 0,\ t\in E^1.\end{cases}$$

In 1983, Charnes et al. investigated a Log-type DEA model[14]. The model transfers the original data $X=(x_1, x_2, \cdots, x_n)$ and $Y=(y_1, y_2, \cdots, y_n)$ into $(\log X, \log Y)$[1)], and then implements the corresponding C^2R or BC^2 additive models. The following is a Log-type generalized additive model

$$(\hat{P})\begin{cases}\max\left(\hat{e}^{\mathrm{T}}s^{-}+e^{\mathrm{T}}s^{+}\right),\\ \sum_{j=1}^{n}(\mathrm{Log}\,x_j)\lambda_j+s^{-}=\mathrm{Log}\,x_0,\\ \sum_{j=1}^{n}(\mathrm{Log}\,y_j)\lambda_j-s^{+}=\mathrm{Log}\,y_0,\\ \delta_1\left(\sum_{j=1}^{n}\lambda_j+\delta_2(-1)^{\delta_3}\lambda_{n+1}\right)=\delta_1,\\ \lambda_j\geqslant 0,\quad j=1,2,\cdots,n,n+1,\\ s^{-}\geqslant 0,\ s^{+}\geqslant 0,\end{cases}$$

and its dual

$$(\hat{D})\begin{cases}\min\left(\omega^{\mathrm{T}}\mathrm{Log}\,x_0-\mu^{\mathrm{T}}\mathrm{Log}\,y_0+\delta_1\mu_0\right),\\ \omega^{\mathrm{T}}\mathrm{Log}\,x_j-\mu^{\mathrm{T}}\mathrm{Log}\,y_j+\delta_1\mu_0\geqslant 0,\ j=1,2,\cdots,n,\\ \omega\geqslant 0,\ \mu\geqslant 0,\\ \delta_1\delta_2(-1)^{\delta_3}\mu_0\geqslant 0.\end{cases}$$

If DMU-j_0 is DEA efficient under the Log-type models, then there exist $\omega^0>0$, $\mu^0>0$, $\delta_1\mu_0^0$, such that

$$\omega^{0\mathrm{T}}\mathrm{Log}\,x_0-\mu^{0\mathrm{T}}\mathrm{Log}\,y_0+\delta_1\mu_0^0=0,$$

that is,

$$\prod_{r=1}^{s}y_{r_0}^{\mu_r^0}=\mathrm{e}^{\delta_1\mu_0^0}\prod_{i=1}^{m}x_{i_0}^{\omega_i^0}.$$

In particular, when s=1, we have ($\beta=\mu_0^0/\mu_1^0$, $\alpha_i=w_i^0/\mu_1^0$, $i=1,2,\cdots,m$)

$$y_{10}=\mathrm{e}^{\delta_1\beta}\prod_{i=t}^{m}x_{i_0}^{\alpha_i}.$$

1) If matrix $C=(c_{ij})_{\substack{1\leqslant i\leqslant m\\1\leqslant j\leqslant n}}>0$, denoe $\mathrm{Log}C=(\log c_{ij})_{\substack{1\leqslant i\leqslant m\\1\leqslant j\leqslant n}}$.

Therefore, the Log-type DEA model is the local approximation of the Cobb-Douglas production function.

4 The equivalence between (weak) DEA efficiency and (weak) Pareto solution

The problem of evaluating the relative efficiency of n DMUs with m inputs and s outputs can be described by a multiobjective programming problem with $m + s$ objectives. That is,

$$(\mathrm{VP})\quad \begin{cases} \mathrm{V\text{-}min}\,(x, -y), \\ (x, y) \in T, \end{cases}$$

where T is the production possibility set expanded by the observed inputs and outputs, and $x=(x_1, x_2, \cdots, x_m)^{\mathrm{T}}$, $y=(y_1, y_2,\cdots, y_s)^{\mathrm{T}}$. A natural question is that when DMU-j_0 is (weak) DEA efficient, whether the corresponding (x_0, y_0) is the (weak) Pareto solution of multiobjective programming problem (VP).

In 1962, when Charnes et al. studied the multiobjective programming problems, they developed the necessary and sufficient conditions for testing whether a feasible solution is a Pareto solution. The condition is called Charnes-Cooper Test. For BC^2 model, Charnes et al.[13] showed the equivalence between the DEA efficiency and the Pareto solution of (VP) by the Charnes-Cooper Test. This work linked the concept of DEA efficiency to Pareto solution which is a well established and fundamental concept in multiobjective programming. As a matter of fact, if the concept of DEA efficiency would have a different meaning from the concept of Pareto solution in the multiobjective programming, then the definition of DEA efficiency, as well as the overall research of DEA field would loss its theoretical and practical ground. The following theorem shows the results.

Theorem 4. For the generalized DEA model (P) and (D), the following hold.

(i) DMU-j_0 is DEA efficient if and only if (x_0, y_0) is the Pareto solution of the multiobjective programming problem (VP)[5, 12].

(ii) If DMU-j_0 is weak DEA efficient, then (x_0, y_0) is the weak Pareto solution of the multiobjective programming problem (VP)[5, 12].

(iii) For $\mathrm{C}^2\mathrm{R}$ DEA model (δ_1=0) and ST DEA model ($\delta_1 = \delta_2 = \delta_3 =1$), if (x_0, y_0) is the weak Pareto solution of (VP), then DMU-j_0 is weak DEA efficient[15].

5 Cone ratio DEA model, $\mathrm{C}^2\mathrm{WH}$ model and its extension

In conventional DEA models ($\mathrm{C}^2\mathrm{R}$, BC^2, FG and ST), m inputs and s outputs are treated as the same important when they are used in evaluating DMUs. There is no restriction on the weighting scales $\omega = (\omega_1, \omega_2,\cdots, \omega_m)^{\mathrm{T}}$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_s)^{\mathrm{T}}$. Since the publication of the first DEA model, it has been pointed out that this is a weak point of DEA efficiency assessment. Such a DEA model is short of the concern on the decision maker's preference on some input-output attributes. In 1974, Yu introduced a concept of nondominated solution for multiobjective programming problems. The work extended from Pareto solution to nondominated solution which can represent the decision maker's preference. It has been shown that the concept of DEA efficiency is equivalent to the Pareto solution of a multiobjective programming (see Theorem 4(i)). However, there were no corresponding results in DEA research reflecting the decision makers' preference up to then.

To deal with this, Charnes et al. extended the $\mathrm{C}^2\mathrm{R}$ model in 1989. They proposed a "cone ratio $\mathrm{C}^2\mathrm{WH}$ model" that can represent the decision maker's preference on attributes[16].

$$(\mathrm{C}^2\mathrm{WH})\quad \begin{cases} \max\ \dfrac{u^{\mathrm{T}} y_0}{v^{\mathrm{T}} x_0}, \\ v^{\mathrm{T}} X - u^{\mathrm{T}} Y \in K, \\ v \in V \setminus \{0\},\ u \in U \setminus \{0\}, \end{cases}$$

where

$X = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$,

$Y = (y_1, y_2, \cdots, y_n)^{\mathrm{T}}$,

$V \subseteq E_+^m$ is a closed convex cone, and $\mathrm{Int}\,V \neq \varnothing$,

$U \subseteq E_+^s$ is a closed convex cone, and $\mathrm{Int}\,U \neq \varnothing$,

$K \subseteq E_+^n$ is a closed convex cone,

$x_j \in -\mathrm{Int}\,V^*,\quad y_j \in -\mathrm{Int}\,U^*,\quad j = 1, 2, \cdots, n.$

It is clear that when $V = E_+^m$, $U = E_+^s$ and $K = E_+^n$, the above model is reduced to the $\mathrm{C}^2\mathrm{R}$ model. Using Charnes-Cooper transformation, model $\mathrm{C}^2\mathrm{WH}$ is transferred into a linear programming cone ratio DEA model,

$$(P_{\mathrm{C^2WH}})\quad \begin{cases} \max\ \mu^{\mathrm{T}} y_0, \\ \omega^{\mathrm{T}} X - \mu^{\mathrm{T}} Y \in K, \\ \omega^{\mathrm{T}} x_0 = 1, \\ \omega \in V,\ \mu \in U. \end{cases}$$

The dual programming problem of this is given by

$$(D_{\mathrm{C^2WH}})\quad \begin{cases} \min\ \theta, \\ X\lambda - \theta x_0 \in V^*, \\ -Y\lambda + y_0 \in U^*, \\ \lambda \in -K^*. \end{cases}$$

Definition 4. If ($P_{\mathrm{C^2WH}}$) has an optimal solution

(ω^0, μ^0) such that $\omega^0 \in \text{Int}\, V$, $\mu^0 \in \text{Int}\, U$, and $\mu^{0\text{T}} y_0 =$ 1, then DMU-j_0 is called DEA efficient (C^2WH).

The cone ratio DEA model also has the corresponding additive model. We have the following theorem.

Theorem 5. Under the "constraint qualification"[16,17], DMU-j_0 is DEA efficient (C^2WH) if and only if (x_0, y_0) is a non-dominated solution of the following multiobjective programming problem respect to $V^* \times U^*$,

$$\begin{cases} V\text{-}\min\,(x,\ -y), \\ (x, y) \in T_{C^2WH}, \end{cases}$$

where the production possibility set

$$T_{C^2WH} = \{(x, y) \mid (x, y) \in (X\lambda, Y\lambda) + (-V^*,\ U^*), \lambda \in -K^*\}.$$

When V and U are polyhedral cones given by the "sum-form" (linear combinations of a set of points), and when $K = E_+^n$, that is $(A = A_{m'\times m},\ B = B_{s'\times s})$,

$$V = \{A^\text{T}\omega' \mid \omega' \geqslant 0\}, \quad U = \{B^\text{T}\mu' \mid \mu' \geqslant 0\},$$

then

$$V^* = \{v \mid Av \leqslant 0\}, \quad U^* = \{u \mid Bu \geqslant 0\}.$$

Thus, (P_{C^2WH}) and (D_{C^2WH}) are rewritten into the following polyhedral cone ratio DEA model[5, 18]:

$$(\overline{P}_{C^2WH}) \begin{cases} \max \mu'^\text{T}\,(By_0), \\ \omega'^\text{T}\,(AX) - \mu'^\text{T}\,(BY) \geqslant 0, \\ \omega'^T\,(Ax_0) = 1, \\ \omega' \geqslant 0, \quad \mu' \geqslant 0, \end{cases}$$

and the dual programming problem

$$(\overline{D}_{C^2WH}) \begin{cases} \min \theta, \\ (AX)\,\lambda \leqslant \theta(Ax_0), \\ (BY)\,\lambda \geqslant (By_0), \\ \lambda \geqslant 0. \end{cases}$$

That is, the polyhedral cone ratio DEA model is a DEA model with the following input and output data, transformed from the original data.

1	2	⋯	n
Ax_1	Ax_2	⋯	Ax_n
By_1	By_2	⋯	By_n

When the polyhedral cone is given by the "intersection form" (a set of facet describing linear inequalities), it is easy to be transformed into the "sum form"[19].

Yu et al. gave a generalized DEA model with cone structure in 1996[12] ($e = (1, 1, \cdots, 1)^\text{T} \in E^n$),

$$(P_{GDEA}) \begin{cases} \max(\mu^\text{T} y_0 - \delta_1 \mu_0) = h_0, \\ \omega^\text{T} X - \mu^\text{T} Y + \mu_0 \delta_1 e^\text{T} \in K, \\ \omega^\text{T} x_0 = 1, \\ \omega \in V, \mu \in U,\ \delta_1 \delta_2 (-1)^{\delta_3} \mu_0 \geqslant 0. \end{cases}$$

And the dual programming problem

$$(D_{GDEA}) \begin{cases} \min \theta, \\ \begin{pmatrix} X\lambda - \theta x_0 \\ -Y\lambda + y_0 \end{pmatrix} \in W^*, \\ \delta_1 \left(e^\text{T}\lambda + \delta_2 (-1)^{\delta_3} \lambda_{n+1}\right) = \delta_1, \\ \lambda \in -K^*,\ \lambda_{n+1} \geqslant 0, \quad \theta \in E^1. \end{cases}$$

In the models above, $\delta_1, \delta_2, \delta_3$ are 0-1 parameters. For different values of $\delta_1, \delta_2, \delta_3$, the above model can be transformed into different models with a cone structure, C^2R, BC^2, FG and ST. In particular, when $W = V \times U$ and $\delta_1 = 0$, it is the C^2WH model. Furthermore, when $V = E_+^m$, $U = E_+^s$ and $K = E_+^n$, it is transformed into the regular DEA model.

Wei et al. conducted a series of researches on the cone structured DEA model (P_{GDEA}) and (D_{GDEA})[9, 12, 20,21]. They showed the equivalence of the DEA efficiency (GDEA) and the non-dominated solution of a multiobjective programming problem respect to W^*, investigated the corresponding additive model and the DEA model with polyhedral cone. It is clear that the research on the generalized DEA model incorporated research results on all other DEA models.

Denote the production possibility set by

$$T_{GDEA} = \left\{ (x, y) \,\middle|\, \begin{pmatrix} X\lambda - x \\ -Y\lambda + y \end{pmatrix} \in W^*, \right.$$
$$\left. \delta_1 \left(e^\text{T}\lambda + \delta_2 (-1)^{\delta_3} \lambda_{n+1}\right) = \delta_1,\ \lambda \in -K^*,\ \lambda_{n+1} \geqslant 0 \right\}.$$

The above DEA model can be written into

$$\text{(GDEA-I)} \begin{cases} \min \theta, \\ (\theta x_0, y_0) \in T_{GDEA}, \quad \theta \in E^1. \end{cases}$$

This model is called an input-oriented generalized DEA model. Similarly, we can obtain the following output-oriented generalized DEA model[3]:

$$\text{(GDEA-O)} \begin{cases} \max z, \\ (x_0, zy_0) \in T_{GDEA}, \quad z \in E^1. \end{cases}$$

Model (GDEA-O) can be discussed parallelly to the model (GDEA-I). For the research on discret DEA models with cone structure, see ref. [22].

6 DEA models with infinitely many DMUs

In 1988, Charnes et al. discussed semi-infinitely multicriteria programming DEA models with infinitely many DMUs C^2W[7] and C^2WY[23]. The semi-infinitely multicriteria programming is a branch of mathematical programming started by Charnes et al. C^2W model is the first nonlinear DEA model. It reveals the mathematical and economic background of DEA. It was recognized as "a perfect research framework which provides a base for statistical analysis". Denote Z as the set of DMUs. Consider

$$(P_{C^2WY})\begin{cases}\max\,(\mu^{T}y(z_0)-\delta_1\mu_0)=h_0,\\ \omega^{T}x(z)-\mu^{T}y(z)+\delta_1\mu_0\geqslant 0,\quad z\in Z,\\ \omega^{T}x(z_0)=1,\\ \omega\in V,\quad \mu\in U,\quad \delta_1\delta_2(-1)^{\delta_3}\mu_0\geqslant 0.\end{cases}$$

And the dual programming problem

$$(D_{C^2WY})\begin{cases}\min\theta,\\ \sum\limits_{z\in Z}x(z)\lambda(z)-\theta x(z_0)\in V^*,\\ -\sum\limits_{z\in Z}y(z)\lambda(z)+y(z_0)\in U^*,\\ \delta_1\left(\sum\limits_{z\in Z}\lambda(z)+\delta_2(-1)^{\delta_3}\lambda_0\right)=\delta_1,\\ \lambda(z)\geqslant 0, z\in Z, \lambda_0\geqslant 0,\quad \theta\in E'.\end{cases}$$

The corresponding production possibility set is

$$T_{C^2WH}=\left\{(x,y)\,\middle|\,\begin{array}{l}\sum\limits_{z\in Z}x(z)\lambda(z)-x\in V^*,\\ \sum\limits_{z\in Z}y(z)\lambda(z)-y\in -U^*,\\ \delta_1\left(\sum\limits_{z\in Z}\lambda(z)+\delta_2(-1)^{\delta_3}\lambda_0\right)=\delta_1,\\ \lambda(z)\geqslant 0,\ \lambda_0\geqslant 0,\ z\in Z.\end{array}\right\}.$$

Definition 5. If (P_{C^2WY}) has the optimal objective value 1, then $(x(z_0), y(z_0))$ is called to be on the production frontier of T_{C^2WY} (that is, DMU-Z_0 is weak DEA efficient).

In particular, when $\delta_1=1,\ \delta_2=1,\ \delta_3=0$ (that is, this is an FG DEA model with infinitely many DMUs), $V=E_+^m, U=E_+^1$, and $Z=E_+^m=\{x\,|\,x\geqslant 0\}$, we can describe the production function background of the DEA model. Consider the format of DEA model given as below[24]:

$x\in E_+^m$

x	x_0
$f(x)$	y_0

where $f(x)$ is a continuously differentiable production function, and $f(x)$ is concave homogeneous function of order k, $k\leqslant 1$. From DEA model (P_{C^2WY}) and (D_{C^2WY}), we can obtain the following results.

Theorem 6. Let $(x_0, y_0)\in T_{C^2WY}$, and $x_0>0$, then

(i) $T_{C^2WY}=\{(x,y)\,|\,f(x)\geqslant y, x\in E_+^m\}$.

(ii) Assume that $(x_0, y_0)\in T_{C^2WY}$, then (x_0, y_0) is on the production frontier T_{C^2WY} if and only if $y_0=f(x_0)$.

(iii) If (x_0, y_0) is on the production frontier of T_{C^2WY}, then we have $\omega^{0T}x_0-\mu^0 y_0=-\mu_0^0$, where (ω^0,μ^0,μ_0^0) is an optimal solution of (P_{C^2WY}), and $\omega^0=f(x_0)^{T}, \mu^0=1,\ \mu_0^0=(1-k)f(x_0)$. In addition, μ_0^0 $>$f and only if (x_0, y_0) has a decreasing return-to-scale, $\mu_0^0=0$ if and only if (x_0, y_0) has a stable return-to-scale.

From the discussion above, we can see that a production possibility set generated from a finite number of DMUs is an approximation to the epigraph of production function $\{(x,y)\,|\,f(x)\geqslant y, x\in E_+^m\}$, and the DEA production frontier is a piece wise linear approximation of the production function surface. Therefore, parameter μ_0^0 in the DEA model can be used to determine the return-to-scale of a DMU. We can thus use DEA model for the assessment of technological progress of organizations or companies[25, 26]. DEA models can be also used to establish non-parametric micro-economic models[10, 27].

7 Chance constrained DEA models

Chance constrained programming is an important and useful area in stochastic programming, and is proposed by Charnes and Cooper in 1959. The stochastic DEA model and the DEA model with a cone structure have been recognized as two important branches in the research of DEA models[28]. Sengupta in 1987[29] and Land et al. in 1993[30] studied the stochastic DEA models. In 1996, Huang et al.[31] investigated the model again. They considered that all inputs and outputs are stochastic variables, proposed the concept of stochastically nondominated point, and obtained a chance constrained DEA model.

Denote $\tilde{x}_j$ the input vector of dimension m, $\tilde{y}_j$ the output vector of dimension s, of DMU-j, for j=1, 2, $\cdots$,

n. We have the following definition.

Definition 6. Denote

$$\tilde{T}=\left\{(\tilde{x},\tilde{y})\left|\sum_{j=1}^{n}\tilde{x}_j\lambda_j\leqslant x,\ \sum_{j=1}^{n}\tilde{y}_j\lambda_j\geqslant y,\right.\right.$$
$$\delta_1\left(\sum_{j=1}^{n}\lambda_j+\delta_2(-1)^{\delta_3}\lambda_{n+1}\right)$$
$$\left.=\delta_1,\ \lambda_j\geqslant 0,\ j=1,2,\ldots,n,n+1\right\}.$$

Then $\tilde{T}$ is called a stochastic production possibility set. For a given closed convex cone Γ and a number $0\leqslant\alpha\leqslant 1$, if $\forall(\tilde{x},\tilde{y})\in\tilde{T}$, it always has $P\{(\tilde{x},-\tilde{y})\in(\tilde{x},-\tilde{y})+\Gamma\}\leqslant\alpha$, then $(\tilde{x}_0,\tilde{y}_0)$ is called an α-stochastically nondominated point respect to Γ, and DMU-j_0 is called α-stochastically nondominated efficiency respect to Γ.

Using the concept of α-stochastically nondominated point respect to Γ, we can further define the α-stochastically efficient frontier of $\tilde{T}$.

For convenience, assume that Γ is a convex polyhedral cone given by "intersection form":

$$\Gamma=V\times U,\ V=\{v\mid Av\leqslant 0\},\ U=\{u\mid Bu\leqslant 0\}.$$

Then $\forall(\tilde{x},\tilde{y})\in\tilde{T}$, $P\{(\tilde{x},-\tilde{y})\in(\tilde{x}_0,-\tilde{y}_0)+\Gamma\}\leqslant\alpha$ is equivalent to

$$P\left\{\sum_{j=1}^{n}(A\tilde{x}_j)\lambda_j\leqslant(A\tilde{x}_0),\ \sum_{j=1}^{n}(B\tilde{y}_j)\lambda_j\geqslant(B\tilde{y}_0)\right\}\leqslant\alpha,$$

where $\lambda_1,\ \cdots,\ \lambda_n,\ \lambda_{n+1}$ satisfy

$$\delta_1\left(\sum_{j=1}^{n}\lambda_j+\delta_2(-1)^{\delta_3}\lambda_{n+1}\right)=\delta_1,\ \lambda_j\geqslant 0, j=1,\cdots,n,n+1.$$

The chance constrained DEA model is given by (stochastic Charnes-Cooper test),

$$(\mathrm{P_{CC}})\begin{cases}\max P\left\{\hat{e}^{\mathrm{T}}\left(\sum_{j=1}^{n}(A\tilde{x}_j)\lambda_j\right)-e^{\mathrm{T}}\left(\sum_{j=1}^{n}(B\tilde{y}_j)\lambda_j\right),\right.\\ \left.\qquad\leqslant\hat{e}^{\mathrm{T}}A\tilde{x}_0-e^{\mathrm{T}}B\tilde{y}_0\right\},\\ P\left\{a_i\left(\sum_{j=1}^{n}\tilde{x}_j\lambda_j\leqslant\tilde{x}_0\right)\right\}\geqslant 1-\varepsilon,\quad i=1,2,\ldots,m',\\ P\left\{b_r\left(\sum_{r=1}^{n}\tilde{y}_j\lambda_j\geqslant\tilde{y}_0\right)\right\}\geqslant 1-\varepsilon,\ r=1,2,\ldots,s',\\ \delta_1\left(\sum_{j=1}^{n}\lambda_j+\delta_2(-1)^{\delta_3}\lambda_{n+1}\right)=\delta_1,\ \lambda_j\geqslant 0,\\ \quad j=1,2,\ldots,n,n+1.\end{cases}$$

In (P_{CC}), a_i is the ith row of A, $i=1,\ \cdots,\ m'$, and b_r is the rth row of B, $r=1,\ \cdots,\ s'$, and ε is a non-Archimedean infinitesimal.

Theorem 7. If DMU-j_0 is α-stochastically nondominated efficient respect to Γ, the (P_{CC}) has its optimal objective value not greater than α.

Huang et al. [31] pointed out that when $\tilde{x}_j$ and $\tilde{y}_j$, $j=1,\ \cdots,\ n$, follow the normal distribution, (P_{CC}) can be rewritten into a nonlinear programming model.

8 DEA model and two-person constrained game

In 1953, Charnes et al[32]. proposed and solved a constrained game. They introduced a concept of chance constrained game by transferring the concept of chance constrained programming into game theory. In 1980, Banker et al. discussed the relationship between the DEA and the two-person, finite, zero-sum game[33, 34]. In 1986, Charnes et al. discussed the relationship between C^2W DEA model with infinitely many DMUs and game models[35]. In early 2000, Wei investigated the relationship between the generalized DEA model and game model, both with cone structures[36]. They established different game models according to the input and output data, and showed that the game value is the efficiency index in the DEA model.

In 1995, Roussean et al.[37, 38] gave a ratio efficiency games model, and studied its equivalency with DEA models. In 2000, Hao et al. discussed the equivalency of the ratio efficient cone constrained game and the DEA model with cone structure[39]. Consider a two-person, infinite and 0-sum game (denoted by G),

$$G=\left\{\mathrm{I},\mathrm{II};\ S_1,S_2,\ \frac{\mu^{\mathrm{T}}Y\lambda}{\omega^{\mathrm{T}}X\lambda+\delta_1\mu_0}\right\},$$

where

$$S_1=\{(\lambda,\delta_1\delta_2\lambda_{n+1})\mid\delta_1(e^{\mathrm{T}}\lambda+\delta_2(-1)^{\delta_3}\lambda_{n+1})$$
$$=\delta_1,\lambda\in-K^*,\lambda_{n+1}\ \geqslant 0\},$$
$$S_2=\left\{(\omega,\mu,\delta_1\mu_0)\mid\omega\in V,\ \mu\in U,\right.$$
$$\left.\delta_1\delta_2(-1)^{\delta_3},\ \mu_0\geqslant 0,\ \frac{\mu^{\mathrm{T}}y_0}{\omega^{\mathrm{T}}x_0+\delta_1\mu_0}=1\right\},$$
$$X=(x_1,x_2,\ldots,x_n),\ Y=(y_1,y_2,\ldots,y_n).$$

The pay-off function is given by

$$f_1(\lambda,\delta_1\delta_2\lambda_{n+1},\omega,\mu,\delta_1\mu_0)$$
$$=\frac{\mu^{\mathrm{T}}Y\lambda}{\omega^{\mathrm{T}}X\lambda+\delta_1\mu_0}$$
$$=-f_2(\lambda,\delta_1\delta_2\lambda_{n+1},\omega,\mu,\delta_1\mu_0).$$

Definition 7. Let $(\lambda^0,\delta_1\delta_2\lambda^0{}_{n+1})\in S_1$, $(\omega^0,\mu^0,$

$\delta_1\mu_0^0) \in S_2$, If $\forall(\lambda, \delta_1\delta_2\lambda_{n+1}) \in S_1$, $\forall(\omega, \mu, \delta_1\mu_0) \in S_2$, the following holds

$$f_1(\lambda, \delta_1\delta_2\lambda_{n+1}, \omega^0, \mu^0\delta_1\mu_0^0) \leqslant f_1(\lambda^0, \delta_1\delta_2\lambda_{n+1}^0, \omega^0, \delta_1\mu_0^0) \leqslant f_1(\lambda^0, \delta_1\delta_2\lambda_{n+1}^0, \omega, \delta_1\mu_0).$$

Then $(\lambda^0, \delta_1\delta_2\lambda_{n+1}^0)$ and $(\omega^0, \mu^0, \delta_1\mu_0^0)$ is called an optimal strategy of player I and player Ⅱ in the game, respectively. Furthermore, the game value

$$G_V = f_1(\lambda^0, \delta_1\delta_2\lambda_{n+1}^0, \omega^0, \delta_1\mu^0{}_0).$$

If $G_V = 1$, DMU-j_0 is called game efficiency.

Recall the production possibility set of DEA model with cone structure ($W = V \times U$):

$$T_{GDEA} = \left\{ (x, y) \left| \begin{array}{l} X\lambda - x \in V^*, \ -Y\lambda + y \in U^*, \\ \delta_1\left(e^T\lambda + \delta_2(-1)^{\delta_3}\lambda_{n+1}\right) = \delta_1, \\ \lambda \in -K^*, \lambda_{n+1} \geqslant 0 \end{array} \right. \right\}.$$

The game G can be explained as follows. When DMU-j_0 has its efficiency index $1\left(\frac{\mu^T y}{\omega^T x\lambda + \delta_1\mu_0} = 1\right)$, player I can be treated as an evaluator. The evaluator determines $(x, y) = (X\lambda, Y\lambda)$ from the production possibility set T_{GDEA}, such that $f_1(\lambda, \delta_1\delta_2\lambda_{n+1}, \omega, \mu, \delta_1\mu_0)$ is maximized. Player Ⅱ can be treated as DMU-j_0. This player selects a weight $(\omega, \mu, \delta_1\mu_0) \in S_2$, such that $f_1(\lambda, \delta_1\delta_2\lambda_{n+1}, \omega, \mu, \delta_1\mu_0)$ is minimized. Clearly, if $G_V > 1$, then DMU-j_0 is not DEA efficient although player Ⅱ selects the most favorable weight, there may still exist a better production state with a higher efficiency rating $G_V > 1$ in the production possibility set).

Theorem 8. Under the constraint qualification[39], the game value G_v of game G equals the efficiency index of the output oriented generalized DEA model with a cone structure (GDEA-O).

9 Inverse DEA model

In 1999, Zhang et al. proposed a new DEA model when they investigated the relative efficiency of evaluation subsystems in China's economic information system[40]. Wei et al. studied the model further, and proposed an inverse DEA model[41]. In later 1999, Yan et al[1)]. extended the work to the DEA model with cone structure, and discussed the issue of resource reallocation and production input-output analysis.

A DEA model with cone structure can be given as below. For DMU-j_0, the output oriented DEA model (GDEA-O) is given as follows:

$$\text{(GDEA-O)} \quad \begin{cases} \max z, \\ X\lambda - x_0 \in V^*, \ -Y\lambda + z\,y_0 \in U^*, \\ \delta_1\left(e^T\lambda + \delta_2(-1)^{\delta_3}\lambda_{n+1}\right) = \delta_1, \\ \lambda \in -K^*, \lambda_{n+1} \geqslant 0, z \in E^1. \end{cases}$$

Let the efficiency index $z_0 > 1$. Along the direction of $-V^*$, we then increase the input x_0 to $\alpha^0 = x_0 + \Delta x$, $\Delta x \in -V^*$, and the output y_0 will be increased to β^0. Under the condition that efficiency index is unchanged (this implies a stable technical conditions), we need to estimate the new output β^0. Consider a multiobjective programming problem

$$(VP)' \quad \begin{cases} \text{V-max}\,(\beta_1, \beta_2, \ldots, \beta_n), \\ X\lambda - \alpha^0 \in V^*, \ -Y\lambda + z^0\beta_0 \in U^*, \\ \beta - y_0 \in -U^*, \\ \delta_1\left(e^T\lambda + \delta_2(-1)^{\delta_3}\lambda_{n+1}\right) = \delta_1, \\ \lambda \in -K^*, \lambda_{n+1} \geqslant 0, \end{cases}$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$.

Theorem 9. Assume that $\alpha^0 - x_0 \in -V^*$, $\beta^0 - y_0 \in -U^*$, then for the new DMU-j_0 corresponding to (α^0, β^0), the efficiency index is still z^0 if and only if β^0 is a non-dominated solution of $(VP)'$ respect to $-\text{Int}\, U^*$.

10 Efficient frontier of production possibility set and its structure

The concept of production frontier was proposed in the first DEA article of Charnes et al. We briefly discussed it further. Consider a production possibility set ($K = K_+^n$):

$$T_{GDEA} = \left\{ (x, y) \left| \begin{array}{l} X\lambda - x \in V^*, -Y\lambda + y \in U^*, \\ \delta_1\left(e^T\lambda + \delta_2(-1)^{\delta_3}\lambda_{n+1}\right) = \delta_1, \\ \lambda \in -K^*, \lambda_{n+1} \geqslant 0 \end{array} \right. \right\}.$$

Definition 8. Let $(\overline{\omega}, \overline{\mu}) \in \text{Int}\, W$, $\delta_1\delta_2(-1)^{\delta_3}\overline{\mu}_0 \geqslant 0$, and

$$\overline{\omega}^T x_j - \overline{\mu}^T y_j + \delta_1\overline{\mu}_0 \geqslant 0, \quad j = 1, 2, \ldots, n.$$

Denote

$$L = \{(x, y) \mid \overline{\omega}^T x - \overline{\mu}^T y + \delta_1\overline{\mu}_0 = 0\},$$

If $T_{GDEA} \cap L \neq \varnothing$, then L is called an efficient frontier of T_{GDEA}.

1) Yan, H., Wei, Q. L., Hao, G., DEA models for resource reallocation and production input/output estimation, City University of Hong Kong, working paper.

Theorem 10. DMU-j_0 is DEA efficient (GDEA) if and only if (x_0, y_0) is on an efficient frontier of T_{GDEA}.

The efficient frontier defined in Definition 8 is a supporting plane of T_{GDEA}. The properly efficient frontier is a facet of T_{GDEA} with direction $(\overline{\omega}, \overline{\mu}) \in \mathrm{Int}\, W$. In 1996, Yu et al. constructed the properly efficient frontier by using the K-cone, a predilection cone, in the generalized DEA model, and obtained constructive theorems of properly efficient frontier under different DEA models (C^2R, BC^2, FG and ST)[9,20].

In 1999, Wei et al.[1)] gave a much simpler method for constructing the properly (weak) efficient frontier, by applying a method of extreme point identification[42]. It is clear that to construct the efficient frontiers of a production possibility set gives an effective way to analyze the DMUs. More insightful management information can be thus obtained.

11 Economics background of DEA efficiency

From Theorem 10, DMU-j_0 is DEA efficient if and only if (x_0, y_0) is on a production frontier of the corresponding production possibility set. In general, if a DMU is on a production frontier of the production possibility set T_{BC^2} generated from the corresponding (P_{BC^2}) and (D_{BC^2}) models (satisfying Postulates 1, 2, 3 and 5), it is called to be technical efficient. That is, a DEA efficient DMU under BC^2 model is technical efficient. From Theorem 2, a DEA efficient DMU under C^2R, FG and ST models is also technical efficient. In addition, if a DMU on a production frontier of the production possibility set T_{C^2R} generated from the corresponding (P_{C^2R}) and (D_{C^2R}) models (satisfying Postulates 1, 2, 3, 4.1 and 5), it is to be scale efficient. It is clear that a DEA efficient DMU under C^2R model is both technical efficient and scale efficient.

Using DEA models to study the return-to-scale of a DMU has long been a widely concerned problem. In 1984, Banker et al. gave conditions for stable, increasing and decreasing return-to-scale under the BC^2 DEA model, provided that the optimal solution is unique[8]. In 1992, Banker et al. gave the necessary and sufficient conditions for determining return-to-scale by BC^2 DEA model[43]. In 1993, Wei et al. studied the generalized DEA model. They discussed an output oriented generalized DEA (GDEA- O) with the background of production function, and defined the stable, increasing and decreasing return-to-scale of a DMU provided that the DMU is weak DEA efficient under the BC^2 model with cone structure (we need to point out that this condition is very important, some articles missed this condition in their discussion). They then gave necessary and sufficient conditions for different returns-to-scale of a DMU, when different DEA models (C^2R, BC^2, FG and ST models with cone structure) are applied. Consider

$$(\text{GDEA-O})' \begin{cases} \min \left(\omega^{\mathrm{T}} x_0 + \delta_1 \mu_0\right), \\ \omega^{\mathrm{T}} X - \mu^{\mathrm{T}} Y + \mu_0 \delta_1 e^{\mathrm{T}} \in K, \\ \mu^{\mathrm{T}} y_0 = 1, \\ \begin{pmatrix} \omega \\ \mu \end{pmatrix} \in W. \end{cases}$$

Let $(\omega^0, \mu^0, \delta_1 \mu_0^0)$ be an optimal solution to $(\text{GDEA-O})'$. $(\text{GDEA-O})'$ has its dual programming problem

$$(\text{GDEA-O}) \begin{cases} \max z, \\ \begin{pmatrix} X\lambda - x_0 \\ -Y\lambda + y_0 \end{pmatrix} \in W^*, \\ \delta_1\left(e^{\mathrm{T}}\lambda + \delta_2(-1)^{\delta_3}\lambda_{n+1}\right) = \delta_1, \\ \lambda \in -K^*,\ \lambda_{n+1} \geqslant 0,\ z \in E^1. \end{cases}$$

Let $(\lambda^0, \delta_1\delta_2\lambda_{n+1}^0)$ be an optimal solution to (GDEA-O). In all theorems below, we assume that DMU-j_0 is weak DEA efficient under the BC^2 model ($\delta_1 = 1$, $\delta_2 = 0$).

Theorem 11 (General model).

(i) The return-to-scale of DMU-j_0 is stable if and only if DMU-j_0 is weak DEA efficient (*FG*) and weak DEA efficient (ST).

(ii) The return-to-scale of DMU-j_0 is increasing if and only if DMU-j_0 is weak DEA efficient (ST) and not weak DEA efficient (FG).

(iii) The return-to-scale of DMU-j_0 is decreasing if and only if DMU-j_0 is weak DEA efficient (FG) and not weak DEA efficient (ST).

Theorem 12 (C^2R model).

(i) The return-to-scale of DMU-j_0 is stable if and only if problem (GDEA-O) has the optimal objective value $z_0 = 1$ (weak DEA efficient (C^2R)).

(ii) The return-to-scale of DMU-j_0 is increasing if and only if problem (GDEA-O) has the optimal objective value $z_0 > 1$ and $\sum_{j=1}^{n} \lambda_j^0 < 1$.

(iii) The return-to-scale of DMU-j_0 is decreasing if and only if problem (GDEA-O) has the optimal objective value $z_0 > 1$ and $\sum_{j=1}^{n} \lambda_j^0 > 1$.

Theorem 13 (BC^2 model).

1) Wei, Q. L., Yan, H., Hao, G, Characteristics and construction method of surface and weak surface of DEA production possibility, The Hong Kong Polytechnic University, working paper.

(i) The return-to-scale of DMU-j_0 is stable if and only if problem (GDEA-O)$'$ has an optimal solution (ω^0, μ^0, μ_0^0) such that $\mu_0^0 = 0$.

(ii) The return-to-scale of DMU-j_0 is increasing if and only problem (GDEA-O)$'$ has an optimal solution (ω^0, μ^0, μ_0^0) such that $\mu_0^0 < 0$.

(iii) The return-to-scale of DMU-j_0 is decreasing if and only problem (GDEA-O)$'$ has an optimal solution (ω^0, μ^0, μ_0^0) such that $\mu_0^0 > 0$.

Theorem 14 (FG model).

(i) The return-to-scale of DMU-j_0 is stable if and only if problem (GDEA-O)$'$ has an optimal solution (ω^0, μ^0, μ_0^0) such that $\mu_0^0 = 0$.

(ii) The return-to-scale of DMU-j_0 is decreasing if and only if problem (GDEA-O)$'$ has an optimal solution (ω^0, μ^0, μ_0^0) such that $\mu_0^0 > 0$.

Theorem 15 (ST model).

(i) The return-to-scale of DMU-j_0 is stable if and only if problem (GDEA-O)$'$ has an optimal solution (ω^0, μ^0, μ_0^0) such that $\mu_0^0 = 0$.

(ii) The return-to-scale of DMU-j_0 is increasing if and only if problem (GDEA-O)$'$ has an optimal solution (ω^0, μ^0, μ_0^0) such that $\mu_0^0 < 0$.

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