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ANALYTIC STUDY OF THE CLASSICAL EQUILIBRIUM OF HIGHLY ROTATING SPHEROIDAL POLYTROPES

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We have considered some aspects of the structural features of the classical (Newtonian) equilibrium of a highly rotating spheroidal polytrope $n = 1$, governed by the equation of state: $P =$ constant ρ^γ (P denotes the pressure, ρ the density and γ the adiabatic constant). Approximate analytical solutions to the equilibrium equations suitable for use in very short computer programs or on small calculators have been given in (u_Θ, v_Θ), (u_P, v_P), (u_ρ, v_ρ) and (ξ_Θ, Θ) planes for $\gamma = 2$ following Padé (2,2) approximation technique. Under certain transformations, the equilibrium equation has been cast into first order differential equations in (u_Θ, v_Θ), (u_P, v_P), (u_ρ, v_ρ), (z_Θ, y_Θ), (z_P, y_P) and (z_ρ, y_ρ) planes. Transformations connecting solutions in these planes have been derived. Graphical material is included showing a comparative study of the runs of u_Θ with v_Θ (Fig. 1), u_P with v_P (Fig. 2), u_ρ with v_ρ (Fig. 3), Θ with ξ_Θ (Fig. 4) and ξ with Δw (Fig. 5) for rotating ($w = 0.05$ and $w = 0.15$) and non-rotating ($w = 0$) configurations. It has been found that the present method of approach is also more suitable for the study of both slowly and highly rotating configurations.

1. Introduction

The study of the properties of polytropes has been a fascinating subject of discussion to applied mathematicians in general and to astrophysicists in particular since long (10^2 yrs according to some estimates). The theory of polytropes is fundamental not only in precise investigations of stellar structure, star formation, galactic dynamics, etc. but also in the rough estimation of some processes in real stars. Most of the stars in the sky are adequately described by Newtonian physics, without taking into account general relativity. Such Newtonian stars deserve some attention here, both because they serve as limiting cases for the more exotic objects that interest general relativists, and also because they guide us in understanding the qualitative properties of these objects. The fundamental problem of the equilibrium of a configuration under its own gravitation with underlaying law

$$P = K\rho^{1+\frac{1}{n}} \tag{1}$$

is almost due to Ritter [1] (K is a disposable constant). The foregoing relation can represent a variety of different possible physical conditions. For example, $n = 0$

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represents a homogeneous liquid, $n = 1$, a not centrally condensed matter (which is quite reasonable approximation for neutron stars of one solar mass or greater; and planets like the Earth are better approximated by a polytrope with $n = 7/10$ (Allen [2]). White dwarfs and main-sequence stars are approximated by polytropes with $1.5 \leq n \leq 3$. An isothermal perfect gas is defined by $n \to \infty$. Mass distribution and velocity dispersion in the bulge halo subsystems correspond approximately to that of a polytrope of index 5 (Mark [3]). Knowledge of polytropic (or isothermal) configurations is useful in the study of gaseous filaments or of spiral arms and globular star clusters.

Considerable amount of work has already been done towards the study of the equilibrium of static, polytropic (or isothermal) configurations (for example, Emden [4]; Eddington [5]. Milne [6]; Chandrasekhar [7]; Ostriker [8]; Taff et al [9]; Lightman [10]; Srivastava [11]; Seidov and Sharma [12]; Sharma [13]; Sharma and Yadav [14]) and rotating configurations (Jeans [15]; Chandrasekhar [16]; Roberts [17]; James [18] Monaghan and Roxburgh [19]; Carl J. Hansen et al [20]; Cunningham [21]; Sharma and Yadav [14]) in classical (Newtonian) theory. Extensive studies have also been made towards the above mentioned configurations under special relativistic treatment (for example, Stoner [22]; Kothari [23]; Chandrasekhar [7], Schatzmann [24], Sharma [13, 25, 26]) and slowly or highly rotating configurations (neutron, supermassive and polytropic stars) under general relativistic treatment (for example, Hartle [27]; Hartle and Thorne [28]; Hartle et al [29]; Hartle and Munn [30]; Sharma [31]).

In most of the above works, particularly, in rotating cases, with which we are presently concerned, the following methods have generally been adopted to solve the equilibrium equations (i) a perturbation approach, (ii) the Roche approximation, (iii) variational principle, (iv) formation of self-consistent density and potential distributions, and (v) numerical methods.

The above mentioned methods are, however, lengthy, cumbersome, and involve considerable mathematical complexities. Hence, these may not be economical for computer programming. Further, one is faced with inherent analytical difficulties for the case of highly rotating polytropes (high angular velocity Ω or ω as it needs developing in powers of Ω, expansions for the departures of the equilibria from Emden spheres). All this could be avoided by employing a much simpler method known as Padé (2,2) approximation technique, as used elsewhere (see for example, Seidov and Sharma [12], Seidov [32], Sharma [13], Sharma and Yadav [14]), to solve the equilibrium equation for rotating spheroidal polytrope of index unity ($n \leq 1$ are only physically admissible values to the present case). The main advantages of the present approach are: (i) it does not involve too much mathematical complexity related with computational work, (ii) it is less time consuming, (iii) it is computationally efficient and economical, and (iv) it is suitable for both cases of slowly and highly rotating polytropes.

First, in Section 2, we will present the structural equation in (ξ_Θ, Θ) plane. In Section 3, we will derive first-order differential equations in (u_Θ, v_Θ), (u_p, v_p), (u_ρ, v_ρ), (z_Θ, y_Θ), (z_p, y_p) and (z_ρ, y_ρ) planes. Section 4 deals with certain transformations connecting the solutions in these planes. Approximate analytical solutions describing the physical structure of the configurations in (u_Θ, v_Θ), (u_p, v_p), (u_ρ, v_ρ) and (ξ_Θ, Θ) planes for $\gamma = 2$ have been given in Section 5. Section 6 throws some light on possible values of the critical angular velocity attainable by the configurations. Concluding remarks are given in Section 7. Results of our calculations are displayed in Figs 1–5. Stability considerations or bifurcation analysis could be another interesting aspect of the problem which we, however, intend to include in future work, as some light has already been thrown towards this aspect by some of the above mentioned authors following different methods than the present one.

2. Structure equation

Structure equation in the (ξ_Θ, Θ) plane

The fundamental equation of classical equilibrium of a highly rotating (spheroidal) mass of fluid obeying a polytropic equation of state (1) is given by

$$K(n+1)\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\rho^{1/n}}{dr}\right) = -\left[\frac{4\pi G}{e}(1-e^2)^{\frac{1}{2}}\sin^{-1}e\right]\rho + 2\Omega^2, \tag{2}$$

where G is the gravitational constant, e the eccentricity, and Ω the angular velocity.

To reduce the foregoing equation to a manageable, dimensionless form, we introduce the dimensionless variables Θ and ξ_Θ defined by

$$\rho = \lambda\Theta^n; \quad r = \left[\frac{e}{(1-e^2)^{\frac{1}{2}}\sin^{-1}e}\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{\frac{1}{2}}\xi_\Theta, \tag{3}$$

and w and v by

$$w = \frac{e}{(1-e^2)^{\frac{1}{2}}\sin^{-1}e}v, \quad v = \frac{\Omega^2}{2\pi G\lambda}. \tag{4}$$

Then, Eq. (2) in (ξ_Θ, Θ) plane is obtained in the form

$$\frac{1}{\xi_\Theta^2}\frac{d}{d\xi_\Theta}\left(\xi_\Theta^2\frac{d\Theta}{d\xi_\Theta}\right) = -\Theta^n + w, \tag{5}$$

which satisfies the initial boundary conditions

$$\Theta(0) = 1, \quad \frac{d\Theta(0)}{d\xi_\Theta} = 0 \quad \text{at} \quad \xi_\Theta = 0. \tag{6}$$

3. First-order differential equations in (u_Θ, v_Θ), (u_p, v_p), (u_ρ, v_ρ), (z_Θ, y_Θ), (z_p, y_p) and (z_ρ, y_ρ) planes

3.1 First-order differential equations in (u_Θ, v_Θ), (u_p, v_p) and (u_ρ, v_ρ) planes

Let the two independent functions u_Θ and v_Θ be related with the variables ξ_Θ and Θ by

$$u_\Theta = -\frac{\xi_\Theta(\Theta^n - w)}{\Theta'}, \quad v_\Theta = -\frac{\xi_\Theta \Theta'}{\Theta}. \tag{7}$$

Then, Eq. (5) reduces to its equivalent first-order differential equation:

$$\frac{du_\Theta}{dv_\Theta} = -\frac{u_\Theta}{v_\Theta}\left[\frac{u_\Theta + n\alpha_\Theta v_\Theta - 3}{u_\Theta + v_\Theta - 1}\right]; \quad \alpha_\Theta = \frac{\Theta^n}{\Theta^n - w}. \tag{8}$$

In (r, P) and (r, ρ) planes, Eq. (5) can be written as

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{P^{\frac{n}{n+1}}} \cdot \frac{dP}{dr}\right) = -P^{\frac{n}{n+1}} + cw; \quad c = \lambda K^{\frac{n}{n+1}}, \tag{9}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n}{n+1}}} \cdot \frac{d\rho}{dr}\right) = -\rho + \lambda w, \tag{10}$$

where the dimensionless variables ξ_P and ξ_ρ are defined by

$$r \equiv \alpha_P \xi_P \equiv \left[(n+1)K^{\frac{n-1}{n+1}}\lambda^{\frac{1}{n}-1}\right]^{\frac{1}{2}} \xi_P \tag{11}$$

and

$$r \equiv \alpha_\rho \xi_\rho \equiv \left[n\lambda^{\frac{1}{n}-1}\right]^{\frac{1}{2}} \xi_\rho. \tag{12}$$

Further, if we define the four independent variables u_P and v_P, u_ρ and v_ρ by equations

$$u_P = -\frac{\gamma(P^{\frac{2n}{n+1}} - CP^{\frac{n}{n+1}}w)}{P'}, \quad v_P = -\frac{rP'}{P} \quad (P' = \frac{dP}{dr}), \tag{13}$$

$$u_\rho = -\frac{r(\rho^{2-\frac{1}{n}} - \lambda\rho^{1-\frac{1}{n}}w)}{\rho'}, \quad v_\rho = -\frac{r\rho'}{\rho} \quad (\rho' = \frac{d\rho}{dr}). \tag{14}$$

Then, we obtain from Eq. (5) the following first-order differential equations

$$\frac{du_P}{dv_P} = -\frac{u_P}{v_P}\left[\frac{u_P + \alpha_P v_P - 3}{u_P + \alpha'_P v_P - 1}\right]; \quad \alpha_P = \left(\frac{n}{n+1}\right)\frac{P^{\frac{n}{n+1}}}{P^{\frac{n}{n+1}} - Cw}, \quad \alpha'_P = \frac{1}{n+1} \tag{15}$$

and

$$\frac{du_\rho}{dv_\rho} = -\frac{u_\rho}{v_\rho}\left[\frac{u_\rho + \alpha_\rho v_\rho - 3}{u_\rho + \alpha'_\rho v_\rho - 1}\right]; \quad \alpha_\rho = \frac{\rho}{\rho - \lambda w}, \quad \alpha'_\rho = \frac{1}{n}. \tag{16}$$

3.2 First-order differential equations in (z_Θ, y_Θ), (z_P, y_P) and (z_ρ, y_ρ) planes

The relations between the variables (z_Θ, y_Θ) and (ξ_Θ, Θ) like those used in the discussion of static polytropic gas spheres are (Chandrasekhar [7])

$$z_\Theta = \log\{n(\Theta^n - w)\} + 2\log\xi_\Theta \tag{17}$$

and

$$y_\Theta = \frac{dz_\Theta}{dt_\Theta} = -\frac{n\Theta^{n-1}}{\Theta^n - w}\xi_\Theta\frac{d\Theta}{d\xi_\Theta} - 2, \tag{18}$$

respectively. In consequence of the foregoing Eqs (17) and (18), Eq. (5) reduces to the form of first order differential equation

$$\left[y_\Theta\frac{dy_\Theta}{dz_\Theta} - y - 2 - u(\Theta, y_\Theta)\right]v(\Theta) + e^{z_\Theta} = 0, \tag{19}$$

where

$$u(\Theta, y_\Theta) = \left\{\frac{(1-\frac{1}{n})(\Theta^n - w)}{\Theta^n} - 1\right\}(y+2)^2, \quad v(\Theta) = \frac{(\Theta^n - w)}{\Theta^{n-1}}.$$

If we further define

$$\begin{aligned} &\text{(i) } z_P = \xi_P^{-m}P, \quad m = -2, \\ &\text{(ii) } y_P = \frac{dz_P}{dt_P} = -\xi_P^{-m+1}\frac{dP}{d\xi_P} + mz_P, \quad \xi_P = e^{-t_P} \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\text{(i) } z_\rho = \xi_\rho^{-m}\rho, \quad m = -2, \\ &\text{(ii) } y_\rho = \frac{dz_\rho}{dt_\rho} = -\xi_\rho^{-m+1}\frac{d\rho}{d\xi_\rho} + mz_\rho, \quad \xi_\rho = e^{-t_\rho}, \end{aligned} \tag{21}$$

then Eqs (9) and (10) get transformed into two similar first-order differential equations

$$\begin{aligned} y_P\frac{dy_P}{dz_P} - y_P + mz_P - \left(\frac{n}{n+1}\right)z_P^{-1}y_P^2 + \left(\frac{m}{n+1}\right)(mz_P - 2y_P) \\ + F_{1,P}(\xi_P, z_P) - CwF_{2,P}(\xi_P, z_P) = 0, \end{aligned} \tag{22}$$

where

$$F_{1,P}(\xi_P, z_P) = \xi_P^{\frac{2-m}{n+1}} z_P^{\frac{2n}{n+1}}$$

and

$$F_{2,P}(\xi_P, z_P) = \xi_P^{\frac{mn}{n+1}} z_P^{\frac{n}{n+1}},$$

$$y_\rho \frac{dy_\rho}{dz_\rho} - y_\rho + mz_\rho + \frac{1}{n}(m^2 z_\rho - 2my_\rho) - (1 - \frac{1}{n})z_\rho^{-1} y_\rho^2 + F_{1,\rho}(\xi_\rho, z_\rho) + F_{2,\rho}(\xi_\rho z_\rho) = 0, \tag{23}$$

where

$$F_{1,\rho}(\xi_\rho, z_\rho) \equiv \xi_\rho^2 z_\rho (\xi_\rho^m z_\rho)^{1-\frac{1}{n}},$$

and

$$F_{2,\rho}(\xi_\rho, z_\rho) \equiv -\lambda w\, \xi_\rho^{2-m} (\xi_\rho^m z_\rho)^{1-\frac{1}{n}}.$$

4. Transformations connecting solutions of polytropic equations in (u_Θ, v_Θ), (u_P, v_P), (u_ρ, v_ρ) planes and (z_Θ, y_Θ), (z_P, y_P) and (z_ρ, y_ρ) planes

Dividing the first equation in (7) by the first equation in (13), and using relations in (3) and (11), we have

$$\frac{u_\Theta}{u_P} = \frac{(1-e^2)^{\frac{1}{2}} \sin^{-1} e \cdot 4\pi G}{e \cdot K^{\frac{2n}{n+1}}}, \tag{24}$$

since

$$P' = \frac{dP}{dr} = \frac{(n+1)K\lambda^{1+\frac{1}{n}} \Theta^n \cdot \Theta'}{\alpha_\Theta}, \quad (\Theta' = \frac{d\Theta}{d\xi_\Theta}) \tag{25}$$

$$\alpha_\Theta = \left[\frac{e(n+1)K}{(1-c^2)^{\frac{1}{2}} \sin^{-1} e\, 4\pi G} \lambda^{\frac{1}{n}-1} \right]^{\frac{1}{2}}.$$

We find from (24)

$$u_\Theta = c_1 u_P, \quad \text{where} \quad c_1 = \frac{(1-e^2)^{\frac{1}{2}} \sin^{-1} e \cdot 4\pi G}{e \cdot K^{\frac{2n}{n+1}}}. \tag{26}$$

Further dividing the first equation in (7) by the first equation in (14), and using relations in (3) and (12), we get

$$u_\Theta = c_2 u_\rho; \quad c_2 = \left(\frac{n}{n+1}\right) \left\{ \frac{(1-e^2)^{\frac{1}{2}} \sin^{-1} e}{e} \right\} \cdot \frac{4\pi G}{K}, \tag{27}$$

because

$$\rho' = \frac{d\rho}{dr} = \frac{n\lambda \Theta^{n-1} \cdot \Theta'}{\alpha_\Theta}.$$

Hence from Eqs (26) and (27), we can easily deduce that

$$\begin{aligned} v_\Theta &= \frac{1}{n+1} v_P, \\ v_\Theta &= \frac{1}{n} v_\rho, \\ v_P &= (1 + \frac{1}{n}) v_\rho. \end{aligned} \tag{28}$$

Similarly, using Eqs (3), (11), (12), (17), (18) and (21), one may obtain the transformations connecting the solutions in (z_Θ, y_Θ), (z_P, y_P) and (z_ρ, y_ρ) planes.

From the viewpoint of astrophysical applications, we are more interested in obtaining approximate analytical solutions of some of the above first-order differential equations, say, Eqs (5), (8), (15) and (16), as given in the following Section 5.

5. Approximate analytical solutions of the structure equations for $\gamma = 2$

5.1. Approximate analytical solutions of Eq. (8)

We assume a series expansion of Eq. (8) of the form

$$u_\Theta = 3 + a^*_\Theta v_\Theta + b^*_\Theta v^2_\Theta + c^*_\Theta v^3_\Theta + d^*_\Theta v^4_\Theta, \tag{29}$$

which satisfies the initial conditions $u_\Theta \to 3$, $v_\Theta \to 0$ as $\xi_\Theta \to 0$. With the help of Eqs (8) and (29), we may determine the coefficients $a^*_\Theta, b^*_\Theta, c^*_\Theta, d^*_\Theta, \ldots$, successively by equating the coefficients of like powers of v_Θ. Thus, we have

$$\begin{aligned} a^*_\Theta &= -\frac{3}{5} n\alpha_\Theta, & b^*_\Theta &= -\frac{1}{7} a^*_\Theta (2a^*_\Theta + 1 + n\alpha_\Theta), \\ c^*_\Theta &= -\frac{1}{9} b^*_\Theta (5a^*_\Theta + 2 + n\alpha_\Theta), & d^*_\Theta &= -\frac{1}{11} \{3b^{*2}_\Theta + c^*_\Theta (6a^*_\Theta + 3 + n\alpha_\Theta)\}. \end{aligned} \tag{30}$$

Now, we may express the function u_Θ as Padé (2.2) approximant:

$$u_\Theta = 3 \cdot \frac{1 + A^*_\Theta v_\Theta + B^*_\Theta v^2_\Theta}{1 + C^*_\Theta v_\Theta + D^*_\Theta v^2_\Theta}, \tag{31}$$

where

$$\begin{aligned} A^*_\Theta &= \frac{1}{3} a^*_\Theta + C^*_\Theta, \quad B^*_\Theta = \frac{1}{3}(b^*_\Theta + a^*_\Theta C^*_\Theta) + D^*_\Theta, \\ C^*_\Theta &= \frac{a^*_\Theta d^*_\Theta - b^*_\Theta c^*_\Theta}{\Delta^*_\Theta}, \quad D = \frac{{c^*_\Theta}^2 - b^*_\Theta d^*_\Theta}{\Delta^*_\Theta}, \\ \Delta^*_\Theta &\equiv b^{*2}_\Theta - a^*_\Theta c^*_\Theta. \end{aligned} \tag{32}$$

Figure 1 shows run of u_Θ with v_Θ for highly rotating ($w = 0.05$ and 0.15) and non-rotating ($w = 0$) configurations.

5.2. Approximate analytical solutions of Eqs (15), (16) and (5)

The series expansion of Eq. (15), near the origin $\xi_P \to 0$, satisfying the initial conditions $u_P \to 3$, $v_P \to 0$, is given by

$$u_P = 3 + a_P v_P + b_P v_P^2 + c_P v_P^3 + d_P v_P^4 + \dots \tag{33}$$

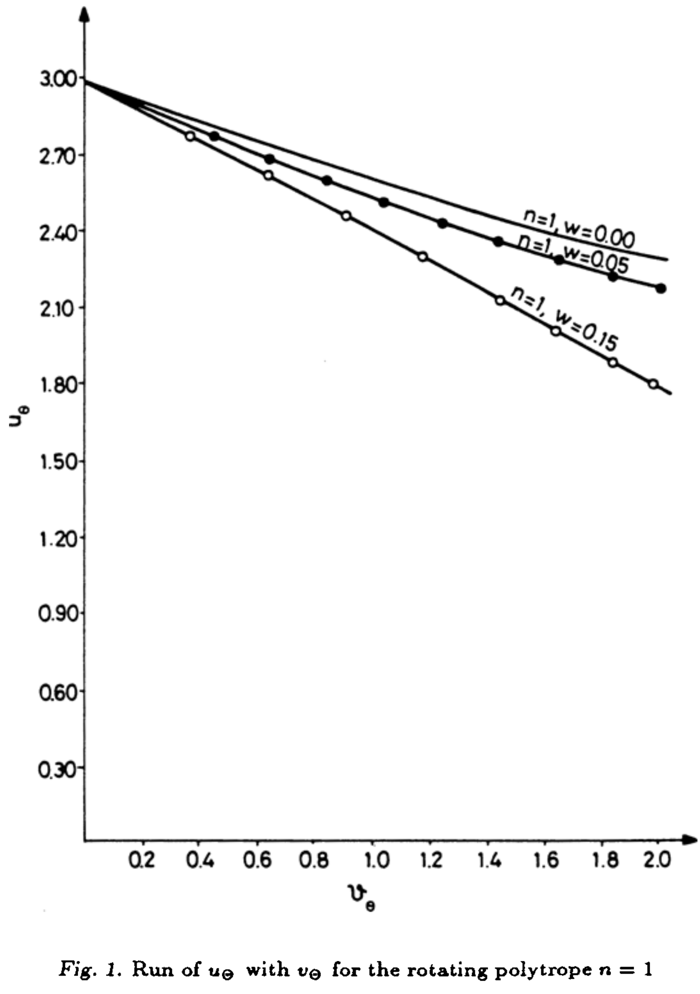

Fig. 1. Run of u_Θ with v_Θ for the rotating polytrope $n = 1$

With the initial conditions $u_\rho \to 3$, $v_\rho \to 0$, near the origin $\xi_\rho \to 0$, we obtain the series solution of Eq. (16):

$$u_\rho = 3 + a_\rho v_\rho + b_\rho v_\rho^2 + c_\rho v_\rho^3 + d_\rho v_\rho^4 + \dots \tag{34}$$

The series solution of Eq. (5), satisfying the initial conditions in (6), can be written as

$$\Theta = 1 + a_\Theta \xi_\Theta^2 + b_\Theta \xi_\Theta^4 + c_\Theta \xi_\Theta^6 + d_\Theta \xi_\Theta^8 + \dots \quad (35)$$

Corresponding to the above three series solutions u_P, u_ρ and Θ [Eqs (33), (34) and (35)], we obtain the following expressions for Padé (2.2) approximant:

$$u_P = 3 \cdot \frac{1 + A_P v_P + B_P v_P^2}{1 + C_P v_P + D_P v_P^2}, \quad (36)$$

$$u_\rho = 3 \cdot \frac{1 + A_\rho v_\rho + B_\rho v_\rho^2}{1 + C_\rho v_\rho + D_\rho v_\rho^2} \quad (37)$$

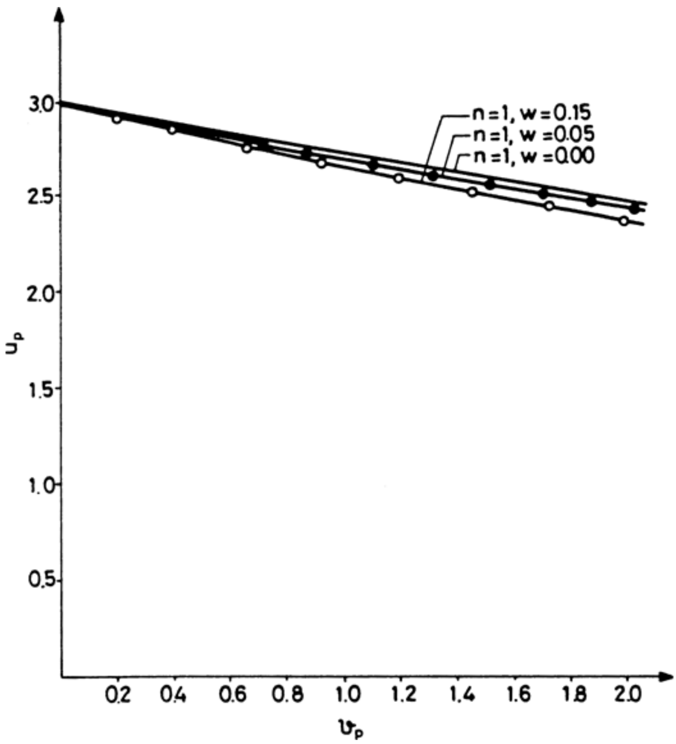

Fig. 2. Run of u_P with v_P for the rotating polytrope $n = 1$

and

$$\Theta = 3 \cdot \frac{1 + A_\Theta \xi_\Theta^2 + B_\Theta \xi_\Theta^4}{1 + C_\Theta \xi_\Theta^2 + D_\Theta \xi_\Theta^4}, \quad (38)$$

respectively.

Results of our calculations are displayed in Figs 2, 3 and 4, respectively, for two chosen values of angular velocity (w = 0.05 and 0.15). For comparison, the non-rotating case ($w = 0$) is shown by smooth curve (for values of A_P, B_P, C_P, D_P, a_P, b_P, c_P, d_P, A_ρ, B_ρ, C_ρ, D_ρ, etc. see Appendix 1).

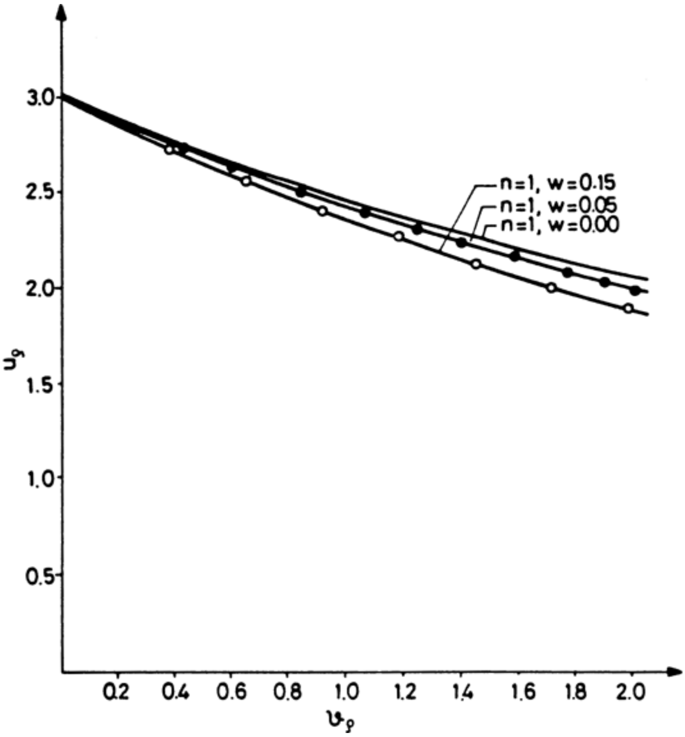

Fig. 3. Run of u_ρ with v_ρ for the rotating polytrope $n = 1$

6. Critical angular velocity

Spheroidal equilibria would bifurcate at $\Omega = \Omega_b$ (subindex 'b' means bifurcation). More explicitly, we may say that bifurcation (possibility of the two equilibria: the 'spheroidal' and 'ellipsoidal') would occur if $\Omega_b \leq \Omega_c$ or equivalently if $w_b \leq w_c$ (subindex 'c' denotes the critical value), and it does not if $w_b > w_c$. The equilibrium is broken at Ω_c. If the angular velocity Ω is increased more and more the matter would flunge away from the equator, and it would form a thin disk.

From our approximate analytical solution in Eq. (38) we may find that the value of the critical angular velocity w_c is $\simeq 0.18$ for $n = 1$ polytrope, for which $\Theta \to 0$ at $\xi = \xi_1 = 4.2976495$. This clearly suggests that due to rotation the geometrical size ξ_1 is increased by 36.94 % over its spherical shape. This result is in good agreement with that of Roberts [17] as found by variational technique.

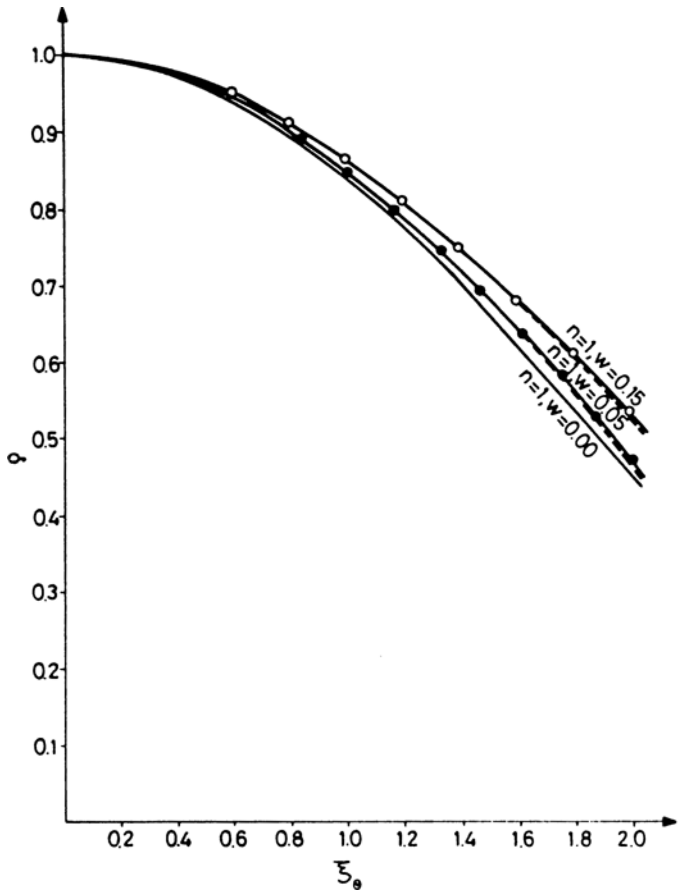

Fig. 4. Variation of density ρ, measured in units of central density λ, plotted as a function of equatorial radius ξ, for rotating polytrope $n = 1$. For comparison, cases of non-rotating and rotating polytropes ($n = 1$), respectively, are shown by solid (Chandrasekhar [7]) and dashed (Roberts [17]) curves

Further, our interest is to calculate small variation Δw ($= w_c - w_P$) in angular velocity for $n = 1$ polytrope for two chosen values of $w_P = 0.05$ and 0.15 (subindex 'P' means particular) by employing the formula.

$$w_c = \alpha^{-n} w_P, \tag{39}$$

where α is the limiting value of Θ when ξ is small. In Fig. 5 Δw is plotted with ξ.

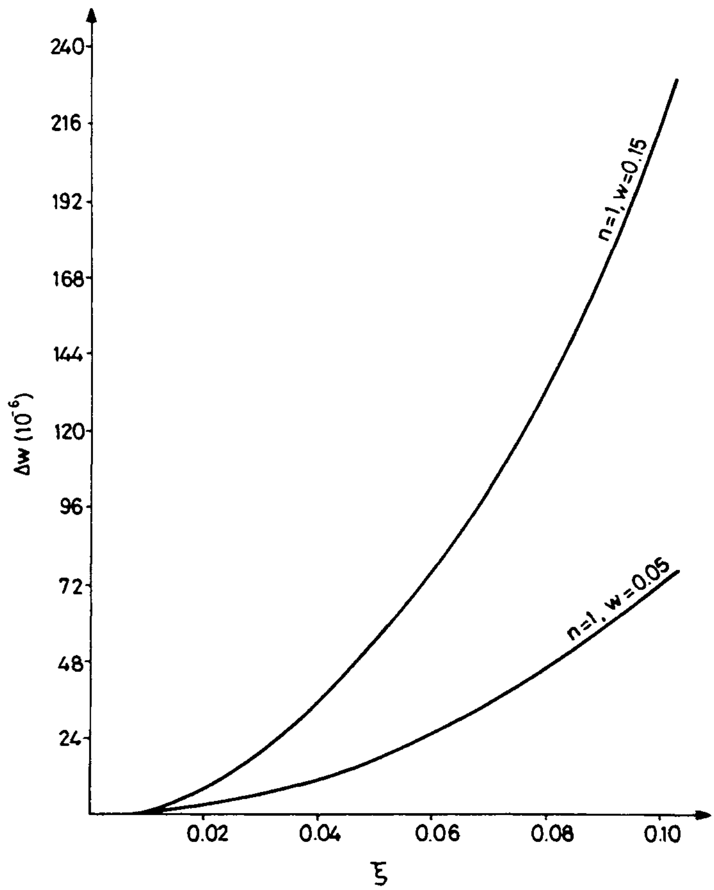

Fig. 5. Run of $\Delta w = (w_c - w_P)$ with ξ for rotating polytrope $n = 1$

7. Conclusions

The present formalism attempts to analyze analytically some structural features of the classical equilibrium of a highly rotating spheroidal $n = 1$ polytrope which obeys an equation of state: $P = K\rho^{\gamma}$. For this purpose, equation of equilibrium (2) has been transformed into first-order differential equations in (u_Θ, v_Θ), (u_P, v_P), (u_ρ, v_ρ), (z_Θ, y_Θ), (z_P, y_P) and (z_ρ, y_ρ) planes (Eqs (8), (15), (16), (19), (22), (23)). Since our previous methods, numerical, variational, perturbation analysis, etc.) would involve mathematical difficulty associated with computational work, we have derived here simple approximate analytical formulae (Eqs (31), (36), (37), (38)) in concise form from which the desired value of the physical parameter can be obtained using even an electronic pocket calculator (without using computer programs). Evidently, therefore, the present method seems more economical on computer machine than the previous ones.

Results of our calculations are displayed in Figs 1–5, for two chosen values of angular velocities, $w_P = 0.05$ and 0.15. Monotonic falls in u_Θ with v_Θ, u_P with v_P, u_ρ with v_ρ, Θ with ξ_Θ and increasing trend in Δw with ξ have been noted. As pointed out in the main body of this paper (Section 6), our present approach, when applied to bifurcation analysis, leads to yielding the value of critical angular velocity $w_c \simeq 0.18$ which is quite a good approximation in view of the previous findings (Roberts [17]).

With the help of our analytical formulae in Eqs (31), (36), (37) and (38), one may also obtain very conveniently solutions for other values of n. Our present approach may find notable applications in the discussion of stability analysis which has, however, not been included in the present work.

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Appendix 1

Values of the coefficients in Eqs (36), (37) and (38):

$$A_P = \frac{1}{3}a_P + C_P, \qquad B_P = \frac{1}{3}(b_P + a_P C_P) + D_P,$$
$$C_P = \frac{a_P d_P - b_P c_P}{\Delta_P}, \qquad D_P = \frac{c_P^2 - b_P d_P}{\Delta_P},$$
$$\Delta_P \equiv b_P^2 - a_P c_P$$

$$a_P = -\frac{3}{5}\alpha_P, \qquad b_P = -\frac{1}{7}a_P(2a_P + \alpha_P + \alpha'_P),$$
$$c_P = -\frac{1}{9}b_P(5a_P + \alpha_P + 2\alpha'_P),$$
$$d_P = -\frac{1}{11}[3b_P^2 + c_P(6a_P + \alpha_P + 3\alpha'_P)],$$

$$A_\rho = \frac{1}{3}a_\rho + C_\rho, \qquad B_\rho = \frac{1}{3}(b_\rho + a_\rho C_\rho) + D_\rho,$$
$$C_\rho = \frac{a_\rho d_\rho - b_\rho c_\rho}{\Delta_\rho}, \qquad D_\rho = \frac{c_\rho^2 - b_\rho d_\rho}{\Delta_\rho},$$
$$\Delta_\rho \equiv b_\rho^2 - a_\rho c_\rho,$$

$$a_\rho = -\frac{3}{5}\alpha_\rho, \qquad b_\rho = -\frac{1}{7}a_\rho(2a_\rho + \alpha_\rho + \alpha'_\rho),$$
$$c_\rho = -\frac{1}{9}b_\rho(5a_\rho + \alpha_\rho + 2\alpha'_\rho),$$
$$d_\rho = -\frac{1}{11}[3b_\rho^2 + c_\rho(6a_\rho + \alpha_\rho + 3\alpha'_\rho)],$$

$$A_\Theta = a_\Theta + C_\Theta, \qquad B_\Theta = b_\Theta + a_\Theta C_\Theta + D_\Theta,$$
$$C_\Theta = \frac{a_\Theta d_\Theta - b_\Theta c_\Theta}{\Delta_\Theta}, \qquad D_\Theta = \frac{c_\Theta^2 - b_\Theta d_\Theta}{\Delta_\Theta},$$
$$\Delta_\Theta \equiv b_\Theta^2 - a_\Theta c_\Theta$$

$$a_\Theta = -\frac{1}{6}(1-w), \qquad b_\Theta = -\frac{A' a_\Theta}{20},$$
$$c_\Theta = -\frac{1}{42}(A' b_\Theta + a_\Theta^2 B'), \quad d_\Theta = -\frac{1}{72}(A' c_\Theta + 2a_\Theta b_\Theta B' + a_\Theta^3 C'),$$
$$A' = \frac{n}{1!}, \quad B' = \frac{n(n-1)}{2!}, \quad C' = \frac{n(n-1)(n-2)}{3!}.$$

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