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On the Numerical Stability of the Method of Fundamental Solution Applied to the Dirichlet Problem

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In this paper we are concerned with analyzing the numerical stability of the method of fundamental solution [2] applied to the Dirichlet problem of Laplace's equation. In the course of the analysis we clarify the mechanism of the propagation of the perturbation in the boundary condition of the problem and develop a numerical technique to examine the numerical stability of the method.

Key words: numerical stability, the method of fundamental solution, Dirichlet problem

1. Introduction

We deal with the approximate solution of the Dirichlet problem

$$\Delta \boldsymbol{u}=0 \quad \text{in} \quad \Omega \tag{1.1}$$

$$\boldsymbol{u}=g \quad \text{on} \quad \partial\Omega , \tag{1.2}$$

where $\Omega=\{\omega\in \boldsymbol{R}^2 \mid \|\omega\|_2<p\}$, obtained by the method of fundamental solution [2] (or charge simulation method in terms of electric engineering [5]). The method approximates the solution $\boldsymbol{u}(x)$ by

$$\boldsymbol{u}_n(x)=\sum_{k=1}^{n} c_k G(x, y_k) , \qquad x\in\Omega \tag{1.3}$$

where $G(x, y)$ is the Green's function for (Δ, Ω),

$$G(x, y)=-\frac{1}{2}\log \|x-y\|_2 , \qquad x, y\in \boldsymbol{R}^2$$

points y_k's, called charge points, are chosen appropriately and c_k's are constants to be determined. The vector $c=(c_1, c_2, \cdots, c_n)^t\in \boldsymbol{R}^n$ is called charge and determined in such a way that $\boldsymbol{u}_n(x)$ satisfies the boundary condition

$$\boldsymbol{u}_n(\hat{x}_j)=g(\hat{x}_j) \qquad j=1, 2, \cdots, n , \tag{1.4}$$

where $\hat{x}_j$'s are properly chosen n collocation points on the boundary. Let the charge points $y_1, y_2, \cdots, y_n$ be on the auxiliary boundary which is the outer circle with radius

R (with "outer" we imply $R>\rho$).

With the collocation points $\hat{x}_k=\rho e^{(2\pi/n)(k-1)i}$ and the charge points $y_k=Re^{(2\pi/n)(k-1)i}$, $k=1, 2, \cdots, n$, the following results are known [4].

a) The approximate solution $\boldsymbol{u}_n$ converges to the solution $\boldsymbol{u}$ exponentially with respect to n. More precisely

$$\|\boldsymbol{u}-\boldsymbol{u}_n\|_\infty \leqq \sup_{\|x\|_2=r_0} |\boldsymbol{u}(x)| \frac{2}{1-\rho/r_0} \tag{1.5}$$
$$\{(1+C(R, \rho))(\rho/r_0)^{n/3}+4(\rho/R)^{n/3}\},$$

where we suppose that the harmonic extension of u exists in $\Omega_{r_0}=\{\omega \mid \|\omega\|_2<r_0\}$ with $\rho<r_0$. $C(R, \rho)$ is a constant depends on R and ρ.

b) The condition number of the coefficients matrix of the equation (1.4) which determines the charge c grows exponentially with respect to n. Approximately the condition number Cond(n, R) can be estimated by

$$\text{Cond}(n, R) \sim \frac{\log R}{2} n \left(\frac{R}{\rho}\right)^{n/2}. \tag{1.6}$$

(1.6) follows from the fact that the coefficient matrix for the particular location of $\hat{x}_k$ and y_k is circulant. For the properties of circulant matrices, see e.g. [2].

From the above facts we can point out an anomaly by overviewing the whole procedure of the method of fundamental solution from practical point of view:

i) If we try to obtain higher accuracy and quick convergence for the approximate solution by the charge simulation method, a) suggests that we should take n (number of collocation points) and R (the radius of auxiliary boundary) larger.

ii) For large n and R the condition number of the coefficient matrix of the equation to determine the charge c becomes large exponentially.

iii) Since the boundary value $g(\hat{x}_j)$ cannot be represented precisely on a floating-point number system, the perturbation Δg of $g(\hat{x})=(g(\hat{x}_1), \cdots, g(\hat{x}_n))$ may introduce arbitrarily large perturbation Δc in c.

iv) Then the approximate solution $\boldsymbol{u}_n(x)$ shall be calculated by the formula (1.3) for all $x\in\Omega$ utilizing the charge which may contain the large perturbation Δc.

In this paper, we examine numerical stability of the method of fundamental solution, that is, how the perturbation Δg in g propagates to the numerical solution $\tilde{\boldsymbol{u}}_n$ by the method. On the basis of the analysis, we propose a numerical technique for testing the numerical stability of the method. Finally, numerical examples are given.

2. Formulation

Though the convergence result a) and the estimate of the condition number b) are only valid for circular domain and auxiliary boundary, the following analysis is applicable to any simply connected convex domain. First, we prepare the following notations.

Let $c=(c_1, c_2, \cdots, c_n)^t$ be charge, $\hat{x}=(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)^t$ be collocation points and $y=(y_1, y_2, \cdots, y_n)^t$ be charge points. We note that $\hat{x}_i, \hat{y}_i \in \boldsymbol{R}^2$ and $c_i \in \boldsymbol{R}$ for $i=1, 2, \cdots, n$. Setting

$$\Gamma(\hat{x}, y)=(\gamma_{jk}) \qquad j, k=1, \cdots, n \tag{2.1}$$

with

$$\gamma_{jk}=G(\hat{x}_j, y_k)\,,$$

we can write the equation (1.4) to determine c

$$\Gamma(\hat{x}, y)c=g(\hat{x}) \tag{2.2}$$

where $g(\hat{x})$ is defined in the previous section. Hence, for any points $x=(x_1, x_2, \cdots, x_n)^t$ on the boundary we have from (1.3)

$$(\boldsymbol{u}_n(x_1), \cdots, \boldsymbol{u}_n(x_n))^t=\Gamma(x, y)c\,. \tag{2.3}$$

Furthermore, we let $\bar{g}(\hat{x})$ be boundary values with perturbation Δg, or $\bar{g}(\hat{x})=g(\hat{x})+\Delta g$. Δg may be viewed as the rounding error for floating-point representation of $g(\hat{x})$. In the course of actual computation, some perturbation $\Delta\Gamma$ for Γ may be introduced due to rounding error. We assume, however, that Γ is computed precisely, or $\Delta\Gamma=0$ for the simplicity of further analysis.

Let $\bar{c}$ denote the perturbed charge, or the solution of $\Gamma(\hat{x}, y)\bar{c}=\bar{g}(\hat{x})$ for perturbed boundary condition $\bar{g}(\hat{x})$. The "numerical solution $\bar{\boldsymbol{u}}_n(x)$" by the method of fundamental solution is defined by

$$\bar{\boldsymbol{u}}_n(x)=\sum_{k=1}^{n} \bar{c}_k G(x, y_k)\,, \qquad x\in\partial\Omega\,. \tag{2.4}$$

We let $\sigma_1 \geqq \sigma_2 \geqq \cdots \geqq \sigma_n > 0$ be singular values of $\Gamma=\Gamma(\hat{x}, y)$ and $\{u_i\}$, $\{v_i\}$ $i=1, 2, \cdots, n$ be singular vectors. We note that $\Gamma v_i=\sigma_i u_i$, $\Gamma^t u_i=\sigma_i v_i$ and $(u_i, u_j)=(v_i, v_j)=\delta_{ij}$, where δ_{ij} denotes Kronecker's delta.

3. The Propagation of the Perturbation Δg to the Charge c

First, we consider the propagation of Δg to the charge c. We denote the perturbation introduced into charge Δc and the charge with the perturbation $\bar{c}(=c+\Delta c)$. Using the notations defined above we have

$$\Gamma(\hat{x}, y)c=g \tag{3.1}$$

and

$$\Gamma(\hat{x}, y)\bar{c}=\bar{g}\,. \tag{3.2}$$

Expanding c and $\bar{c}$ by the singular vectors $\{v_i\}$ $i=1, 2, \cdots, n$, we have

$$\bar{c}=\sum_{i=1}^{n} (\bar{c}, v_i)v_i \quad \text{and} \quad c=\sum_{i=1}^{n} (c, v_i)v_i\,.$$

From the relations of Fourier coefficients

$$\sigma_i(\bar{c}, v_i)=(\bar{g}, u_i) \quad \text{and} \quad \sigma_i(c, v_i)=(g, u_i)\,,$$

we can write

$$\Delta c=\bar{c}-c$$

$$=\sum_{i=1}^{n} \frac{1}{\sigma_i}(\Delta g, u_i)v_i\,. \tag{3.3}$$

Thus, we have from orthonormality of $\{v_i\}$

$$\|\Delta c\|^2=\sum_{i=1}^{n} \frac{1}{\sigma_i^2}(\Delta g, u_i)^2\,. \tag{3.4}$$

(3.4) implies that $\|\Delta c\|^2$ becomes arbitrarily large as $n\to\infty$ and $\sigma_n\to 0$, that is, the perturbation introduced in the charge shall be of great concern. Next, we examine how this "large" perturbation Δc due to Δg propagates to the numerical solution $\bar{\boldsymbol{u}}_n$.

4. The Propagation of Δg to the Numerical Solution $\bar{\boldsymbol{u}}_n$

An advantage to employing the method of fundamental solution involves the fact that the error $\bar{\boldsymbol{u}}_n(x)-\boldsymbol{u}(x)$ is harmonic in Ω. By the maximum principle, this enables us to evaluate the maximum norm of the error only on the boundary. So we can restrict ourselves to consider evaluating the error only on the boundary. We prepare the following notations.

We define n distinct evaluating points $\bar{x}=(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)^t$ by

$$\bar{x}(\theta)=\hat{x}e^{\theta i}\,, \qquad -\pi/n\leqq\theta<\pi/n \tag{4.1}$$

namely $\bar{x}(\theta)$ rotates $\hat{x}$ by the angle θ counter-clockwise. Setting

$$\bar{\Gamma}=\Gamma(\bar{x}(\theta), y)=(\bar{\gamma}_{ij})$$

with

$$\bar{\gamma}_{ij}=G(\bar{x}(\theta)_i, y_j)\,, \tag{4.2}$$

we can write the numerical solution $\bar{\boldsymbol{u}}_n(\bar{x})=(\bar{\boldsymbol{u}}_n(\bar{x}_1), \bar{\boldsymbol{u}}_n(\bar{x}_2), \cdots, \bar{\boldsymbol{u}}_n(\bar{x}_n))^t$,

$$\bar{\boldsymbol{u}}_n(\bar{x})=\Gamma(\bar{x}, y)\bar{c}\,. \tag{4.3}$$

Similar to the previous section we denote the singular values of $\bar{\Gamma}(\bar{x}, y)$ by $\bar{\sigma}_1\geqq\bar{\sigma}_2\geqq\cdots\geqq\bar{\sigma}_n>0$ and the singular vectors $\{\bar{u}_i\}$, $\{\bar{v}_i\}$, $i=1, 2, \cdots, n$. We again note that $(\bar{u}_i, \bar{u}_j)=(\bar{v}_i, \bar{v}_j)=\delta_{ij}$ and $\bar{\Gamma}\bar{v}_i=\bar{\sigma}_i\bar{u}_i$, $\bar{\Gamma}^t\bar{u}_i=\bar{\sigma}_i\bar{v}_i$.

Now we examine the propagation of Δg to the numerical solution $\bar{\boldsymbol{u}}_n(\bar{x})$. The numerical error $\Delta\bar{\boldsymbol{u}}_n(\bar{x})=\bar{\boldsymbol{u}}_n(\bar{x})-\boldsymbol{u}_n(\bar{x})$ at $\bar{x}$ can be written as

$$\Delta\bar{\boldsymbol{u}}_n(\bar{x})=\Gamma(\bar{x}(\theta), y)\bar{c}-\Gamma(\bar{x}(\theta), y)c\,. \tag{4.4}$$

THEOREM 4.1 (Propagation of Δg to the numerical solution).

(4.5) $$\|\Delta\bar{\boldsymbol{u}}_n\|_\infty \leqq \sqrt{n}\,\|\Xi * \Delta\|_F\|\Delta g\|_\infty\,,$$

where

$$\Xi=(\xi_{ij}),\qquad \xi_{ij}=\bar{\sigma}_i/\sigma_j\,,$$
$$\Delta=(\delta_{ij})\,,\qquad \delta_{ij}=(\bar{v}_i, v_j)\,,$$

$\Xi * \Delta$ stands for the Hadamard product of the matrix Ξ and Δ, that is, $\Xi * \Delta=(h_{ij})$, $h_{ij}=\xi_{ij}\delta_{ij}$, and $\|\cdot\|_F$ represents the Frobenius norm.

REMARK: In the ordinary notation, $\|\Xi * \Delta\|_F$ can be written as (Trace $(\Xi\Delta^t))^{1/2}$, but we prefer the former notation which shall be more intuitively appealing in the later discussion.

Proof. Firstly, expanding Δc by $\{\bar{v}_i\}$ $i=1, 2, \cdots, n$, we have

(4.6) $$\Delta c=\sum_{i=1}^{n} (\Delta c, \bar{v}_i)\bar{v}_i\,.$$

On the other hand expanding Δc by $\{v_i\}$ $i=1, 2, \cdots, n$, we also have

(4.7) $$\Delta c=\sum_{j=1}^{n} (\Delta c, v_j)v_j\,.$$

Putting (4.7) into (4.6), we have

(4.8) $$\Delta c=\sum_{i=1}^{n}\left(\sum_{j=1}^{n} (\Delta c, v_j)v_j, \bar{v}_i\right)\bar{v}_i\,.$$

Using the relations of Fourier coefficients

(4.9) $$(\Delta c, v_i)=(1/\sigma_i)(\Delta g, u_i),\qquad i=1, 2, \cdots, n\,,$$

we have

(4.10) $$\Delta c=\sum_{i=1}^{n}\left\{\left(\sum_{j=1}^{n}\frac{1}{\sigma_j}(\Delta g, u_j)v_j, \bar{v}_i\right)\bar{v}_i\right\}.$$

Now, since the numerical error $\Delta\bar{\boldsymbol{u}}_n$ satisfies

(4.11) $$\Delta\bar{\boldsymbol{u}}_n=\bar{\Gamma}\bar{c}-\bar{\Gamma}c=\bar{\Gamma}\Delta c\,,$$

we have

$$\Delta\bar{\boldsymbol{u}}_n=\sum_{i=1}^{n}\bar{\Gamma}\bar{v}_i\left(\sum_{j=1}^{n}\frac{1}{\sigma_j}(\Delta g, u_j)v_j, \bar{v}_i\right)$$

(4.12) $$=\sum_{i=1}^{n}\bar{\sigma}_i\bar{u}_i\left(\sum_{j=1}^{n}\frac{1}{\sigma_j}(\Delta g, u_j)v_j, \bar{v}_i\right).$$

Since $\{\bar{u}_i\}$ forms orthonormal basis, we have

$$\|\Delta\bar{\boldsymbol{u}}_n\|_2^2=\sum_{i=1}^n \bar{\sigma}_i^2\left\{\sum_{j=1}^n \frac{1}{\sigma_j}(\Delta g, u_j)(v_j, \bar{v}_i)\right\}^2 .$$

We obtain from Schwarz's inequality

$$\|\Delta\bar{\boldsymbol{u}}_n\|_2^2\leqq \sum_{i=1}^n \bar{\sigma}_i^2\sum_{j=1}^n \frac{1}{\sigma_j^2}(v_j, \bar{v}_i)^2 \sum_{j=1}^n (\Delta g, u_j)^2$$

$$(4.13)\qquad =\sum_{i=1}^n \bar{\sigma}_i^2\sum_{j=1}^n \frac{1}{\sigma_j^2}(\bar{v}_i, v_j)^2\|\Delta g\|_2^2 .$$

Setting $\Xi=(\xi_{ij})$, $\xi_{ij}=\bar{\sigma}_i/\sigma_j$ and $\Delta=(\delta_{ij})$, $\delta_{ij}=(\bar{v}_i, v_j)$, $i, j=1, 2, \cdots, n$, we rewrite (4.13) to

$$(4.14)\qquad \|\Delta\bar{\boldsymbol{u}}_n\|_2^2\leqq \|\Xi * \Delta\|_F^2\|\Delta g\|_2^2 .$$

Thus, we have

$$\|\Delta\bar{\boldsymbol{u}}_n\|_\infty\leqq n^{1/2}\|\Xi * \Delta\|_F\|\Delta g\|_\infty . \qquad \text{Q.E.D.}$$

We call the matrix Ξ the explosive factor matirx and Δ the distorsion coefficients matrix. These two matrices play the essential role in propagating the perturbation Δg and have a vivid intuitive image. We should note that

1) The magnifying factor $\|\Xi * \Delta\|_F$ in (4.5) does not depend on the problem or given boundary condition.

2) Ξ and Δ as well as $\|\Xi * \Delta\|_F$ are readily computable by utilizing the singular value decomposition (S.V.D.) algorithm, see e.g. [3].

In the next section we discuss the meaning of the matrices Ξ and Δ and how to make use of them in practice.

5. Mechanism of the Propagation of Δg and the Meaning of Ξ and Δ

By tracing the proof of Theorem 4.1, we can understand the mechanism of the propagation Δg fairly clearly. First, we note that for $i, j=1, 2, \cdots, n$ $\{u_j\}$ forms an orthonormal basis for the boundary value g, $\{v_j\}$ and $\{\bar{v}_i\}$ form basis for the charge c and $\{\bar{u}_i\}$ forms a basis for the numerical solution $\bar{u}_n$. Here, we examine how $(\Delta g, u_j)$, the u_j-component of Δg, propagates to $(\bar{\boldsymbol{u}}_n(\bar{x}), \bar{u}_i)$, the $\bar{u}_i$-component of $\bar{u}_n(\bar{x})$:

1) The u_j-component of Δg propagates to the v_j-component of Δc and shall be magnified by the factor of $1/\sigma_j$.

2) The v_j-component of Δc propagates to the $\bar{v}_i$-component of Δc on the auxiliary boundary and multiplied by the coefficient of $(\bar{v}_i, v_j)$.

3) The $\bar{v}_i$-component of Δc propagates to the $\bar{u}_i$-component of $\Delta\bar{\boldsymbol{u}}_n(\bar{x})$ and is multiplied by the factor $\bar{\sigma}_i$.

Since the indices i and j run through $1, 2, \cdots, n$, n^2 combinations of paths from u_j to $\bar{u}_i$ are conceivable. For each path the factors $1/\sigma_j$ and $\bar{\sigma}_i$ represent the possibility of how the perturbation can be magnified. When the ratio $\bar{\sigma}_i/\sigma_j$ is very large the path from u_j to $\bar{u}_i$ has the possibility to destroy the numerical solution significantly. So, the explosive factor matrix Ξ

$$\Xi=\begin{pmatrix} \bar{\sigma}_1/\sigma_1 & \bar{\sigma}_1/\sigma_2 & \cdots & \bar{\sigma}_1/\sigma_n \\ \bar{\sigma}_2/\sigma_1 & \bar{\sigma}_2/\sigma_2 & \cdots & \bar{\sigma}_2/\sigma_n \\ \vdots & \vdots & & \vdots \\ \bar{\sigma}_n/\sigma_1 & \cdot & \cdots & \bar{\sigma}_n/\sigma_n \end{pmatrix} \tag{5.1}$$

shows us how "explosive" each path can magnify the perturbation Δg. We can easily recognize which path can be dangerous by constructing the matrix (5.1). For our present problem, since (1.6) of b) in the first section implies that $\sigma_n \downarrow 0$ very rapidly (exponentially), the upper right corner of Ξ where $\bar{\sigma}_i$'s are relatively large and σ_j's are close to 0 can be the most threatening area.

This factor, however, not always fully magnifies the components of Δg. This heavily depends on another factor of distortion coefficients matrix Δ defined by

$$\Delta=\begin{pmatrix} (\bar{v}_1, v_1) & (\bar{v}_1, v_2) & \cdots & (\bar{v}_1, v_n) \\ (\bar{v}_2, v_1) & (\bar{v}_2, v_2) & \cdots & (\bar{v}_2, v_n) \\ \vdots & \vdots & & \vdots \\ (\bar{v}_n, v_1) & (\bar{v}_n, v_2) & \cdots & (\bar{v}_n, v_n) \end{pmatrix} \tag{5.2}$$

whose element $(\bar{v}_i, v_j)$ determines the ratio of the propagation through the path from u_j to $\bar{u}_i$. Since $\{\bar{v}_i\}$ and $\{v_i\}$ $i=1, 2, \cdots, n$ form orthonormal bases, $|(\bar{v}_i, v_j)|\leq 1$. For the present problem $\{v_j\}$ and $\{\bar{v}_i\}$ $i,j=1, 2, \cdots, n$ are singular vectors of $\Gamma(\hat{x}, y)$ and $\bar{\Gamma}(\bar{x}(\theta), y)$ respectively, where $x(\theta)$ rotates $\bar{x}$ by the angle θ with $-\pi/n\leq\theta<\pi/n$. In this sense, we can regard $\bar{\Gamma}(\bar{x}(\theta), y)$ as a perturbed matrix of $\Gamma(\hat{x}, y)$ and accordingly $\{\bar{v}_i\}$ as a perturbed singular vector of $\{v_i\}$. If $\theta=0$, $\bar{v}_i$ coincide with v_i $i=1, 2, \cdots, n$ and we have, from the orthogonality, $(\bar{v}_j, v_i)=0$ for $i\neq j$. This orthogonality, however, shall be distorted by the rotation θ, and $(\bar{v}_i, v_j)$ represents the degree of the distortion. In this sense, we call Δ the distortion coefficients matrix.

We can obtain the total magnifying ratio of the path from u_j to $\bar{u}_i$ by multiplying the explosive factor $\bar{\sigma}_i/\sigma_j$ and the distortion coefficient $(\bar{v}_i, v_j)$. Since the largest explosive factors locates at the upper right corner of Ξ, we can examine whether the whole numerical procedure is extremely unstable or not by checking the corresponding area of Δ of (5.2). That is, if the upper right corner of Δ has elements whose absolute values are close to unity the perturbation Δg can be magnified significantly and the numerical method shall be very unstable. Furthermore, we can recognize other reasons why a numerical algorithm does not work well by examining each element of Ξ and Δ carefully. Some examples shall be presented in the next section.

6. Numerical Examples

Making use of the singular value decomposition, we can construct matrices Ξ and Δ very easily. In the following numerical examples we set the radius ρ of the domain Ω unity, the charge points $y_j = r\exp(2\pi(j-1)i/n)$ and the collocation points $\hat{x}_j = \exp(2\pi(j-1)i/n)$, $j=1, 2, \cdots, n$, where i represents the imaginary unit. The boundary $\partial\Omega$ is the unit disc, the auxiliary boundary is an outer circle with radius $R>1$ and the charge points are scattered at similar position to the collocation points. R and n, we write the pair (n, R), are varied from $(4, \sqrt[3]{2})$ to $(51, 32)$. The condition numbers of $\Gamma(\hat{x}, y; n, R)$ are visualized in the Fig. 6.1, where the surface of the cond$(\Gamma(n, R))$ is cut by the plane of Cond$(\Gamma(n, R)) = 10^6$ which is critical for our numerical experiments.

Since it is impossible to present the matrices Ξ and Δ for all $n=4, 5, \cdots, 51$ and $R \in [\sqrt[3]{2}, 32]$, we pick up a couple of combinations of (n, R) which present typical patterns of Ξ and Δ. Furthermore, we set the evaluation points $\bar{x}_j(\theta) = \bar{x}_j(\pi/n) = \exp(\pi/n(j-1)i)$, or the middle point of $\hat{x}_j$ and $\hat{x}_{j+1}$ on the boundary arc.

Example 1. We observe the case of $(n, R) = (16, 4.0)$ where the numerical

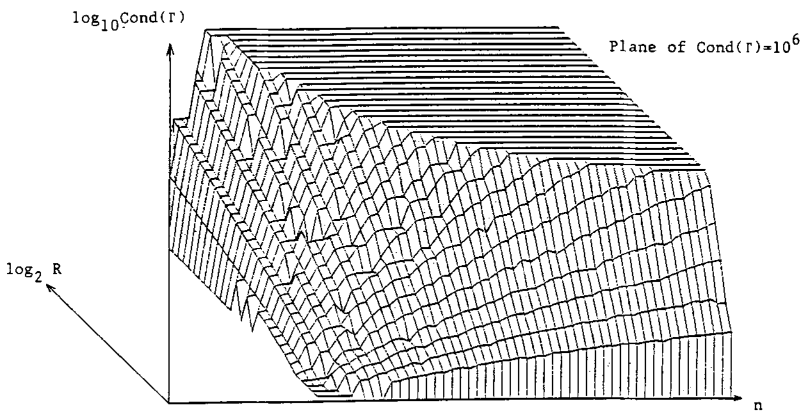

Fig. 6.1. Surface of condition number of $\Gamma(n, R)$.

0	1	1	2	2	3	3	3	3	4	4	5	5	5	5	6
-1	0	0	1	1	2	2	2	2	3	3	4	4	4	4	5
-1	0	0	1	1	2	2	2	2	3	3	4	4	4	4	5
-2	-1	-1	0	0	1	1	2	2	2	2	4	3	3	4	4
-2	-1	-1	0	0	1	1	2	2	2	2	4	3	3	4	4
-3	-2	-2	-1	-1	0	0	1	1	1	1	3	2	2	3	3
-3	-2	-2	-1	-1	0	0	1	1	1	1	3	2	2	3	3
-3	-2	-2	-2	-2	-1	-1	0	0	1	1	2	1	1	2	2
-3	-2	-2	-2	-2	-1	-1	0	0	1	1	2	1	1	2	2
-4	-3	-3	-2	-2	-1	-1	-1	-1	0	0	1	1	1	1	2
-4	-3	-3	-2	-2	-1	-1	-1	-1	0	0	1	1	1	1	2
-5	-4	-4	-4	-4	-3	-3	-2	-2	-1	-1	0	-1	-1	0	0
-5	-4	-4	-3	-3	-2	-2	-1	-1	-1	-1	1	0	0	1	1
-5	-4	-4	-3	-3	-2	-2	-1	-1	-1	-1	1	0	0	1	1
-5	-4	-4	-4	-4	-3	-3	-2	-2	-1	-1	0	-1	-1	0	0
-6	-5	-5	-4	-4	-3	-3	-2	-2	-2	-2	0	-1	-1	0	0

Fig. 6.2. Explosive factor matrix for $(n, R) = (16, 4)$.

solution attains relatively high accuracy. We should note that $\mathrm{Cond}(\Gamma(n, R)) = \mathrm{cond}(\Gamma(16, 4.0)) \approx 10^6$ ($\sigma_1 \approx 1.77$ and $\sigma_{16} \approx 2.38 * 10^{-6}$) and are severely ill-conditioned. The numerical solution $\tilde{u}_{16}(\bar{x})$, however, attains the accuracy of 5 decimal digits ($\|u_n(\bar{x}) - u(\bar{x})\|_\infty \approx 1.2 * 10^{-5}$) for $u = x^2 + y - y^2$. In this case, the explosive factor matrix Ξ reflects the ill-conditioning of $\Gamma(n, R)$ and the upper right corner of Ξ is fairly large. (Fig. 6.2) The elements of Ξ are rewritten as $[\log_{10}\xi_{ij} + 0.5]$, or rounding off the fractions of $\log_{10}\xi_{ij}$ for simple notation.

Now, we examine the corresponding distortion coefficients matrix Δ (Fig. 6.3). Similar to Ξ, the elements of $\Delta = (\delta_{ij})$ are rewritten as rounding off the fractions of $\log_{10}\delta_{ij}$. We can recognize that the upper right corner of Δ is small enough to cancel the explosive factor. This implies that the propagation of the explosive factor due to the ill-conditioning of $\Gamma(n, R)$ is shut off by the coefficients $\delta_{ij} = (\bar{v}_i, v_j)$. We can also observe that the distortion of the orthogonality can be seen near diagonal elements and the lower right corner in which the singular vectors corresponding to small singular values are involved.

0	-7	-6	-7	-6	-7	-6	-6	-7	-7	-6	-6	-6	-7	-7	-6
-6	0	-1	-7	-6	-6	-6	-6	-7	-7	-6	-6	-6	-7	-6	-6
-6	-1	0	-7	-7	-6	-7	-6	-6	-6	-6	-7	-6	-7	-7	-7
-6	-6	-8	0	0	-6	-5	-5	-6	-6	-5	-5	-5	-6	-7	-5
-7	-7	-7	0	0	-5	-5	-6	-6	-6	-5	-5	-5	-6	-6	-5
-6	-6	-6	-5	-5	0	0	-5	-5	-5	-5	-5	-5	-5	-5	-5
-7	-8	-6	-5	-5	0	0	-4	-5	-5	-4	-4	-4	-5	-5	-4
-6	-6	-6	-5	-5	-5	-5	0	0	-4	-3	-4	-3	-4	-5	-4
-6	-6	-7	-5	-5	-4	-5	0	0	-5	-4	-4	-4	-4	-6	-4
-6	-8	-6	-5	-5	-5	-4	-4	-4	0	0	-3	-3	-4	-4	-3
-6	-7	-6	-5	-5	-5	-5	-4	-4	0	0	-4	-4	-3	-4	-4
-6	-6	-8	-5	-6	-5	-5	-4	-5	-4	-3	-3	0	0	-3	-2
-6	-6	-7	-6	-7	-5	-6	-4	-6	-3	-3	-1	-2	-3	0	-1
-6	-6	-6	-5	-5	-4	-4	-4	-4	-3	-3	-2	0	0	-3	-2
-6	-6	-6	-5	-5	-4	-4	-4	-4	-3	-3	0	-2	-2	-1	-2
-6	-6	-6	-5	-5	-5	-5	-4	-4	-3	-3	-3	-2	-2	-1	0

Fig. 6.3. Distortion coefficients matrix for $(n, R) = (16, 4)$.

0	2	2	3	3	5	5	6	6	7	6	7	7	8	8	8	8	8	8	8	8	8	8	10
-2	0	0	2	2	3	3	4	4	5	4	6	6	6	6	6	6	6	6	6	6	6	6	8
-2	0	0	2	2	3	3	4	4	5	4	6	6	6	6	6	6	6	6	6	6	6	6	8
-3	-2	-2	0	0	1	1	3	3	4	3	4	4	4	4	4	4	4	4	4	4	4	5	7
-3	-2	-2	0	0	1	1	3	3	4	3	4	4	4	4	4	4	4	4	4	4	4	5	7
-5	-3	-3	-1	-1	0	0	2	1	2	1	3	3	3	3	3	3	3	3	3	3	3	3	6
-5	-3	-3	-1	-1	0	0	2	1	2	1	3	3	3	3	3	3	3	3	3	3	3	3	6
-6	-4	-4	-3	-3	-1	-1	0	0	1	0	1	1	1	1	1	1	1	2	2	2	2	2	4
-6	-4	-4	-3	-3	-1	-1	0	0	1	0	1	1	1	1	1	1	1	1	1	1	2	2	4
-6	-4	-4	-3	-3	-1	-1	0	0	1	0	1	1	1	1	1	1	1	2	2	2	2	2	4
-7	-5	-5	-3	-3	-2	-2	0	-1	0	-1	1	1	1	1	1	1	1	1	1	1	1	1	4
-8	-6	-6	-4	-4	-3	-3	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	1	3
-8	-6	-6	-4	-4	-3	-3	-1	-1	0	-1	0	0	0	0	0	0	0	0	0	0	0	1	3
-8	-6	-6	-4	-4	-3	-3	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	3
-8	-6	-6	-4	-4	-3	-3	-1	-2	-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	3
-8	-6	-6	-4	-4	-3	-3	-1	-2	-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	3
-8	-6	-6	-4	-4	-3	-3	-1	-2	-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	3
-8	-6	-6	-4	-4	-3	-3	-1	-2	-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	3
-8	-6	-6	-4	-4	-3	-3	-1	-2	-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	3
-8	-6	-6	-4	-4	-3	-3	-1	-2	-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	3
-8	-6	-6	-4	-4	-3	-3	-1	-2	-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	3
-8	-6	-6	-4	-4	-3	-3	-2	-2	-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	2
-8	-6	-6	-4	-4	-3	-3	-1	-2	-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	3
-8	-6	-6	-5	-5	-3	-3	-2	-2	-1	-2	-1	-1	0	-1	0	0	0	0	0	0	0	0	2

Fig. 6.4. Explosive factor matrix for $(n, R) = (24, 16)$.

0	-7	-7	-7	-7	-7	-7	-7	-6	-7	-7	-7	-8	-8	-7	-7	-7	-7	-7	-7	-7	-7	-7	-7
-7	0	-1	-6	-6	-6	-6	-6	-7	-6	-6	-7	-6	-6	-7	-6	-6	-6	-7	-6	-6	-6	-7	-6
-8	-1	0	-6	-6	-7	-6	-5	-6	-6	-6	-6	-7	-6	-6	-6	-7	-7	-6	-7	-6	-6	-7	-6
-6	-6	-6	-1	0	-5	-5	-5	-5	-5	-6	-5	-5	-5	-4	-6	-5	-5	-5	-5	-5	-5	-5	-5
-6	-6	-6	0	-1	-4	-5	-4	-4	-4	-6	-5	-5	-6	-5	-5	-5	-4	-5	-6	-5	-4	-4	-5
-6	-7	-6	-5	-5	0	0	-3	-3	-3	-4	-4	-4	-4	-3	-4	-5	-3	-3	-4	-4	-4	-4	-3
-6	-6	-6	-4	-5	0	0	-3	-3	-3	-3	-3	-3	-4	-3	-3	-3	-5	-4	-4	-4	-3	-4	-3
-7	-6	-6	-4	-5	-3	-3	0	-2	-2	0	-4	-1	-2	-3	-2	-1	-1	-3	-1	-2	-1	-1	-1
-7	-6	-5	-4	-4	-3	-3	0	-1	-1	0	-2	-2	-1	-3	-1	-1	-1	-2	-1	-1	-1	-1	-2
-6	-7	-6	-4	-5	-3	-3	-1	0	-1	-1	-2	-2	-3	-2	-3	-2	-2	-2	-2	-2	-2	-3	-2
-7	-6	-6	-4	-6	-5	-4	-1	-1	-1	-1	-3	-1	-1	-1	-1	-1	-1	-1	0	-1	0	0	-1
-6	-6	-7	-4	-5	-4	-3	-1	-4	0	-2	0	-1	-1	0	-1	0	-1	-1	-1	-1	-1	-1	-1
-7	-8	-6	-4	-5	-4	-4	-1	-3	0	-2	-1	-1	-1	0	-1	-1	-1	-2	0	0	-1	-2	-1
-8	-7	-6	-5	-5	-3	-4	-1	-2	-1	-1	-1	0	-2	-1	-1	-1	-1	0	-1	0	-1	-1	-3
-7	-6	-7	-6	-5	-4	-3	-1	-3	-1	-2	-3	0	0	-1	0	-1	-1	-1	-1	-1	-1	-1	-1
-8	-7	-6	-4	-5	-3	-3	-1	-2	0	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0
-7	-7	-6	-6	-5	-3	-4	-2	-2	-1	-1	-3	0	0	0	-1	-1	-1	0	-2	-1	0	-1	0
-7	-8	-7	-6	-5	-3	-4	-1	-2	-1	-3	-2	-1	-1	-1	-2	0	0	-1	0	0	-1	-1	-1
-7	-6	-7	-5	-5	-5	-3	-1	-2	0	-1	-1	-1	0	-2	-1	-1	-1	-1	0	-1	-2	0	-1
-7	-8	-8	-5	-6	-4	-3	-2	-1	-1	-4	-1	-1	0	-1	0	-1	-1	0	0	-1	-1	-1	-1
-7	-6	-7	-5	-5	-4	-5	-3	-2	-1	-2	0	0	-1	0	-1	-1	-2	0	-1	-1	-1	-1	-1
-7	-6	-6	-4	-5	-3	-3	-2	-2	-2	-2	0	-1	-1	-1	-1	-1	-2	0	-1	-2	0	-1	-1
-7	-6	-7	-5	-5	-3	-4	-2	-3	-1	-2	-1	0	-1	-1	0	0	-1	0	-1	-1	0	-1	-1
-7	-6	-7	-5	-6	-3	-3	-1	-2	-1	-3	-1	-1	-1	-1	0	-1	0	-1	-2	-1	-1	-1	0

Fig. 6.5. Distortion coefficients matrix for $(n, R)=(24, 16)$.

Example 2. We examine the case of $(n, R)=(24, 16)$. Though the convergence result a) in the first section asserts that the larger n and R are, the more accuracy the approximate solution $\boldsymbol{u}_n$ attains, this is not true for the numerical solution $\bar{\boldsymbol{u}}_n(\bar{x})$. In this case, the condition number of $\Gamma(n, R)=\Gamma(24, 16)$ is far larger than 10^6 ($\sigma_1 \approx 5.30$ and $\sigma_{24} \approx 2.11 * 10^{-10}$) and the accuracy of the numerical solution $\bar{\boldsymbol{u}}_n(\bar{x})$ is about 3 decimal digits ($\|\boldsymbol{u}_n(x)-\boldsymbol{u}(x)\|_\infty \approx 1.2 * 10^{-3}$), which is worse than the case of Example 1. But still, the numerical solution is not catastrophically destroyed. In the same logarithmic notation Fig. 6.4 represents the matrix Ξ. We can observe that the explosive area, where the elements of the matrix of Fig. 6.4 are larger than 5, is much wider than that of Example 1. As in Example 1, these corresponding elements of Δ in Fig. 6.5 at the upper right corner are as small as $\delta_{ij}=10^{-6} \sim 10^{-8}$ in size, but they are not small enough to cancel the explosive factor ξ_{ij} whose maximal elements are $10^8 \sim 10^{10}$. In this case, the magnifying factors $\delta_{ij}\xi_{ij} \approx 10^2 \sim 10^3$ remains.

We can also observe that the lower right part of Δ has relatively large elements of $10^{-2} \sim 10^0$ for $i, j \geqq 8$. The corresponding elements ξ_{ij}, $i, j \geqq 8$ of Ξ also contain relatively large factors of $10^2 \sim 10^4$ in size. In this way, we have the magnifying factors $\xi_{ij}\delta_{ij}$ of $10^0 \sim 10^2$ at the lower right part $(i, j \geqq 8)$ and they accumulate into the magnifying coefficient $\|\Xi * \Delta\|_F \approx 1.6 * 10^2$. This explains why numerical solutions for larger (n, R) may give worse results in accuracy and do not always follow the convergence result. This is the case when (n, R) is too large. We should not make (n, R) too large so that the lower right part with large distortion coefficients of δ_{ij}'s shall not spread widely.

7. Concluding Remarks

We can utilize the matrices Ξ and Δ to examine the numerical stability of the numerical method. It can also be used to study optimal locations of the charge points

y, or to check if the method of regularization in stabilizing the numerical scheme is effective or not. The technique developed here is applicable to other problems in examining the stability of numerical methods which consist of two steps that (i) one solves an equation $\Gamma c=g$ where Γ may be severely ill-conditioned, and (ii) one utilizes the solution c to obtain the final result $\boldsymbol{u}$ in the form of $\boldsymbol{u}=\Lambda c$, where Γ and Λ might be discretizations of some linear operators.

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