STABILITY OF TWO SUPERPOSED HOMOGENEOUS FLUIDS

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The stability of two homogeneous fluids under gravitational force has been discussed. A general perturbation in the horizontal plane $z = 0$ has been taken with wave number k_x, k_y along and perpendicular to the streaming motion, respectively. It is found that critical wave number k^* lies on an ellipse in the first quadrant of k_x, k_y plane.

Introduction

Initially, Jeans [1] studied the problem of gravitational instability of a static infinite homogeneous medium. He found a critical wave number $k^*[= 4\pi G\rho/C^2]^{1/2}$ and showed that the system becomes unstable for all perturbations of wave numbers less than k^*. Here C stands for velocity of sound ρ for density of medium and G for gravitational constant. Ledoux [2] considered this problem of stability in an infinite isothermal medium and showed that the medium in unstable for perturbations propagating parallel to the plane of symmetry of the medium. Ficke [3] discussed this problem with effect of rotation. Chandrasekhar [4,5,6] reviewed the work of Jeans and showed that, when the medium is rotating with an angular velocity Ω and perturbation is propagating in perpendicular direction then the critical wave number k^* is given by

$$k^* = \left[\frac{4\pi G\rho - 4\Omega^2}{C^2}\right]^{1/2}.$$

Later Sharma and Thakur [7] considered the problem of two fluids in porous medium. Here, we propose to discuss the problem of instability of two superposed homogeneous fluids for general perturbation in horizontal plane $z = 0$. A general dispersion relation be obtained. Critical wave number k^* will be derived and some special cases will be discussed.

Mathematical formulation of the problem

The two streams of different densities are separated by the plane $z = 0$, such that in the region $z > 0$ the system is of density ρ_1 and in the region $z < 0$ the system is of density ρ_2. The streams are moving along the x axis with velocity V_1 in region $z > 0$ and V_2 in region $z < 0$. The external force on the system is the gravitational force.

Following Chandrasekhar [8] the linearized perturbation equations are

$$\rho_r\left(\frac{\partial}{\partial t}+V_r\frac{\partial}{\partial x}\right)u_r=-\frac{\partial}{\partial x}\delta p_r+\rho_r\frac{\partial}{\partial x}\delta\varphi_r, \tag{1}$$

$$\rho_r\left(\frac{\partial}{\partial t}+V_r\frac{\partial}{\partial x}\right)v_r=-\frac{\partial}{\partial y}\delta p_r+\rho_r\frac{\partial}{\partial y}\delta\varphi_r, \tag{2}$$

$$\rho_r\left(\frac{\partial}{\partial t}+V_r\frac{\partial}{\partial x}\right)w_r=-\frac{\partial}{\partial z}\delta p_r+\rho_r\frac{\partial}{\partial z}\delta\varphi_r, \tag{3}$$

$$\left(\frac{\partial}{\partial t}+V_r\frac{\partial}{\partial x}\right)\delta\rho_r=-\rho_r\left(\frac{\partial u_r}{\partial x}+\frac{\partial v_r}{\partial y}+\frac{\partial w_r}{\partial z}\right), \tag{4}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\delta\varphi_r=-4\pi G\delta\rho_r, \tag{5}$$

$$\left(\frac{\partial}{\partial t}+V_r\frac{\partial}{\partial x}\right)\delta p_r=C_r^2\left(\frac{\partial}{\partial t}+V_r\frac{\partial}{\partial x}\right)\delta\rho_r. \tag{6}$$

Here the suffix r stands for the two regions. For $r = 1$ we have the region $z > 0$ and for $r = 2$ we have $z < 0$. (u, v, w) are components of perturbation of velocity along x, y, z axes, respectively. C_r is the velocity of sound in the medium and V_r is the streaming velocity in the region along x axis. Other symbols have their usual meanings.

We ascribe to all quantities describing the perturbation a dependence on x, y and t of the form

$$\Psi(z)e^{i(k_x x+k_y y)+nt}, \tag{7}$$

where

$$k=\sqrt{k_x^2+k_y^2}. \tag{8}$$

Here k_x, k_y are the real numbers denoting the wave numbers of the propagation of the disturbance along x and y axes, respectively. k given by (8) is the wave number of the disturbance. $i = \sqrt{-1}$, t is symbol for time and $\psi(z)$ denotes some functions of z. n is a constant, in general a complex number, of the form $n = n_R + in_I$.

For the perturbation of the form (7) we have

$$\frac{\partial}{\partial t}=n,\ \frac{\partial}{\partial x}=ik_x,\ \frac{\partial}{\partial y}=ik_y,\ \frac{\partial}{\partial z}=D$$

and

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=D^2-k^2. \tag{9}$$

Now writing

$$\sigma_r = n + ik_x V_r \tag{10}$$

and taking the perturbation of the form (7) we get linearized Eqs (1) to (6) as

$$\rho_r \sigma_r u_r = - ik_x \delta p_r + ik_x \rho_r \delta\phi_r, \tag{11}$$

$$\rho_r \sigma_r v_r = - ik_y \delta p_r + ik_y \rho_r \delta\phi_r, \tag{12}$$

$$\rho_r \sigma_r w_r = - D\, \delta p_r + \rho_r D\, \delta\phi_r, \tag{13}$$

$$\sigma_r \delta\rho_r = - \rho_r (ik_x u_r + ik_y v_r + Dw_r), \tag{14}$$

$$(D^2 - k^2)\delta\varphi_r = - 4\pi G \delta\rho_r, \tag{15}$$

$$\delta p_r = C_r^2 \delta\rho_r. \tag{16}$$

Substituting the value of u_r, v_r and w_r from (11) and (13), respectively, in (14) and eliminating δp_r with the help of (16) we get

$$[\sigma_r^2 - (D^2 - k^2)C_r^2]\delta\rho_r = -\rho_r (D^2 - k^2)\delta\varphi_r. \tag{17}$$

Eliminating $\delta\rho_r$ from (15) and (17) we get a fourth order differential equation in $\delta\varphi_r$ as

$$(D^2 - k^2)(D^2 - \alpha_r^2)\delta\varphi_r = 0, \tag{18}$$

where

$$\alpha_r^2 = \frac{\sigma_r^2 + k^2 C_r^2 - 4\pi G \rho_r}{C_r^2}. \tag{19}$$

Solutions of the differential equation

The differential equations are solved subject to the physical conditions of the problem. The solutions are to be bounded in the two regions. This leads to the solution of (18) in the two regions giving $\delta\varphi_1$ in the region $z > 0$ and $\delta\varphi_2$ in the region $z < 0$, as

$$\delta\varphi_1 = \lambda_1 e^{-kz} + \mu_1 e^{-\alpha_1 z}, \tag{20}$$

$$\delta\varphi_2 = \lambda_2 e^{kz} + \mu_2 e^{\alpha_2 z}, \tag{21}$$

where α_1, α_2 are non-negative quantities. λ_1, μ_1, λ_2 and μ_2 are arbitrary constants in the above equations, to be determined with the help of the four boundary conditions.

Boundary condition 1:

Perturbed gravitational potential $\delta\varphi$ is continuous at $z = 0$, i.e.

$$\delta\varphi_1 = \delta\varphi_2 \quad \text{at} \quad z = 0,$$

this gives,

$$\lambda_1 + \mu_1 = \lambda_2 + \mu_2,$$

or

$$\lambda_1 + \mu_1 - \lambda_2 - \mu_2 = 0. \tag{22}$$

Boundary condition 2:
Normal derivative of the perturbed potential is continuous at $z = 0$, i.e.

$$D\delta\varphi_1 = D\delta\varphi_2 \quad \text{at} \quad z = 0,$$

this gives,

$$-k\lambda_1 - \alpha_1\mu_1 = k\lambda_2 + \alpha_2\mu_2,$$

or

$$k\lambda_1 + \alpha_1\mu_1 + k\lambda_2 + \alpha_2\mu_2 = 0. \tag{23}$$

Boundary condition 3:
Total perturbed pressure is continuous at $z = 0$, i.e.

$$\delta p_1 = \delta p_2 \quad \text{at} \quad z = 0,$$

this gives,

$$C_1^2\delta\rho_1 = C_2^2\delta\rho_2 \quad \text{at} \quad z = 0,$$

i.e.

$$C_1^2(D^2 - k^2)\delta\varphi_1 = C_2^2(D^2 - k^2)\delta\varphi_2 \quad \text{at} \quad z = 0,$$

i.e.

$$o\lambda_1 + C_1^2(\alpha_1^2 - k^2)\mu_1 + o\lambda_2 - C_2^2(\alpha_2^2 - k^2)\mu_2 = 0. \tag{24}$$

Boundary condition 4:
Normal displacement of any point is unique at the interface $z = 0$ equivalently,

$$\frac{w_1}{\sigma_1} = \frac{w_2}{\sigma_2} \quad \text{at} \quad z = 0.$$

Now from Eqs (1.13), (1.15) and (1.16) eliminating δp_r, and $\delta\rho_r$ we get

$$w_r = \frac{1}{\sigma_r}\left[1 + \frac{C_r^2}{4\pi G\rho_r}(D^2 - k^2)\right] D\delta\varphi_r.$$

Hence the above condition gives

$$\frac{1}{\sigma_1^2}\left[1 + \frac{C_1^2}{4\pi G\rho_1}(D^2 - k^2)\right] D\delta\varphi_1 = \frac{1}{\sigma_2^2}\left[1 + \frac{C_1^2}{4\pi G\rho_2}(D^2 - k^2)\right] D\delta\varphi_2 \quad \text{at} \quad z = 0,$$

i.e.

$$\frac{k}{\sigma_1^2}\lambda_1 + \frac{\alpha_1}{\sigma_1^2}\left[1+\frac{C_1^2}{4\pi G\rho_1}(\alpha_1^2-k^2)\right]\mu_1 + \frac{k}{\sigma_2^2}\lambda_2 + \frac{\alpha_2}{\sigma_2^2}\left[1+\frac{C_2^2}{4\pi G\rho_2}(\alpha_2^2-k^2)\right]\mu_2 = 0 \tag{25}$$

Writing the above linear equations in matrix form we get,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \mu_1 \\ \lambda_2 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

or, symbolically as

$$[a_{ij}][X_j] = [0], \tag{26}$$

where

$$\begin{array}{llll} X_1 = \lambda_1, & X_2 = \mu_1, & X_3 = \lambda_2, & X_4 = \mu_2, \\ a_{11} = 1, & a_{12} = 1, & a_{13} = -1, & a_{14} = -1, \\ a_{21} = k, & a_{22} = \alpha_1, & a_{23} = k, & a_{24} = \alpha_2, \\ a_{31} = 0, & a_{32} = C_1^2(\alpha_1^2 - k^2), & a_{33} = 0, & a_{34} = -C_2^2(\alpha_2^2 - k^2), \\ a_{41} = \frac{k}{\sigma_1^2}, & a_{42} = \frac{\alpha_1}{\sigma_1^2}\left[1+\frac{C_1^2}{4\pi G\rho_1}(\alpha_1^2-k^2)\right], & & \\ a_{43} = \frac{k}{\sigma_2^2}, & a_{44} = \frac{\alpha_2}{\sigma_2^2}\left[1+\frac{C_2^2}{4\pi G\rho_2}(\alpha_2^2-k^2)\right]. & & \end{array}$$

For non trivial solutions of the Eq. (26) we must have the determinant of the coefficient equal to zero, i.e.

$$|a_{ij}| = 0. \tag{27}$$

Simplifying the above determinant we get

$$\begin{vmatrix} 2C_1^2(\alpha_1+k), & C_2^2(\alpha_2+k) + C_1^2(\alpha_1+k) \\ \left(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right) - \frac{2C_1^2\alpha_1(\alpha_1+k)}{4\pi G\rho_1\sigma_1^2}, & \left(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right) + \frac{1}{4\pi G}\left[\frac{C_2^2(\alpha_2+k)}{\rho_2\sigma_2^2} - \frac{C_1^2\alpha_1(\alpha_1+k)}{\rho_1\sigma_1^2}\right] \end{vmatrix} =$$

or

$$\left(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right)\left[\frac{1}{C_1^2(\alpha_1+k)} - \frac{1}{C_2^2(\alpha_2+k)}\right] = \frac{1}{2\pi G}\left[\frac{\alpha_2}{\rho_2\sigma_2^2} + \frac{\alpha_1}{\rho_1\sigma_1^2}\right]. \tag{28}$$

Equation (28) is the dispersion relation for the problem in the most general case. Solving this and putting $n = 0$ we can get the critical wave number k^*. This k^* determines the criterion for instability. However, it is not possible, in general, to get the value of $k = k^*$ from the dispersion relation in closed form. The numerical value of k^* can be obtained in a specific physical problem. In order to get a feel of the solution, we do this in special cases of physical interest.

Special case 1:

Let the two streams be moving in opposite directions with equal velocities parallel to x axis, i.e. $\mathbf{V}_1 = V_1\mathbf{i}$, $\mathbf{V}_2 = -V_2\mathbf{i}$.

In this case

$$\sigma_1^2 = \sigma_2^2 = (n + ik_xV)^2 .$$

Putting this value in Eq. (28) we get

$$\frac{\alpha_1}{\rho_1} + \frac{\alpha_2}{\rho_2} = 0. \tag{29}$$

Simplifying this equation and putting the value of α_1^2 and α_2^2 from (19) we have the above dispersion relation as

$$n(n + i2k_xV)\left[\rho_2^2C_2^2 - \rho_1^2C_1^2\right] + \left[\rho_2^2C_2^2(C_1^2 - V^2) - \rho_1^2C_1^2(C_2^2 - V^2)\right]k_x^2$$
$$+ C_1^2C_2^2\left[\rho_2^2 - \rho_1^2\right]k_y^2 = 4\pi G\rho_1\rho_2\left[\rho_2C_2^2 - \rho_1C_1^2\right].$$

For discussing the marginal state when the instability sets in we put $n = 0$ in the above dispersion relation and obtain the critical wave number $k^*\left(= \sqrt{k_x^2 + k_y^2}\right)$. Thus

$$[\rho_2^2C_2^2(C_1^2 - V^2) - \rho_1^2C_1^2(C_2^2 - V^2)]k_x^2 + C_1^2C_2^2[\rho_2^2 - \rho_1^2]k_y^2 = 4\pi G\rho_1\rho_2[\rho_2C_2^2 - \rho_1C_1^2],$$

i.e.

$$\frac{k_x^2}{\Delta_1^2} + \frac{k_y^2}{\Delta_2^2} = 1, \tag{30}$$

where

$$\Delta_1^2 = \frac{4\pi G\rho_1\rho_2[\rho_2C_2^2 - \rho_1C_1^2]}{\rho_2^2C_2^2(C_1^2 - V^2) - \rho_1^2C_1^2(C_2^2 - V^2)},$$

$$\Delta_2^2 = \frac{4\pi G\rho_1\rho_2[\rho_2C_2^2 - \rho_1C_1^2]}{C_1^2C_1^2[\rho_2^2 - \rho_1^2]}.$$

We observe that when the perturbation propagates along both the axes x and y with wave numbers k_x and k_y, respectively, then the value of the critical wave number k^* lies on the elliptic orbit in the first quadrant given by Eq. (30) whose axes are $k_y = 0$ and $k_x = 0$. Thus the positive k_x, k_y plane is divided in two regions by the marginal state elliptic curve (30). One is the unstable region where $k < k^*$ and the other is the stable region where $k > k^*$.

We also see that if we have horizontal wave propagation of the perturbation along and perpendicular to the streaming motion then the criterion for the stability is different from those as it would be when the perturbation is propagating only along the streaming motion or perpendicular to the streaming motion. k_x, k_y play a combined role in deciding k^*, and it is not just by simple addition but by the rule of Eq. (30). For a given set of k_x, k_y the critical wave number $k^* = \sqrt{k_x^2 + k_y^2}$ does not mean that k_x and k_y are separately critical numbers. It is critical only when one of them is zero, i.e. when $k_x = 0$, $k^* = k_y$ or when $k_y = 0$, $k^* = k_x$.

Particular cases

1. When the perturbation is propagating only along the streaming motion, then $k_y = 0$ and $k_x = k^*$ is given by

$$k^* = \left[\frac{4\pi G\rho_1\rho_2(\rho_2 C_2^2 - \rho_1 C_1^2)}{\rho_2^2 C_2^2(C_1^2 - V^2) - \rho_1^2 C_1^2(C_2^2 - V^2)}\right]^{1/2}. \tag{31}$$

The above expression clearly shows that the determination of the stability is dependent on streaming velocity and it has destabilizing effect on the stability. k^* also depends on density and sound velocity in the medium. Similar results have been obtained by Sengar and Khare.

2. When the perturbation is propagating perpendicular to the direction of the streaming motion in the horizontal plane, then $k_x = 0$ and $k_y = k^*$ is given by

$$k^* = \left[\frac{4\pi G\rho_1\rho_2(\rho_2 C_2^2 - \rho_1 C_1^2)}{C_1^2 C_2^2(\rho_2^2 - \rho_1^2)}\right]. \tag{32}$$

This expression is free from streaming velocity showing that in this case the stability of the system is unaffected by the streaming motion.

Further considering Eq. (29) since α_1 and α_2 are non-negative, it follows that

$$\alpha_1 = \alpha_2 = 0, \tag{33}$$

i.e.

$$\frac{-k_x^2 V^2 + k^2 C_1^2 - 4\pi G\rho_1}{C_1^2} = \frac{-k_x^2 V^2 + k^2 C_2^2 - 4\pi G\rho_2}{C_2^2} = 0,$$

giving

$$\frac{k_x^2}{\frac{4\pi G\rho_1}{C_1^2 - V^2}} + \frac{k_y^2}{\frac{4\pi G\rho_1}{C_1^2}} = 1 \tag{34}$$

and

$$\frac{k_x^2}{\frac{4\pi G\rho_2}{C_2^2 - V^2}} + \frac{k_y^2}{\frac{4\pi G\rho_2}{C_2^2}} = 1. \tag{35}$$

Thus the two media become disentangled for the stability conditions in this case and the system becomes unstable for the wave number $k\left(\sqrt{k_x^2 + k_y^2}\right)$ whenever it is less than k^* given by (34) or (35) for the two regions, respectively. Particularly, when the perturbation is propagating only along the streaming motion, i.e. $k_x = k^*$, $k_y = 0$ we have

$$k^* = \frac{4\pi G\rho_1}{C_1^2 - V^2} \quad \text{and} \quad \frac{4\pi G\rho_2}{C_2^2 - V^2}. \tag{36}$$

When the perturbation is propagating in the perpendicular direction to the streaming motion in its plane, i.e. $k_y = k^*$, $k_x = 0$ we have

$$k^* = \frac{4\pi G\rho_1}{C_1^2} \quad \text{and} \quad \frac{4\pi G\rho_2}{C_2^2}. \tag{37}$$

The results (36) clearly show that k^* dependence is only on streaming velocity, medium density and velocity of sound in the medium. It is independent of the other medium density and sound velocity. Thus the values of k^* show that the system is decomposed.

Similarly in the case of (37) the system is decomposed with k^* depending only on the medium density and sound velocity. The streaming velocity has no effect on k^*.

This decomposition of the system in two separate media suggests that the two media may be treated independently for various results under consideration.

Special case 2:

For a single homogeneous medium when the two streams are moving in opposite direction with equal velocity v, (i.e. $\rho_1 = \rho_2 = \rho$, $C_1 = C_2 = C$, $V_1 = V$, $V_2 = -V$) we have Eq. (33) as

$$\alpha_1 = \alpha_2 = 0,$$

giving

$$\frac{k_x^2}{\frac{4\pi G\rho}{C^2 - V^2}} + \frac{k_y^2}{\frac{4\pi G\rho}{C^2}} = 1, \tag{38}$$

which determines the critical wave number k^* satisfying (38) and $k^* = \sqrt{k_x^2 + k_y^2}$.

In particular, for the perturbation along the streaming motion $k_y = 0$ and

$$k_x = k^* = \frac{4\pi G\rho}{C^2 - V^2} \tag{39}$$

and for the perturbation perpendicular to the streaming motion $k_x = 0$ and

$$k_y = k^* = \left[\frac{4\pi G\rho}{C^2}\right]^{1/2}. \tag{40}$$

Obviously, for the single static homogeneous medium

$$k^* = \left[\frac{4\pi G\rho}{C^2}\right]^{1/2}, \tag{41}$$

which is Jeans result. Talwar and Kalra have obtained a similar result.

Special case 3:

Let the two media be at rest, i.e. $V_1 = V_2 = 0$. Then $\sigma_r^2 = n^2$. Putting in dispersion relation (28) we get,

$$\frac{\alpha_1}{\rho_1} + \frac{\alpha_2}{\rho_2} = 0,$$

where

$$\alpha_r^2 = \frac{n^2 - 4\pi G\rho_r + k^2 C_r^2}{C_r^2}.$$

From these two relations we get

$$\rho_2^2 C_2^2 [n^2 - 4\pi G\rho_r + k^2 C_1^2] = \rho_1^2 C_1^2 [n^2 - 4\pi G\rho_2 + k^2 C_2^2],$$

i.e.

$$n^2 = \frac{4\pi G(\rho_1\rho_2^2 C_2^2 - \rho_2\rho_1^2 C_1^2) - k^2 C_1^2 C_2^2 (\rho_2^2 - \rho_1^2)}{(\rho_2^2 C_2^2 - \rho_1^2 C_1^2)}.$$

For the critical wave number k^* we put $n = 0$ in the above equation and get

$$k^* = \sqrt{k_x^2 + k_y^2} = \left[\frac{4\pi G\rho_1\rho_2[\rho_2 C_2^2 - \rho_1 C_1^2]}{C_1^2 C_2^2(\rho_2^2 - \rho_1^2)}\right]^{1/2}, \tag{42}$$

which shows that k^* follows a circular path of radius

$$\left[\frac{4\pi G\rho_1\rho_2[\rho_2 C_2^2 - \rho_1 C_1^2]}{C_1^2 C_2^2(\rho_2^2 - \rho_1^2)}\right]^{1/2},$$

i.e. in every direction of perturbation propagation for the wave number $k < k^*$ given by (42) the system is unstable and for $k > k^*$ it is stable.

Conclusion

A general dispersion relation for horizontal propagation has been derived. The limitation of obtaining a general solution for k^* has been discussed and results obtained in special cases. It is suggested that numerical calculation may be made to get some results.

For some special cases the critical wave number has been obtained. In particular, we discussed the stability criteria for the perturbation propagation along the streaming motion, and perpendicular to the streaming motion separately. We found that the streaming motion has destabilizing effect when the perturbation is propagating along the streaming motion, but for perpendicular propagation the instability criterion is unaffected by the streaming motion. In general, k^* follows an elliptic path in first quadrant. The value of critical wave number k^* can be found for the perturbation propagation in any direction in horizontal plane $z = 0$. In case of static medium, i.e. in the absence of streaming motion, the value of k^* is the same in every direction. In other words, we find a circular path in first quadrant for k^*, of a radius equal to Jeans critical wave number k_j^*. But in the presence of streaming motion, because of destabilizing effect of streaming velocity, the value of k^* is increased from k_j^* for perturbation propagation in every direction of the horizontal plane $z = 0$ other than the transversal. As a result, the circular orbit changes into an elliptic one. We further observe that the stabilizing tendency is dependent on the wave number, therefore the system has maximum stabilizing tendency for the transversal perturbation propagation and minimum for the perturbations propagating parallel to the streaming velocity.

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