Convergence on the iteration of Halley family in weak conditions

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WE have obtained the convergence theorems[1, 2] of the iteration of Halley family by the point estimate of Smale[3]. In the point estimate, map f which is desired to be solved is presumed to be analytic in some proper neighborhood at the initial value z_0. From the viewpoint of literature in numerical functional, this is a hypothesis of the strong conditions although it is necessary in the study of real computational complexity. For an iteration containing the kth derivative (or difference) in the iterative map, what the weak condition means is that f is assumed to have a continuous $(k+1)$th derivative in some neighborhood at z_0, as well as Kantorovich's classical work[4] on Newton method. The biggest difficulty to establish a convergence theorem with weak conditions is the determination of the existence of the positive root of the dominating mapping. This is a completely resolved problem[5] in the theory of algorithms because the dominating map is often chosen as a polynomial. But for a successful convergence theorem it is required that the convergence condition be expressed clearly and explicitly, while it is not sufficient to give a determining algorithm only. This is very obvious by contrasting the success of ref. [6] to the disadvantage of ref. [7]. This is also the main difficulty why the convergence theorem of iteration of Halley family has not been established with the weak conditions for a long time. Some convergence theorems have been established for the original Halley iteration but we cannot say which is better.

It is unexpected that the theory of the point estimate results in our weak condition which makes the iteration of Halley family converge. This weak condition is not only harmonical and natural, but also the convergence result obtained under this condition is very straightforward.

1 γ-condition of order k

Let E and F be real or complex Banach spaces. For $z_0 \in E$ and a positive number ρ, let $B(z_0, \rho)$ denote a ρ-neighborhood at z_0, i.e. the set of $z \in E$ satisfying $\| z - z_0 \| < \rho$ and $\overline{B(z_0, \rho)}$ denote its closure. Suppose f is a map from a domain containing $\overline{B(z_0, \rho)}$ to F.

Definition 1. We say that f satisfies γ-condition of order k about $\overline{B(z_0, \rho)}$ at z_0. If f

has a continuous $(k+1)$-derivative in $\overline{B(z_0,\rho)}$, $Df(z_0)^{-1}$ exists (i.e. $Df(z_0)^{-1}$ is a continuous linear operator) and satisfies

$$\| Df(z_0)^{-1}D^if(z_0) \| \leqslant i!\,\gamma^{i-1},\ i = 2,\cdots,k;$$

$$\| Df(z_0)^{-1}D^if(z) \| \leqslant (k+1)!\,\gamma^k(1-\gamma\| z - z_0 \|)^{-k-2},\ \forall z \in \overline{B(z_0,\rho)}.$$

Naturally, if f is analytic in $\overline{B(z_0,\rho)}$ and satisfies

$$\sup_{i\geqslant 2}\left\| \frac{1}{i!}Df(z_0)^{-1}D^if(z_0)\right\|^{\frac{1}{i-1}} = \gamma,$$

then we say f satisfies γ-condition of order ∞ about $\overline{B(z_0,\rho)}$ at z_0.

We have

Theorem 1. *For any $k > j \geqslant 1$, if f satisfies γ-condition of order k about $\overline{B(z_0,\rho)}$ at z_0, then f certainly satisfies γ-condition of order j about $\overline{B(z_0,\rho)}$ at z_0.*

Proof. Suppose that f satisfies γ-condition of order k about $\overline{B(z_0,\rho)}$ at z_0. For $z \in \overline{B(z_0,\rho)}$, we have

$$Df(z_0)^{-1}D^kf(z) = Df(z_0)^{-1}D^kf(z_0) + \int_0^1 Df(z_0)^{-1}D^{k+1}f(z_0 + \tau(z - z_0))\mathrm{d}\tau.$$

Hence,

$$\| Df(z_0)^{-1}D^kf(z) \| \leqslant k!\,\gamma^{k-1} + \int_0^1 (k+1)!\,\gamma^k(1 - \tau\gamma\| z - z_0 \|)^{-k-2}\mathrm{d}\tau$$
$$= k!\,\gamma^{k-1}(1 - \gamma\| z - z_0 \|)^{-k-1}.$$

That is to say, f satisfies γ-condition of order $k-1$ about $\overline{B(z_0,\rho)}$ at z_0. So the theorem is proved for integer k. As to the case $k = \infty$, this theorem has been proved in reference [8].

2 Iterative map

The kth iterative map of Halley iterative family is defined by

$$\begin{cases} H_{k,f}(z) = z + \delta_k(z), \\ \delta_1(z) = -Df(z)^{-1}f(z), \\ \delta_j(z) = \left\{ I + \sum_{i=2}^{j}\frac{1}{i!}Df(z)^{-1}D^if(z)\delta_{j-i+1}(z)\delta_{j-i+2}(z)\cdots\delta_{j-1}(z)\right\}^{-1}\delta_1(z), \\ \qquad j = 2,\cdots,k. \end{cases}$$

The first two elements of Halley iterative family are Newton and Halley methods:

$$H_{1,f}(z) = z - Df(z)^{-1}f(z),$$

$$H_{2,f}(z) = z - \left\{ I - \frac{1}{2}Df(z)^{-1}D^2f(z)Df(z)^{-1}f(z)\right\}^{-1}Df(z)^{-1}f(z).$$

When $E = F = C$ or $= R$,

$$H_{k,f}(z) = z + k\frac{(f(z)^{-1})^{(k-1)}}{(f(z)^{-1})^{(k)}},$$

which is the zero of Pade approximation of order $[1/(k-1)]$ of f at z_0. When f is a polynomial, $\{H_{k,f}(z)\}$ is a Bernoulli sequence approaching the zero ζ of f which is nearest to z, providing that ζ is single and unique.

We call iteration

$$z_{n+1}^{(k)} = H_{k,f}(z_n^{(k)}),\ n = 0, 1, \cdots,\ z_0^{(k)} = z_0$$

the iteration H_k of f with initial value z_0. The sequence produced by it is denoted by $\{z_n^{(k)}\}_{n=0}^{\infty}$.

The sequence $\{z_n^{(k)}\}$ defined by

$$z_1^{(n)} = H_{n,\,f}(z_0),\ n = 1,\ 2,\cdots,\ z_1^{(0)} = z_0$$

is called (generalized) Bernoulli sequence.

3 Convergence theorem

When f satisfies γ-condition of order k, let

$$\alpha = \alpha(f,\ z_0) = \beta\cdot\gamma,$$
$$\beta = \| Df(z_0)^{-1}f(z_0)\|.$$

The real map h defined by

$$h(t) = \beta - t + \gamma t^2(1-\gamma t)^{-1}$$

is called the dominating map of f under iteration H_k. For the dominating map h and the initial value $t_0^{(k)} = 0$, the sequence $\{t_n^{(k)}\}_{n=0}^{\infty}$ defined by

$$t_{n+1}^{(k)} = H_{k,\,h}(t_n^{(k)}),\ n = 0,\ 1,\cdots,\ t_0^{(k)} = 0$$

is a dominating sequence of $\{z_n^{(k)}\}_{n=0}^{\infty}$. Similarly, $\{t_1^{(k)}\}$ is a dominating sequence corresponding to Bernoulli sequence $\{z_1^{(n)}\}$. When $\alpha\leqslant 3-\sqrt{2}$, the dominating map h has two zeros

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{1+\alpha\mp\sqrt{(1+\alpha)^2-8\alpha}}{4\gamma}.$$

Both satisfy the inequality

$$\beta\leqslant r_1\leqslant\left(1+\frac{1}{\sqrt{2}}\right)\beta\leqslant\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\gamma}\leqslant r_2\leqslant\frac{1}{2\gamma}.$$

Theorem 2. *Suppose that f satisfies γ-condition of order 1 about $\overline{B(z_0,\rho)}$ at z_0, i.e. f has a continuous second derivative in $\overline{B(z_0,\rho)}$ and satisfies the inequality*

$$\| Df(z_0)^{-1}D^2f(z)\| \leqslant 2\gamma^2(1-\gamma\| z-z_0\|)^{-3}.$$

Then as $\alpha < 3-2\sqrt{2}$ and $r_1\leqslant\rho< r_2$, or $\alpha = 3-2\sqrt{2}$ and $\rho = r_1$, f has a unique zero $\zeta\in\overline{B(z_0,\ r_1)}$ in $\overline{B(z_0,\ \rho)}$.

Theorem 3. *Suppose that f satisfies γ-condition of order k about $\overline{B(z_0,\rho)}$ at z_0. Then as $\alpha<3-2\sqrt{2}$ and $r_1\leqslant\rho$, for all $1\leqslant j\leqslant k$, iteration H_j is well defined for all f and the initial value z_0 and the sequence $\{z_n^{(j)}\}_{n=0}^{\infty}$ produced by it converges to ζ. Moreover, for $n>n_0\geqslant 0$, we have*

$$\| \zeta - z_n^{(j)}\| \leqslant \frac{1-q^{(j+1)^{n_0}-1}\eta}{1-q^{(j+1)^{n}-1}\eta}\, q^{(j+1)^n-(j+1)^{n_0}}\| \zeta - z_{n_0}^{(j)}\|,$$

where $q = \xi^{\frac{1}{j}}\eta,\ \xi = \dfrac{1-\gamma r_2}{1-\gamma r_1},\ \eta = \dfrac{r_1}{r_2}$.

For $k=\infty$, besides every iteration of Halley family being convergent for f and z_0, Bernoulli sequence $\{z_1^{(n)}\}$ is also convergent and for $n>0$ we have

$$\| \zeta - z_1^{(n)}\| \leqslant \frac{1-\eta}{1-\xi\eta^{n+1}}\xi\eta^n \| \zeta - z_0\|.$$

Moreover, for the smaller zero r_1 and the dominating sequences $\{t_j^{(j)}\}_{n=0}^{\infty}$ and $\{t_1^{(n)}\}$, the equalities hold in the corresponding inequalities above.

Remark 1. The error estimates given in the above two inequalities are the relative ones. In the relative error estimates, the absolute error estimates can be obtained if we notice that ζ

$\in \overline{B(z_0, r_1)}$. For example, for the former inequality, letting $n_0 = 0$ and substituting it by $\| \zeta - z_0 \| \leqslant r_1$, we obtain

$$\| \zeta - z_n^{(j)} \| \leqslant \frac{1-\eta}{1-q^{(j+1)^n-1}\eta} q^{(j+1)^n-1} r_1.$$

Remark 2. The proof of two theorems uses the control principle. Generally speaking, they can be obtained by taking h as the dominating map here in Theorem ⅫX.1.4 in reference [4].

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