Japan J. Indust. Appl. Math., 18 (2001), 161–180

A Competition-Diffusion System Approximation to the Classical Two-Phase Stefan Problem

To the memory of Professor Masaya Yamaguti

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Received August 5, 2000
Revised December 5, 2000

A new type of competition-diffusion system with a small parameter is proposed. By singular limit analysis, it is shown that any solution of this system converges to the weak solution of the two-phase Stefan problem with reaction terms. This result exhibits the relation between an ecological population model and water-ice solidification problems.

Key words: competition-diffusion systems, singular limit analysis, two-phase Stefan problem

1. Introduction

For the theoretical understanding of spatial patterns arising in population dynamics, several free boundary problems have been proposed. They model the dynamics of patterns such as segregation and aggregation of biological individuals [7, 15, for instance]. Among them, Mimura, Yamada and Yotsutani [13, 14] proposed a free boundary problem for two competing species which are regionally segregated. The problem can be stated as follows: Let $u(x,t)$ and $v(x,t)$ be respectively the densities of the competing species at position x and time t and let $\Omega_u(t)$ and $\Omega_v(t)$ be the habitats for u and v in a bounded region Ω in $\mathbf{R}^N$, that is,

$$\Omega_u(t) = \{x \in \Omega \mid u(x,t) > 0 \ \text{ and } \ v(x,t) = 0\} \tag{1.1}$$

and

$$\Omega_v(t) = \{x \in \Omega \mid v(x,t) > 0 \ \text{ and } \ u(x,t) = 0\}. \tag{1.2}$$

The evolution equations for u and v are given by

$$\begin{cases} u_t = d_1 \Delta u + f(u) & \text{in } \ \Omega_u(t), \\ v_t = d_2 \Delta v + g(v) & \text{in } \ \Omega_v(t), \end{cases} \tag{1.3}$$

where $f(u)$ and $g(v)$ are the growth terms for u and v, respectively. An example of f is $f(u) = r(1 - u/K)u$ with the intrinsic growth rate r and the carrying capacity K, which are both positive constants. The function g is defined in a similar way. Let $\Gamma(t)$ be the interface between $\Omega_u(t)$ and $\Omega_v(t)$, namely

$$\Gamma(t) = \Omega \setminus (\Omega_u(t) \cup \Omega_v(t)),$$

which is a free boundary (also called a *segregating boundary*). On the interface, it is assumed that

$$u = v = 0 \quad \text{on} \quad \Gamma(t) \tag{1.4}$$

and that the normal velocity of the interface V_n from $\Omega_u(t)$ to $\Omega_v(t)$ is given by

$$V_n = -\frac{d_1}{k_1}\frac{\partial u}{\partial n} - \frac{d_2}{k_2}\frac{\partial v}{\partial n} \quad \text{on} \quad \Gamma(t). \tag{1.5}$$

Here k_1 and k_2 are some positive constants which indicate respectively the magnitude of the competition through fluxes onto the interface; n is the unit normal vector on $\Gamma(t)$ oriented from $\Omega_u(t)$ to $\Omega_v(t)$ and $\partial u/\partial n$ (resp. $\partial v/\partial n$) is regarded as a boundary value on $\partial\Omega_u(t)$ (resp. $\partial\Omega_v(t)$).

One can interpret the problem (1.3)–(1.5) from an ecological viewpoint as follows: Suppose that the competition between two species is very strong, then one can expect that the regional segregation occurs for two competing species u and v so that the habitat will be divided into two subregions $\Omega_u(t)$ and $\Omega_v(t)$ by an interface $\Gamma(t)$ where (1.4) holds. It is plausible that the dynamics of u (resp. v) in $\Omega_u(t)$ (resp. $\Omega_v(t)$) is described by (1.3). The remaining problem is to formulate the equation describing the motion of $\Gamma(t)$. They assumed that the struggle of the two species for obtaining their habitats is represented by the difference of the normal fluxes of u and v onto the interface, which can be understood as a kind of competition effect for two species on the segregating boundary. In particular when $k_1 = k_2 = \lambda$, the equation (1.5) is known as the classical two-phase Stefan condition which describes solidification, if $u - v$ is regarded as the temperature, where the constant λ is the *latent heat*. For the one-dimensional problem corresponding to (1.3)–(1.5), qualitative behavior of solutions was almost completely analyzed by the authors [13, 14].

Apart from (1.3)–(1.5), a well known reaction-diffusion (RD) equation model for two competing species is proposed in mathematical ecology. It is described by

$$\begin{cases} u_t = d_1\Delta u + f(u) - \dfrac{s_1 uv}{\epsilon}, & x \in \Omega,\ t > 0, \\ v_t = d_2\Delta v + g(v) - \dfrac{s_2 uv}{\epsilon}, & x \in \Omega,\ t > 0, \end{cases} \tag{1.6}$$

where s_1/ϵ, s_2/ϵ are the interspecific competition rates between u and v. If s_1/s_2 is small, for instance, the influence of competition on u is weaker than that on v. From a modelling viewpoint, the following question arises: Is there any relation between

(1.3)–(1.5) and (1.6)? If $\epsilon > 0$ in (1.6), the two species coexist everywhere in Ω by the effect of diffusion. However, if the competition rate is so large (ϵ is so small), one can expect that the two species hardly coexist and are spatially segregated. Recently, taking the limit $\epsilon \to 0$, Dancer, Hilhorst, Mimura and Peletier [4] have shown the following: these systems (1.3)–(1.5) and (1.6) are quite close. That is, in the limit where ϵ tends to zero, the habitats of u and v are completely separated by an interface $\Gamma(t)$ and that (1.3) and (1.4) truly hold, but (1.5) is replaced by

$$0 = -\frac{d_1}{s_1}\frac{\partial u}{\partial n} - \frac{d_2}{s_2}\frac{\partial v}{\partial n} \quad \text{on } \Gamma(t). \tag{1.7}$$

They have concluded that the free boundary problem (1.3), (1.4) and (1.7) is an approximation to the competition-diffusion system (1.6) when the interspecific competition is very large. Conversely speaking, the RD system (1.6) is a good approximation to the classical two-phase Stefan type free boundary problem with reaction terms when the latent heat vanishes.

Motivated by the results above, we naturally address the following question: Are there any RD system approximations to the Stefan problem with positive latent heat (1.3)–(1.5) which was proposed in [13, 14]? The aim of this paper is to answer this question. We propose here the following RD system for three unknowns (u, v, w) with a small positive parameter ϵ:

$$\begin{cases} u_t = d_1\Delta u + f(u) - \dfrac{s_1uv}{\epsilon} - \dfrac{k_1(1-w)u}{\epsilon}, & x \in \Omega,\ t > 0, \\ v_t = d_2\Delta v + g(v) - \dfrac{s_2uv}{\epsilon} - \dfrac{k_2wv}{\epsilon}, & x \in \Omega,\ t > 0, \\ w_t = \dfrac{(1-w)u}{\epsilon} - \dfrac{wv}{\epsilon}, & x \in \Omega,\ t > 0, \end{cases} \tag{1.8}$$

where $k_i = \lambda s_i$ $(i = 1, 2)$ for some positive constant λ. When $\lambda = 0$, (1.8) is obviously reduced to (1.6). The third variable w is regarded as an approximation of the characteristic function of the habitat of the species u. We suppose that the initial distributions for u and v are completely segregated and impose

$$w(x, 0) = W(x), \qquad x \in \Omega, \tag{1.9}$$

where $W(x) = 1$ if $u(x, 0) > 0$ and $W(x) = 0$ if $u(x, 0) = 0$. We numerically demonstrate in Figure 1 how solutions of (1.8) depend on ϵ. We expect that w just becomes the characteristic function of $\Omega_u(t)$ as $\epsilon \to 0$ and then show that (1.3)–(1.5) can be derived from (1.8). We emphasize that, when $f = g = 0$, the two-phase Stefan problem can be derived from (1.8). It should be noted that the RD system (1.8) with small ϵ can be regarded as a variant of penalty methods to solve the two-phase Stefan problem [12].

It is interesting to interpret (1.8) from the ecological viewpoint. Let us intro-

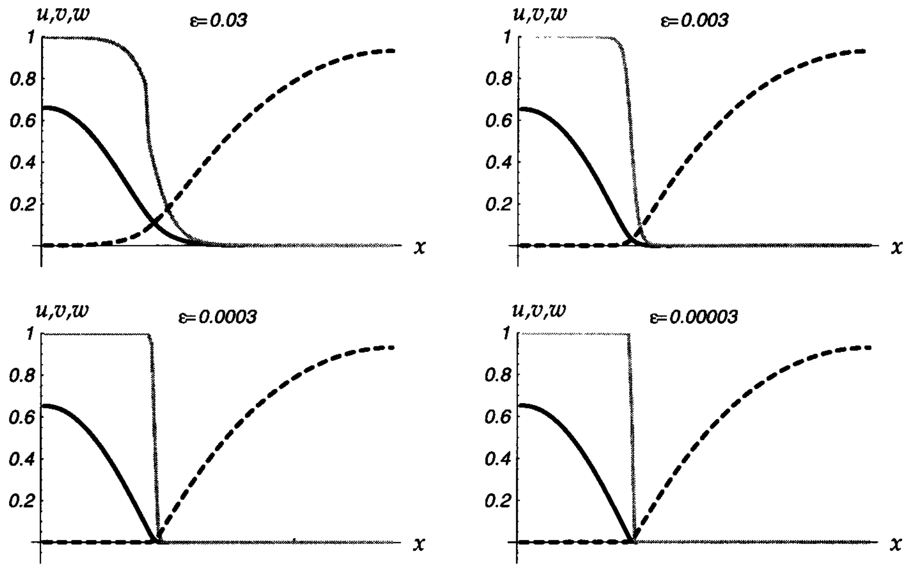

Fig. 1. Dependence on ϵ of the spatial profiles of one-dimensional solutions of (1.8); solid curve: u, dotted curve: v, grey curve: w.

duce a fourth variable p into (1.8) and rewrite it as

$$\begin{cases} u_t = d_1 \Delta u + f(u) - \dfrac{s_1 uv}{\epsilon} - \dfrac{\lambda s_1 pu}{\epsilon}, & x \in \Omega, \quad t > 0, \\ v_t = d_2 \Delta v + g(v) - \dfrac{s_2 uv}{\epsilon} - \dfrac{\lambda s_2 wv}{\epsilon}, & x \in \Omega, \quad t > 0, \\ w_t = \dfrac{pu}{\epsilon} - \dfrac{wv}{\epsilon}, & x \in \Omega, \quad t > 0, \\ p_t = -\dfrac{pu}{\epsilon} + \dfrac{wv}{\epsilon}, & x \in \Omega, \quad t > 0. \end{cases} \tag{1.10}$$

The initial condition for p is

$$p(x,0) = P(x), \qquad x \in \Omega,$$

where $P(x) = 1$ if $v(x,0) > 0$ and $P(x) = 0$ if $v(x,0) = 0$. Because of the complete segregation of initial distributions of u and v, one knows that $W(x) + P(x) = 1$ for each $x \in \Omega$. (Here we assume that the initial segregating boundary $\Gamma(0)$ is a smooth hypersurface with one codimension in Ω.) Obviously $(w + p)_t = 0$, so that

$$w(x,t) + p(x,t) = w(x,0) + p(x,0) = W(x) + P(x) = 1.$$

Therefore it turns out that (1.10) coincides with (1.8). The system (1.10) can be ecologically interpreted as follows: u and v are the densities of two competing species with their own habitats $\Omega_u(t)$ and $\Omega_v(t)$, whose shapes are respectively described by the characteristic-like functions w and p (in fact, as ϵ tends to zero, both of them become the corresponding characteristic functions of the habitats $\Omega_u(t)$ and $\Omega_v(t)$). There are two different types of interactions between u and v. One is the directly competitive interaction (due to the term uv), for obtaining their common resource. The other is the struggle interaction (due to $\lambda s_1 pu$ and $\lambda s_2 wv$), for constructing their own habitats, where λs_1 (resp. λs_2) is the cost rate when u (resp. v) attacks the habitat $\Omega_v(t)$ (resp. $\Omega_u(t)$). For this reason, we may say that (1.10) is not a conventional competition-diffusion model, but a new RD equation of two competing species which move by diffusion.

To state our main result, we impose the assumptions on f, g and the initial datum (u_0, v_0, w_0).

A1 (Assumption on f and g)
There exist C^1-functions $\tilde{f}(u)$ and $\tilde{g}(u)$ and positive constants K_1 and K_2 such that

$$\begin{aligned} &f(u) = \tilde{f}(u)u, \quad g(u) = \tilde{g}(u)u, \\ &\tilde{f}(u) \leq 0 \ \text{ for } \ u \geq K_1, \quad \tilde{g}(u) \leq 0 \ \text{ for } \ u \geq K_2. \end{aligned}$$

A2 (Assumption on the initial datum)

$$\begin{aligned} &(u_0, v_0, w_0) \in C(\overline{\Omega}) \times C(\overline{\Omega}) \times L^\infty(\Omega), \\ &0 \leq u_0(x) \leq \alpha, \quad 0 \leq v_0(x) \leq \beta, \quad 0 \leq w_0(x) \leq 1 \ \text{ in } \ \Omega, \\ &u_0 v_0 = (1 - w_0)u_0 = w_0 v_0 = 0 \ \text{ in } \ \Omega \end{aligned}$$

for some positive constants α and β.

Set

$$\begin{aligned} &\Gamma_0 := \{x \in \Omega \mid u_0(x) = v_0(x) = 0\}, \\ &Q_T := \Omega \times [0, T]. \end{aligned}$$

THEOREM 1.1. *Let T be any positive number and Ω a bounded domain in $\mathbf{R}^N$ $(N \geq 1)$ with C^2-boundary $\partial\Omega$. Assume* **A1** *and* **A2**. *Denote by $(u^\epsilon, v^\epsilon, w^\epsilon)$ the solution of* (1.8) *in Q_T with*

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \ \ \partial\Omega \times (0, T], \tag{1.11}$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x) \quad \text{in} \ \ \Omega, \tag{1.12}$$

where ν is the outward normal vector to $\partial\Omega$. Then there exists $(u, v, w) \in L^2(0, T; H^1(\Omega)) \times L^2(0, T; H^1(\Omega)) \times L^2(Q_T)$ such that

$$\begin{aligned} &u^\epsilon \longrightarrow u, \quad v^\epsilon \longrightarrow v \qquad \text{in} \ \ L^2(Q_T), \\ &w^\epsilon \longrightarrow w \qquad \text{weakly in} \ \ L^2(Q_T) \end{aligned}$$

as $\epsilon \to +0$, *and*

$$\Omega_u \cap \Omega_v = \emptyset,$$

$$w = \begin{cases} 1 & in \ \ \Omega_u, \\ 0 & in \ \ \Omega_v, \end{cases}$$

where

$$\Omega_u := \{(x,t) \in Q_T \mid u(x,t) > 0\} = \{(x,t) \in Q_T \mid x \in \Omega_u(t),\ 0 \le t \le T\},$$
$$\Omega_v := \{(x,t) \in Q_T \mid v(x,t) > 0\} = \{(x,t) \in Q_T \mid x \in \Omega_v(t),\ 0 \le t \le T\}.$$

Moreover, if

$$\Gamma := Q_T \setminus (\Omega_u \cup \Omega_v) = \{(x,t) \in Q_T \mid x \in \Gamma(t),\ 0 \le t \le T\}$$

is a smooth hypersurface satisfying $\Gamma(t) \Subset \Omega$ *for* $0 \le t \le T$ *and if* u *(resp.* v*) is smooth on* $\overline{\Omega_u}$ *(resp.* $\overline{\Omega_v}$*), then* (Γ, u, v) *is the unique solution of the free boundary problem*

$$\begin{cases} u_t = d_1 \Delta u + f(u) & in \ \ \Omega_u(t), \\ v_t = d_2 \Delta v + g(v) & in \ \ \Omega_v(t), \\ u = v = 0 & on \ \ \Gamma(t), \\ \lambda V_n = -\dfrac{d_1}{s_1}\dfrac{\partial u}{\partial n} - \dfrac{d_2}{s_2}\dfrac{\partial v}{\partial n} & on \ \ \Gamma(t) \end{cases} \tag{1.13}$$

under the boundary condition (1.11) *and the initial conditions*

$$\Gamma(0) = \Gamma_0, \tag{1.14}$$

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad in \ \ \Omega. \tag{1.15}$$

This theorem is derived as a corollary of Theorems 3.6 and 3.7 which deal with more general situations.

REMARK 1.2. The conclusion of the latter part of Theorem 1.1 holds true even if $\Gamma(t)$ transversally intersects $\partial\Omega$ for $0 \le t \le T$. On the other hand, the triple of functions (u, v, w) is always a weak solution to (1.13), whether Γ is a smooth hypersurface or not. See Section 3 (especially Definition 3.3) for the definition of weak solutions.

This theorem implies that we can derive the classical two-phase Stefan problem from the RD system (1.8) in the absence of the reaction terms f and g, taking the limit ϵ tends to zero. The parameter λ in (1.13) corresponds to the latent heat in the Stefan problem. For sufficiently small ϵ, one can expect that u and v exhibit corner layers on the interface $\Gamma(t)$, while w has a sharp transition layer, which clearly indicates a segregating boundary between u and v. It is noted that no transition layer appears in (1.6) (see [4]). These results seem to indicate that the latent heat vanishes without transition layers. Along the same line, the one-phase Stefan problem can be discussed. We refer to the papers by Hilhorst, van der Hout and Peletier [9, 10, 11] and by Eymard, Hilhorst, van der Hout and Peletier [6].

2. Formulation of the Problem and Some Basic Properties

In this section we formulate the reaction-diffusion system which we study and derive a number of basic properties of the solutions. As was announced in the introduction, our problem is

$$\begin{cases} u_t = d_1 \Delta u + f(u) - \dfrac{s_1 uv}{\epsilon} - \dfrac{\lambda s_1 (1-w) u}{\epsilon}, & x \in \Omega, \quad t > 0, \\ v_t = d_2 \Delta v + g(v) - \dfrac{s_2 uv}{\epsilon} - \dfrac{\lambda s_2 wv}{\epsilon}, & x \in \Omega, \quad t > 0, \\ w_t = \dfrac{(1-w)u}{\epsilon} - \dfrac{wv}{\epsilon}, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0^\epsilon(x), \ v(x,0) = v_0^\epsilon(x), \ w(x,0) = w_0^\epsilon(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where ν denotes the outward normal vector to $\partial\Omega$. Note that the initial data depend on a small positive parameter ϵ: hereafter we consider a more general setting than that stated in Section 1. In what follows we impose **A1** on the functions f, g and make the following hypotheses about the initial data u_0^ϵ, v_0^ϵ and w_0^ϵ instead of **A2**.

A2'

$$\begin{aligned} & u_0^\epsilon, v_0^\epsilon \in C(\overline{\Omega}), \quad w_0^\epsilon \in L^\infty(\Omega), \\ & 0 \le u_0^\epsilon(x) \le \alpha, \quad 0 \le v_0^\epsilon(x) \le \beta, \quad 0 \le w_0^\epsilon(x) \le 1 \quad \text{in} \ \ \Omega, \\ & u_0^\epsilon \longrightarrow u_0, \ v_0^\epsilon \longrightarrow v_0, \ w_0^\epsilon \longrightarrow w_0 \ \ \text{weakly in} \ \ L^2(\Omega) \ \ \text{as} \ \ \epsilon \longrightarrow 0 \end{aligned}$$

for some positive constants α, β and for some functions $u_0, v_0, w_0 \in L^\infty(\Omega)$.

REMARK 2.1. In **A2'** we do not assume $u_0 v_0 = (1 - w_0)u_0 = w_0 v_0 = 0$. In particular, we do not impose that the supports of u_0 and v_0 are disjoint.

Hereafter **A1** and **A2'** are always assumed.

By a solution of (2.1) in Q_T $(T > 0)$ we mean a triple of functions $(u, v, w) \in C([0,T]; C(\overline{\Omega}) \times C(\overline{\Omega}) \times L^\infty(\Omega))$ such that

$$u, v \in C^1((0,T]; C(\overline{\Omega})) \cap C((0,T]; W^{2,p}(\Omega)), \quad w \in C^1([0,T]; L^\infty(\Omega))$$

for each $p \in (1, \infty)$ and (u, v, w) satisfies the equation (2.1).

LEMMA 2.2. *There exists a positive number* $T = T(\|u_0^\epsilon\|_{C(\overline{\Omega})}, \|v_0^\epsilon\|_{C(\overline{\Omega})}, \|w_0^\epsilon\|_{L^\infty(\Omega)})$ *such that* (2.1) *possesses a unique solution* $(u^\epsilon, v^\epsilon, w^\epsilon)$ *in* Q_T.

The result of Lemma 2.2 does not immediately follow from the theory of analytic semigroups because of the lack of diffusion for w. In particular, if w_0^ϵ is discontinuous at some point, w^ϵ as well as $\triangle u^\epsilon$ and $\triangle v^\epsilon$ may be discontinuous at that point at later times as well. We refer to Appendix for a sketch of the proof of Lemma 2.2.

LEMMA 2.3. *Let* $(u^\epsilon, v^\epsilon, w^\epsilon)$ *be a solution of* (2.1) *in* Q_T. *Then*

$$0 \leq u^\epsilon(x,t) \leq \max\{\alpha, K_1\}, \quad 0 \leq v^\epsilon(x,t) \leq \max\{\beta, K_2\}, \quad 0 \leq w^\epsilon(x,t) \leq 1$$

for $(x,t) \in Q_T$.

Proof. We deduce from the maximum principle that u^ϵ, $v^\epsilon \geq 0$. Let $x \in \Omega$ be such that $w_0^\epsilon(x)$ is defined. Then the condition $0 \leq w_0^\epsilon(x) \leq 1$ implies that $0 \leq w^\epsilon(x,t) \leq 1$ for all $t > 0$. Indeed suppose that at a time $t = \bar{t}$, $w^\epsilon(x,\bar{t}) = 0$, then $w_t^\epsilon(x,\bar{t}) \geq 0$; similarly if at a time $\tilde{t}$, $w^\epsilon(x,\tilde{t}) = 1$, then $w_t^\epsilon(x,\tilde{t}) \leq 0$. Finally we apply again the maximum principle to deduce that $u^\epsilon \leq \max\{\alpha, K_1\}$ and $v^\epsilon \leq \max\{\beta, K_2\}$. □

Without loss of generality, we can assume that

$$\alpha \geq K_1 \quad \text{and} \quad \beta \geq K_2$$

by choosing α and β so large that the above inequalities hold. Lemmas 2.2 and 2.3 ensure that the solution $(u^\epsilon, v^\epsilon, w^\epsilon)$ exists globally in time and satisfies

$$0 \leq u^\epsilon(x,t) \leq \alpha, \quad 0 \leq v^\epsilon(x,t) \leq \beta, \quad 0 \leq w^\epsilon(x,t) \leq 1 \quad \text{for} \quad (x,t) \in \overline{\Omega} \times [0,\infty). \tag{2.2}$$

Set

$$M_f := \max\{f(u) \mid 0 \leq u \leq \alpha\}, \quad M_g := \max\{g(u) \mid 0 \leq u \leq \beta\}.$$

LEMMA 2.4. *For any positive number* T *there exist positive constants* C_i $(i = 1, \ldots, 5)$ *independent of* ϵ *and* λ *such that*

$$\int\int_{Q_T} (s_1 + s_2) u^\epsilon v^\epsilon dx dt \leq C_1 \epsilon,$$

$$\int\int_{Q_T} \lambda s_1 (1 - w^\epsilon) u^\epsilon dx dt \leq C_2 \epsilon,$$

$$\int\int_{Q_T} \lambda s_2 w^\epsilon v^\epsilon dx dt \leq C_3 \epsilon,$$

$$\int\int_{Q_T} d_1 |\nabla u^\epsilon|^2 dx dt \leq C_4,$$

$$\int\int_{Q_T} d_2 |\nabla v^\epsilon|^2 dx dt \leq C_5.$$

In this paper, positive constants which do not depend on ϵ are denoted by C_i for simplicity of notation.

Proof. Integration of the equation for u in Q_T yields

$$\int\int_{Q_T}\Big(\frac{s_1u^\epsilon v^\epsilon}{\epsilon}+\frac{\lambda s_1(1-w^\epsilon)u^\epsilon}{\epsilon}\Big)dxdt$$
$$=\int_\Omega(u_0^\epsilon(x)-u^\epsilon(x,T))dx+\int\int_{Q_T}f(u^\epsilon)dxdt\le(\alpha+TM_f)|\Omega|,$$

which implies the second estimate. The first and third ones can be shown similarly. Next we multiply the equation for u^ϵ by u^ϵ and integrate by parts on Ω. This yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(u^\epsilon)^2dx+d_1\int_\Omega|\nabla u^\epsilon|^2dx+\int_\Omega\Big(\frac{s_1(u^\epsilon)^2v^\epsilon}{\epsilon}+\frac{\lambda s_1(1-w^\epsilon)(u^\epsilon)^2}{\epsilon}\Big)dx\le|\Omega|\alpha M_f,$$

which we integrate on $(0,T)$ to deduce the fourth estimate. The last estimate can be proved similarly. □

LEMMA 2.5. *Let T be any positive number and set*

$$\Omega_\xi:=\{x\in\Omega\mid x+r\xi\in\Omega\ \ for\ \ 0\le r\le1\}$$

with $\xi\in\mathbf{R}^N$. Then there exists positive constants C_6 and C_7 independent of ϵ and λ such that

$$\int_0^T\int_{\Omega_\xi}(u^\epsilon(x+\xi,t)-u^\epsilon(x,t))^2dxdt\le\frac{C_4}{d_1}|\xi|^2, \tag{2.3}$$

$$\int_0^T\int_{\Omega_\xi}(v^\epsilon(x+\xi,t)-v^\epsilon(x,t))^2dxdt\le\frac{C_5}{d_2}|\xi|^2, \tag{2.4}$$

$$\int_0^{T-\tau}\int_\Omega(u^\epsilon(x,t+\tau)-u^\epsilon(x,t))^2dxdt\le C_6\tau, \tag{2.5}$$

$$\int_0^{T-\tau}\int_\Omega(v^\epsilon(x,t+\tau)-v^\epsilon(x,t))^2dxdt\le C_7\tau \tag{2.6}$$

for $\xi\in\mathbf{R}^N$ and $\tau>0$. Here

$$C_6 = 2C_4+\alpha M_fT|\Omega|+\alpha C_1+\alpha C_2,$$
$$C_7 = 2C_5+\beta M_gT|\Omega|+\beta C_1+\beta C_3.$$

Proof. The first and second inequalities (2.3), (2.4) can follow immediately from Lemma 2.4. Indeed, we have

$$\int_0^T\int_{\Omega_\xi}(u^\epsilon(x+\xi,t)-u^\epsilon(x,t))^2dxdt$$
$$=\int_0^T\int_{\Omega_\xi}\Big\{\int_0^1\nabla u^\epsilon(x+r\xi,t)\cdot\xi\,dr\Big\}^2dxdt$$
$$\le\frac{C_4}{d_1}|\xi|^2.$$

Similarly the second one can be shown. Next we prove (2.5) and (2.6). We have

$$
\begin{aligned}
&\int_0^{T-\tau}\int_\Omega (u^\epsilon(x,t+\tau)-u^\epsilon(x,t))^2dxdt\\
&=\int_0^{T-\tau}\int_\Omega (u^\epsilon(x,t+\tau)-u^\epsilon(x,t))\int_0^\tau u_t(x,t+r)drdxdt\\
&=\int_0^{T-\tau}\int_\Omega (u^\epsilon(x,t+\tau)-u^\epsilon(x,t))\int_0^\tau \Big\{d_1\triangle u^\epsilon(x,t+r)+f(u^\epsilon(x,t+r))\\
&\quad-\frac{s_1u^\epsilon(x,t+r)v^\epsilon(x,t+r)+\lambda s_1(1-w^\epsilon(x,t+r))u^\epsilon(x,t+r)}{\epsilon}\Big\}drdxdt.
\end{aligned}
$$

We estimate the three terms on the right-hand side. The first term can be estimated as follows:

$$
\begin{aligned}
&\Big|\int_0^{T-\tau}\int_\Omega (u^\epsilon(x,t+\tau)-u^\epsilon(x,t))\int_0^\tau d_1\triangle u^\epsilon(x,t+r)drdxdt\Big|\\
&=d_1\Big|\int_0^\tau\int_0^{T-\tau}\int_\Omega (\nabla u^\epsilon(x,t+\tau)-\nabla u^\epsilon(x,t))\cdot\nabla u^\epsilon(x,t+r)dxdtdr\Big|\\
&\le 2d_1\tau\int_0^T\int_\Omega |\nabla u^\epsilon(x,t)|^2dxdt\\
&\le 2C_4\tau.
\end{aligned}
$$

Secondly, we see that

$$
\Big|\int_0^{T-\tau}\int_\Omega (u^\epsilon(x,t+\tau)-u^\epsilon(x,t))\int_0^\tau f(u^\epsilon(x,t+r))drdxdt\Big|\le \alpha M_fT|\Omega|\tau.
$$

Finally we have

$$
\begin{aligned}
&\Big|\int_0^{T-\tau}\int_\Omega (u^\epsilon(x,t+\tau)-u^\epsilon(x,t))\\
&\quad\times\int_0^\tau \frac{s_1u^\epsilon(x,t+r)v^\epsilon(x,t+r)+\lambda s_1(1-w^\epsilon(x,t+r))u^\epsilon(x,t+r)}{\epsilon}drdxdt\Big|\\
&\le\alpha\int_0^\tau\int_0^{T-\tau}\int_\Omega \frac{s_1u^\epsilon(x,t+r)v^\epsilon(x,t+r)+\lambda s_1(1-w^\epsilon(x,t+r))u^\epsilon(x,t+r)}{\epsilon}dxdtdr\\
&\le\alpha\tau\int_0^T\int_\Omega \frac{s_1u^\epsilon(x,t)v^\epsilon(x,t)+\lambda s_1(1-w^\epsilon(x,t))u^\epsilon(x,t)}{\epsilon}dxdt\\
&\le\alpha(C_1+C_2)\tau.
\end{aligned}
$$

Thus we have shown that

$$
\int_0^{T-\tau}\int_\Omega (u^\epsilon(x,t+\tau)-u^\epsilon(x,t))^2dxdt\le(2C_4+\alpha M_fT|\Omega|+\alpha C_1+\alpha C_2)\tau. \quad (2.7)
$$

Similarly, we can prove the following estimate:

$$\int_0^{T-\tau}\int_\Omega (v^\epsilon(x,t+\tau)-v^\epsilon(x,t))^2dxdt \le (2C_5+\beta M_g T|\Omega|+\beta C_1+\beta C_3)\tau. \quad (2.8)$$

Thus, all the inequalities in this lemma are proved. □

3. The Problem Obtained as the Singular Limit

Choose a positive number T arbitrarily and fix it. We deduce from Lemmas 2.3 and 2.4 that the families $\{u^\epsilon\}$ and $\{v^\epsilon\}$ are bounded in $L^2(0,T;H^1(\Omega))$ and that the family $\{w^\epsilon\}$ is bounded in $L^\infty(Q_T)$. Furthermore it follows from Lemma 2.5 and the Riesz-Fréchet-Kolmogoroff theorem [3, Theorem IV.25 and Corollary IV.26] that the families $\{u^\epsilon\}$ and $\{v^\epsilon\}$ are precompact in $L^2(Q_T)$. Therefore there exist subsequences $\{u^{\epsilon_n}\}$ and $\{v^{\epsilon_n}\}$ and $\{w^{\epsilon_n}\}$ and functions $u^*, v^* \in L^2(0,T;H^1(\Omega))$ and $w^* \in L^2(Q_T)$ such that

$$\begin{aligned}&u^{\epsilon_n} \longrightarrow u^*, \quad v^{\epsilon_n} \longrightarrow v^*\\ &\qquad \text{strongly in } L^2(Q_T), \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ and a.e. in } Q_T, \quad (3.1)\end{aligned}$$

and

$$w^{\epsilon_n} \longrightarrow w^* \quad \text{weakly in } L^2(Q_T) \quad (3.2)$$

as $\epsilon_n \longrightarrow 0$. It follows from (2.2) that

$$u^* \ge 0, \quad v^* \ge 0, \quad 0 \le w^* \le 1 \quad \text{on } \overline{Q_T}. \quad (3.3)$$

Hence we deduce from Lemma 2.4 that

$$u^*v^* = (1-w^*)u^* = w^*v^* = 0. \quad (3.4)$$

In what follows we will show that (u^*, v^*, w^*) given above is uniquely determined by the unique solution of a Stefan type problem — see (3.6).

LEMMA 3.1. *Let T be an arbitrary positive number. The triple of functions (u^*, v^*, w^*) given in* (3.1) *and* (3.2) *satisfy*

$$\begin{aligned}\int\int_{Q_T}\Big\{\Big(\frac{u^*}{s_1}-\frac{v^*}{s_2}+\lambda w^*\Big)\zeta_t - \nabla\Big(\frac{d_1u^*}{s_1}-\frac{d_2v^*}{s_2}\Big)\cdot\nabla\zeta \\ + \Big(\frac{f(u^*)}{s_1}-\frac{g(v^*)}{s_2}\Big)\zeta\Big\}dxdt \\ = -\int_\Omega\Big(\frac{u_0}{s_1}-\frac{v_0}{s_2}+\lambda w_0\Big)\zeta(x,0)dx \quad (3.5)\end{aligned}$$

for all functions $\zeta \in C^\infty(\overline{Q_T})$ such that $\zeta(x,T)=0$.

Proof. We deduce from (2.1) that

$$\left(\frac{u^\epsilon}{s_1} - \frac{v^\epsilon}{s_2} + \lambda w^\epsilon\right)_t = \frac{d_1 \Delta u^\epsilon}{s_1} - \frac{d_2 \Delta v^\epsilon}{s_2} + \frac{f(u^\epsilon)}{s_1} - \frac{g(v^\epsilon)}{s_2},$$

and multiply it by a test function $\zeta \in C^\infty(\overline{Q_T})$ with $\zeta(x,T) = 0$ and integrate by parts to obtain the identity

$$\int\int_{Q_T} \left\{ -\left(\frac{u^\epsilon}{s_1} - \frac{v^\epsilon}{s_2} + \lambda w^\epsilon\right)\zeta_t + \nabla\left(\frac{d_1 u^\epsilon}{s_1} - \frac{d_2 v^\epsilon}{s_2}\right)\cdot \nabla \zeta - \left(\frac{f(u^\epsilon)}{s_1} - \frac{g(v^\epsilon)}{s_2}\right)\zeta \right\} dxdt$$
$$= \int_\Omega \left(\frac{u_0^\epsilon}{s_1} - \frac{v_0^\epsilon}{s_2} + \lambda w_0^\epsilon\right)\zeta(x,0)dx.$$

Letting $\epsilon = \epsilon_n \to 0$, we deduce (3.5). □

We will formulate (3.5) as a weak form of the following parabolic boundary value problem:

$$\begin{cases} Z_t = \Delta d(\phi(Z)) + h(\phi(Z)), & x \in \Omega, \quad 0 < t \le T, \\ \dfrac{\partial d(\phi(Z))}{\partial \nu} = 0, & x \in \partial\Omega, \quad 0 < t \le T, \\ Z(x,0) = Z_0(x), & x \in \Omega, \end{cases} \tag{3.6}$$

where

$$d(r) := \begin{cases} d_1 r & (r \ge 0), \\ d_2 r & (r < 0), \end{cases}$$

$$\phi(r) := \begin{cases} r - \lambda & (r > \lambda), \\ 0 & (0 \le r \le \lambda), \\ r & (r < 0), \end{cases}$$

$$h(r) := \begin{cases} \dfrac{f(s_1 r)}{s_1} & (r \ge 0), \\ -\dfrac{g(-s_2 r)}{s_2} & (r < 0). \end{cases}$$

We also use the following notation:

$$H(r) := \begin{cases} 1 & (r > 0), \\ [0,1] & (r = 0), \\ 0 & (r < 0), \end{cases}$$

$$r_+ := \max\{r, 0\}, \quad r_- := -\min\{r, 0\}.$$

LEMMA 3.2. *If* $w \in H(z)$, *then* $\phi(z + \lambda w) = z$. *In particular, the functions* u^*, v^* *and* w^* *which are given in* (3.1) *and* (3.2) *satisfy*

$$u^* = s_1\phi(Z^*)_+, \quad v^* = s_2\phi(Z^*)_- \quad \text{and} \quad w^* = \frac{Z^* - \phi(Z^*)}{\lambda}, \tag{3.7}$$

where

$$Z^* := \frac{u^*}{s_1} - \frac{v^*}{s_2} + \lambda w^*. \tag{3.8}$$

Proof. The former claim of this lemma follows from the definitions of ϕ and H. We can deduce

$$w^* \in H\left(\frac{u^*}{s_1} - \frac{v^*}{s_2}\right)$$

from (3.3) and (3.4). Hence we have

$$\phi(Z^*) = \frac{u^*}{s_1} - \frac{v^*}{s_2},$$

which, together with (3.3) and (3.4), implies (3.7). □

DEFINITION 3.3. *A function* $Z \in L^\infty(Q_T)$ *is a weak solution of* (3.6) *with an initial datum* $Z_0 \in L^\infty(\Omega)$ *if*

$$d(\phi(Z)) \in L^2(0, T; H^1(\Omega)),$$

and

$$\int\int_{Q_T} Z\zeta_t dxdt + \int_\Omega Z_0(x)\zeta(x,0)dx = \int\int_{Q_T} \{\nabla d(\phi(Z)) \cdot \nabla\zeta - h(\phi(Z))\zeta\} dxdt \tag{3.9}$$

for all functions $\zeta \in C^\infty(\overline{Q_T})$ *such that* $\zeta(x, T) = 0$.

REMARK 3.4. If Z is a weak solution of (3.6), then $\phi(Z)$ is continuous on $\overline{\Omega} \times [\delta, T]$ for each $\delta \in (0, T]$. Cf. [5].

It is known that the classical two-phase Stefan problem under the Neumann condition can be formulated as the nonlinear system (3.6) with $h \equiv 0$. Then Z and $\phi(Z)$ correspond to the internal energy and the temperature respectively. We note that (3.6) can also deal with the case where the interface fattens. We set

$$\begin{cases} \Omega_+(t) := \{x \in \Omega \mid \phi(Z(x,t)) > 0\}, \\ \Omega_-(t) := \{x \in \Omega \mid \phi(Z(x,t)) < 0\}, \\ \Gamma(t) := \Omega \setminus (\Omega_+(t) \cup \Omega_-(t)) \end{cases} \tag{3.10}$$

for $t \in [0,T]$ and also use the notation

$$\begin{cases} \Omega_+ := \bigcup_{0\le t\le T} \Omega_+(t) \times \{t\}, \\ \Omega_- := \bigcup_{0\le t\le T} \Omega_-(t) \times \{t\}, \\ \Gamma := \bigcup_{0\le t\le T} \Gamma(t) \times \{t\}. \end{cases} \tag{3.11}$$

We could think that $\Omega_+(t)$ and $\Omega_-(t)$ symbolize two distinct phases, and $\Gamma(t)$ represents a *phase boundary* (or an *interface*) at time t.

LEMMA 3.5. *The function Z^* defined by* (3.8) *is a weak solution of Problem* (3.6) *with an initial datum $Z_0 = u_0/s_1 - v_0/s_2 + \lambda w_0$.*

Proof. It follows from Lemma 2.3 that $Z^* \in L^\infty(Q_T)$. We observe that (3.7) implies

$$d(\phi(Z^*)) = \frac{d_1u^*}{s_1} - \frac{d_2v^*}{s_2}.$$

In particular, $d(\phi(Z^*)) \in L^2(0,T;H^1(\Omega))$ holds true. We also notice that

$$h(\phi(Z^*)) = \frac{f(u^*)}{s_1} - \frac{g(v^*)}{s_2}.$$

Therefore (3.5) can be rewritten as (3.9) with $Z = Z^*$ and $Z_0 = u_0/s_1 - v_0/s_2 + \lambda w_0$. This completes the proof of this lemma. □

The uniqueness of the weak solution of the Stefan problem (3.6) for $Z_0 \in L^1(\Omega)$ follows from Hilhorst, Mimura and Schätzle [8]. Thus (u_0, v_0, w_0) uniquely determines Z^*, which uniquely gives (u^*, v^*, w^*) by (3.7). Namely the limits u^*, v^* and w^* in (3.1) and (3.2) do not depend on what subsequence $\{\epsilon_n\}$ is chosen. Consequently we have proved the following result.

THEOREM 3.6. *The function Z^* defined by* (3.8) *is the unique weak solution of the Stefan problem* (3.6) *with an initial datum $u_0/s_1 - v_0/s_2 + \lambda w_0$. As $\epsilon \to 0$,*

$$u^\epsilon \longrightarrow u^*, \quad v^\epsilon \longrightarrow v^* \quad \textit{strongly in } L^2(Q_T) \textit{ and weakly in } L^2(0,T;H^1(\Omega)),$$
$$w^\epsilon \longrightarrow w^* \quad \textit{weakly in } L^2(Q_T).$$

Finally we state a result about the relation between (3.6) and (1.13).

THEOREM 3.7. *Let Z be the unique weak solution of* (3.6) *with initial datum Z_0 and let $\Omega_\pm(t)$ and $\Gamma(t)$ be the sets defined by* (3.10). *Suppose that (each component of) $\Gamma(t)$ is a smooth, closed and orientable hypersurface satisfying $\Gamma(t)\cap\partial\Omega = \emptyset$ for all $t \in [0,T]$. Let n be the unit normal vector on $\Gamma(t)$ oriented from $\Omega_+(t)$ to*

$\Omega_-(t)$. Also assume that $\Gamma(t)$ smoothly moves with a velocity V_n in the direction of n and that the functions

$$u := s_1\phi(Z)_+ \quad \text{and} \quad v := s_2\phi(Z)_-$$

are smooth on $\overline{\Omega_+}$ and $\overline{\Omega_-}$ respectively. Then (Γ, u, v) satisfies

$$\begin{cases} u_t = d_1\Delta u + f(u) & \text{in } \Omega_+(t), \\ v_t = d_2\Delta v + g(v) & \text{in } \Omega_-(t), \\ \lambda V_n = -\dfrac{d_1}{s_1}\dfrac{\partial u}{\partial n} - \dfrac{d_2}{s_2}\dfrac{\partial v}{\partial n} & \text{on } \Gamma(t), \\ u = v = 0 & \text{on } \Gamma(t), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.12}$$

for $t \in (0, T]$ and

$$\begin{cases} \Gamma(0) = \{x \in \Omega \mid \phi(Z_0(x)) = 0\}, \\ u(x,0) = s_1(\phi(Z_0(x)))_+, \quad v(x,0) = s_2(\phi(Z_0(x)))_-, \qquad x \in \Omega. \end{cases} \tag{3.13}$$

Here $\partial u/\partial n$ (resp. $\partial v/\partial n$) is regarded as a boundary value on $\partial\Omega_+(t)$ (resp. $\partial\Omega_-(t)$).

Proof. It follows from the definition of u and v that

$$u = v = 0 \quad \text{on } \Gamma,$$
$$\phi(Z) = Z - \lambda = \frac{u}{s_1} \text{ in } \Omega_+, \quad \phi(Z) = Z = -\frac{v}{s_2} \text{ in } \Omega_-,$$

which are used in some of the calculations below.

Next we derive the parabolic equations for u and v as well as the Stefan condition on the interface Γ. First we rewrite the first term on the left-hand side of (3.9). We see that

$$\int\!\!\int_{Q_T} Z\zeta_t dxdt = \int_0^T \left\{ \int_{\Omega_+(t)} \left(\frac{u}{s_1} + \lambda\right)\zeta_t dx - \int_{\Omega_-(t)} \frac{v}{s_2}\zeta_t dx \right\} dt.$$

From $u|_{\Gamma(t)} = 0$, it follows that

$$\left[\int_{\Omega_+(t)} u\zeta dx\right]_{t=0}^{t=T} = \int_0^T \frac{d}{dt}\left(\int_{\Omega_+(t)} u\zeta dx\right) dt = \int_0^T \int_{\Omega_+(t)} (u\zeta_t + u_t\zeta) dxdt;$$

moreover,

$$\left[\int_{\Omega_+(t)} \zeta dx\right]_{t=0}^{t=T} = \int_0^T \int_{\Omega_+(t)} \zeta_t dxdt + \int_0^T \int_{\Gamma(t)} V_n\zeta d\sigma dt,$$

and similarly since $v|_{\Gamma(t)} = 0$, we have that

$$\left[\int_{\Omega_-(t)} v\zeta dx\right]_{t=0}^{t=T} = \int_0^T \int_{\Omega_-(t)} (v\zeta_t + v_t\zeta) dxdt.$$

Therefore we have that for the test functions ζ which vanish at $t = 0, T$

$$\begin{aligned} \iint_{Q_T} Z\zeta_t dxdt = & -\frac{1}{s_1}\int_0^T \int_{\Omega_+(t)} u_t\zeta dxdt \\ & + \frac{1}{s_2}\int_0^T \int_{\Omega_-(t)} v_t\zeta dxdt - \int_0^T \int_{\Gamma(t)} \lambda V_n \zeta d\sigma dt. \end{aligned} \tag{3.14}$$

On the other hand we have that

$$\begin{aligned} & \iint_{Q_T} \{-\nabla d(\phi(Z)) \cdot \nabla\zeta + h(\phi(Z))\zeta\} dxdt \\ = & -\frac{1}{s_1}\int_0^T \int_{\Omega_+(t)} d_1 \nabla u \cdot \nabla\zeta dxdt + \frac{1}{s_2}\int_0^T \int_{\Omega_-(t)} d_2 \nabla v \cdot \nabla\zeta dxdt \\ & + \frac{1}{s_1}\int_0^T \int_{\Omega_+(t)} f(u)\zeta dxdt - \frac{1}{s_2}\int_0^T \int_{\Omega_-(t)} g(v)\zeta dxdt \\ = & \int_0^T \int_{\Gamma(t)} \left\{-\frac{d_1}{s_1}\frac{\partial u}{\partial n} - \frac{d_2}{s_2}\frac{\partial v}{\partial n}\right\} \zeta d\sigma dt + \int_0^T \int_{\partial\Omega} \left\{-\frac{d_1}{s_1}\frac{\partial u}{\partial \nu} + \frac{d_2}{s_2}\frac{\partial v}{\partial \nu}\right\} \zeta d\sigma dt \\ & + \frac{1}{s_1}\int_0^T \int_{\Omega_+(t)} \{d_1 \triangle u + f(u)\}\zeta dxdt - \frac{1}{s_2}\int_0^T \int_{\Omega_-(t)} \{d_2 \triangle v + g(v)\}\zeta dxdt. \end{aligned} \tag{3.15}$$

We substitute (3.14) and (3.15) into (3.9). This gives

$$\begin{aligned} & \int_0^T \int_{\Gamma(t)} \left\{-\lambda V_n - \frac{d_1}{s_1}\frac{\partial u}{\partial n} - \frac{d_2}{s_2}\frac{\partial v}{\partial n}\right\} \zeta d\sigma dt \\ & + \frac{1}{s_1}\int_0^T \int_{\Omega_+(t)} \{-u_t + d_1 \triangle u + f(u)\}\zeta dxdt \\ & + \frac{1}{s_2}\int_0^T \int_{\Omega_-(t)} \{v_t - d_2 \triangle v - g(v)\}\zeta dxdt \\ & + \int_0^T \int_{\partial\Omega} \left\{-\frac{d_1}{s_1}\frac{\partial u}{\partial \nu} + \frac{d_2}{s_2}\frac{\partial v}{\partial \nu}\right\} \zeta d\sigma dt = 0 \end{aligned} \tag{3.16}$$

for all $\zeta \in C^\infty(\overline{Q_T})$ such that $\zeta(x,0) = \zeta(x,T) = 0$. Considering successively in (3.16) test functions with compact support in Ω_+ and test functions with compact support in Ω_-, we deduce the parabolic equations for u and v. Then, without loss of generality, we may assume that $\Gamma(t) = \partial\Omega_+(t)$ for $t \in [0,T]$. Taking in (3.16) test functions which vanish on $\partial\Omega \times [0,T]$ and do not vanish on Γ, we deduce the

Stefan condition describing the interface motion. Since $\partial\Omega \cap \overline{\Omega_+(t)} = \emptyset$, it is clear that

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T).$$

Thus, taking in (3.16) test functions which do not vanish on $\partial\Omega \times (0,T)$, we can deduce that

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T).$$

Finally consider (3.9) with test functions which do not vanish at $t = 0$. Then we can replace (3.14) with

$$\begin{aligned}\int\!\!\int_{Q_T} Z\zeta_t dxdt &= -\int_\Omega Z(x,0)\zeta(x,0)dx - \frac{1}{s_1}\int_0^T \int_{\Omega_+(t)} u_t\zeta dxdt \\ &+ \frac{1}{s_2}\int_0^T \int_{\Omega_-(t)} v_t\zeta dxdt - \int_0^T \int_{\Gamma(t)} \lambda V_n \zeta d\sigma dt. \end{aligned} \tag{3.17}$$

Substituting (3.17) and (3.15) into (3.9) and using the fact that both the partial differential equations for u and v as well as the Stefan condition for the interface motion and the Neumann boundary conditions are satisfied, we deduce that

$$-\int_\Omega \{Z(x,0) - Z_0(x)\}\zeta(x,0)dx = 0$$

for all $\zeta(\cdot,0) \in L^2(\Omega)$. Hence

$$Z(x,0) = Z_0(x) \quad \text{a.e. in} \quad \Omega.$$

Thus we obtain (3.13). $\square$

Proof of Theorem 1.1. Let (u_0, v_0, w_0) be a triple of functions satisfying **A2** and let Z^* be the unique weak solution of (3.6) with an initial datum $u_0/s_1 - v_0/s_2 + \lambda w_0$. We use the notation $\Omega_u(t)$ and Ω_u (resp. $\Omega_v(t)$ and Ω_v) instead of $\Omega_+(t)$ and Ω_+ (resp. $\Omega_-(t)$ and Ω_-) which are defined by (3.10) and (3.11) with $Z = Z^*$. Set $(u_0^\epsilon, v_0^\epsilon, w_0^\epsilon) := (u_0, v_0, w_0)$ for all $\epsilon > 0$. Since $(u_0^\epsilon, v_0^\epsilon, w_0^\epsilon)$ satisfies **A2'**, the solution $(u^\epsilon, v^\epsilon, w^\epsilon)$ to (1.8) under (1.11) and (1.12) converges to (u^*, v^*, w^*) in the sense of Theorem 3.6. It follows from (3.4), (3.7) and the definition of $\Omega_\pm$ that

$$\Omega_u \cap \Omega_v = \emptyset,$$

$$w^* = \begin{cases} 1 & \text{in} \quad \Omega_u, \\ 0 & \text{in} \quad \Omega_v. \end{cases}$$

Suppose that Γ is a smooth hypersurface satisfying $\Gamma(t) \Subset \Omega$ for $0 \le t \le T$ and that u^* (resp. v^*) is smooth on $\overline{\Omega_u}$ (resp. $\overline{\Omega_v}$). Theorem 3.7 ensures that

(Γ, u^*, v^*) satisfies (1.13) and (1.11) with $(u, v) = (u^*, v^*)$. Observing that $w_0 \in H(u_0/s_1 - v_0/s_2)$ in Ω and applying Lemma 3.2, we can deduce from (3.13) that

$$\Gamma(0) = \left\{ x \in \Omega \;\middle|\; \frac{u_0(x)}{s_1} = \frac{v_0(x)}{s_2} \right\}, \tag{3.18}$$

$$\frac{u^*(x,0)}{s_1} = \left(\frac{u_0(x)}{s_1} - \frac{v_0(x)}{s_2} \right)_+, \quad \frac{v^*(x,0)}{s_2} = \left(\frac{u_0(x)}{s_1} - \frac{v_0(x)}{s_2} \right)_- \quad \text{in } \Omega. \tag{3.19}$$

Thus, by virtue of **A2**, we obtain (1.14) and (1.15) with $(u, v) = (u^*, v^*)$. □

REMARK 3.8. Our proof of the convergence of $(u^\epsilon, v^\epsilon, w^\epsilon)$ to (u^*, v^*, w^*) as $\epsilon \to 0$ is valid as long as $(u_0^\epsilon, v_0^\epsilon, w_0^\epsilon)$ satisfies **A2'**. However **A2'** is not enough for deducing (1.14) and (1.15). For instance, consider the case where the support of u_0 overlaps that of v_0. As the proof of Theorem 1.1 shows, we can still obtain (3.18) and (3.19) if $w_0 \in H(u_0/s_1 - v_0/s_2)$. But we can no longer obtain (1.14) and (1.15).

4. Concluding Remarks

We have proposed a reaction-diffusion system with a small parameter ϵ which describes the competitive interaction between two ecological species. In the limit where $\epsilon \to 0$, we have derived the classical two-phase Stefan problem with reaction terms. This result implies that the singular limit analysis as $\epsilon \to 0$ reveals the relation between an ecological system for competing species and the solidification problem for ice and water. On the other hand, our RD system (1.8) is regarded as a phase field approximation to the Stefan problem where the variable w is an order parameter which indicates sharp interfaces between two phases. Note that in the case that $u_0^\epsilon(x) \equiv u_0(x) \equiv 0$ or in the case that $v_0^\epsilon(x) \equiv v_0(x) \equiv 0$ Problem (2.1) involves a system where a single parabolic equation is coupled to an ordinary differential equation and that Problem (3.12) reduces to a one-phase Stefan problem, so that (1.8) can be an approximation not only to a two-phase Stefan problem but also to a one-phase Stefan problem.

5. Appendix

We show Lemma 2.2. For a positive number d let us denote by $E_d(t; x, y)$ the Green function which is associated with the boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = d \triangle u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Define a one-parameter family $\{E_d(t)\}_{t \geq 0}$ of linear operators on $L^1(\Omega)$ by

$$\left(E_d(t)\psi\right)(x) := \begin{cases} \displaystyle\int_\Omega E_d(t; x, y)\psi(y)\, dy, & t > 0, \\ \psi(x), & t = 0 \end{cases}$$

for $\psi \in L^1(\Omega)$. Set

$$\begin{aligned} N_1(u,v,w) &= f(u) - \frac{s_1 uv}{\epsilon} - \frac{\lambda s_1(1-w)u}{\epsilon}, \\ N_2(u,v,w) &= g(v) - \frac{s_2 uv}{\epsilon} - \frac{\lambda s_2 wv}{\epsilon}, \\ N_3(u,v,w) &= \frac{(1-w)u}{\epsilon} - \frac{wv}{\epsilon}. \end{aligned}$$

We consider the system of integral equations

$$\begin{cases} u(t) = E_{d_1}(t)u_0^\epsilon + \displaystyle\int_0^t E_{d_1}(t-\tau)N_1(u(\tau),v(\tau),w(\tau))\,d\tau, \\ v(t) = E_{d_2}(t)v_0^\epsilon + \displaystyle\int_0^t E_{d_2}(t-\tau)N_2(u(\tau),v(\tau),w(\tau))\,d\tau, \\ w(t) = w_0^\epsilon + \displaystyle\int_0^t N_3(u(\tau),v(\tau),w(\tau))\,d\tau \end{cases} \tag{5.1}$$

in $C([0,\infty);C(\overline{\Omega}) \times C(\overline{\Omega}) \times L^\infty(\Omega))$. Although $\{E_d(t)\}_{t\geq 0}$ is an analytic C_0-semigroup on $C(\overline{\Omega})$, this fact is not enough to prove the local existence for (5.1). In fact the discontinuity of w_0^ϵ causes that $N_1(u(\tau),v(\tau),w(\tau))$, $N_2(u(\tau),v(\tau),w(\tau)) \notin C(\overline{\Omega})$ for $\tau \in [0,t_0]$, where t_0 is a positive number. However, the following estimates of $E_d(t)$ as an operator from $L^\infty(\Omega)$ into $C(\overline{\Omega})$ are useful here.

LEMMA 5.1. *For $\psi \in L^\infty(\Omega)$ the function $E_d(t)\psi$ of t possesses the following properties:*

(i) $E_d(\cdot)\psi \in C^1((0,\infty);C(\overline{\Omega})) \cap C((0,\infty);W^{2,p}(\Omega)), \quad p \in (1,\infty)$,
$\triangle E_d(\cdot)\psi \in C((0,\infty);C(\overline{\Omega}))$,
$\dfrac{\partial}{\partial \nu}E_d(t)\psi = 0$ *on* $\partial\Omega$ *for* $t > 0$;

(ii) $\|E_d(t)\psi\|_{C(\overline{\Omega})} \leq \|\psi\|_{L^\infty(\Omega)}, \quad t > 0$;

(iii) $\left\|\dfrac{d}{dt}E_d(t)\psi\right\|_{C(\overline{\Omega})} = \|\triangle E_d(t)\psi\|_{C(\overline{\Omega})} \leq \dfrac{M}{t}\|\psi\|_{L^\infty(\Omega)}, \quad t > 0,$

where the positive constant M is independent of t and ψ.

This lemma is a result from some properties of the Green function $E_d(t;x,y)$ such as

$$\begin{aligned} &E_d(t;x,y) > 0 \quad (t>0;\ x,y \in \Omega), \\ &\int_\Omega E_d(t;x,y)\,dy = 1 \quad (t>0;\ x \in \Omega), \\ &\left|\left(\frac{\partial}{\partial t}\right)^k E_d(t;x,y)\right| \leq \frac{C}{t^{N/2+k}}\exp\Bigl(-c\frac{|x-y|^2}{t} + \omega t\Bigr) \quad (k=0,1;\ t>0;\ x,y \in \Omega), \end{aligned}$$

where the constants $C > 0$, $c > 0$, $\omega \in \mathbf{R}$ are independent of t, x, y. For these estimates see, e.g., [16, §5.3 and §5.5].

Due to Lemma 5.1 and the fact that

$$E_{d_1}(\cdot)u_0^\epsilon, E_{d_2}(\cdot)v_0^\epsilon \in C([0,\infty); C(\overline{\Omega})) \quad \text{for} \quad u_0^\epsilon, v_0^\epsilon \in C(\overline{\Omega}),$$

we can show that the right-hand sides of (5.1) define a contraction mapping on a closed ball in $C([0,T]; C(\overline{\Omega}) \times C(\overline{\Omega}) \times L^\infty(\Omega))$ if T is sufficiently small. Thus we obtain the following lemma.

LEMMA 5.2. *For any* $(u_0^\epsilon, v_0^\epsilon, w_0^\epsilon) \in C(\overline{\Omega}) \times C(\overline{\Omega}) \times L^\infty(\Omega)$ *there exists a positive number* $T = T(\|u_0^\epsilon\|_{C(\overline{\Omega})}, \|v_0^\epsilon\|_{C(\overline{\Omega})}, \|w_0^\epsilon\|_{L^\infty(\Omega)})$ *such that* (5.1) *possesses a unique solution* (u, v, w) *in* $C([0,T]; C(\overline{\Omega}) \times C(\overline{\Omega}) \times L^\infty(\Omega))$.

With the help of Lemma 5.1 we can derive the regularity of the solution of (5.1) in a similar manner to the standard application of analytic semigroups. Hence we can conclude Lemma 2.2.

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