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Destabilization of Periodic Solutions Arising in Delay-Diffusion Systems in Several Space Dimensions

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We consider a diffusion equation with time delay having a stable spatially homogeneous periodic solution bifurcating from a steady state. We show that under certain circumstances the bifurcating periodic solution loses its stability very near the bifurcation point if the diffusion coefficients are sufficiently small. Such a destabilization phenomenon also occurs when in place of the diffusion coefficients, the shape of the domain is varied instead. Sufficient conditions for the occurrence of such phenomena, along with some specific examples, will be presented.

Key words: destabilization, spatially homogeneous periodic solution, functional differential equation, delay-diffusion equation

Introduction

Differential-delay equations arise in a variety of fields such as biology, optimal control etc., and have been studied by many authors including [2], [3], [5], [6], [13]. Hutchinson [1], for instance, introduced the following delay equation as a biological model which describes an oscillatory phenomenon occurring in the growth process of a single species:

$$\text{(E1)} \qquad \frac{d}{dt}y(t)=\alpha\left(1-\frac{y(t-r)}{K}\right)y(t),$$

where α, r, K are positive constants. Letting $v(t)=y(rt)/K-1$, the equation (E1) is transformed into,

$$\text{(E2)} \qquad \frac{d}{dt}v(t)=-a(1+v(t))v(t-1),$$

where $a=\alpha r$ and the steady state, $y\equiv K$ of (E1), corresponds to $v\equiv 0$ of (E2). It is known that (E2) has a periodic solution for $a>\pi/2$, as was first proved by Jones [2]. Furthermore, Chow and Mallet-Paret have shown in [4] that a Hopf bifurcation occurs from the steady state, $v\equiv 0$, when the parameter a passes through the value $\pi/2$ and that the bifurcating periodic solution is stable.

Here we shall couple the equation (E2) with a diffusion term. More precisely, we consider the following initial-boundary value problem:

$$\text{(E3)}\quad \begin{cases} \dfrac{\partial}{\partial t}v(t,x)=d\Delta v(t,x)-a(1+v(t,x))v(t-1,x), & (t,x)\in(0,\infty)\times\Omega, \\ \dfrac{\partial v}{\partial n}=0, & (t,x)\in(0,\infty)\times\partial\Omega, \\ v(\theta,x)=[\Phi_0(\theta)](x), & (\theta,x)\in[-1,0]\times\Omega, \end{cases}$$

where Ω is a bounded domain in $\boldsymbol{R}^N$ with a smooth boundary $\partial\Omega$, $\partial/\partial n$ denotes the outer normal derivative to $\partial\Omega$ and Δ stands for $\sum_{i=1}^{N}\partial^2/\partial x_i^2$, Φ_0 is an initial function chosen from a certain function space to be specified later (see (3.1) and (5.1)).

It is clear that for $a>\pi/2$ the equation (E3) has a periodic solution corresponding to that of (E2) mentioned above. This periodic solution to (E3) is a spatially homogeneous periodic solution independent of spatial variables.

As for stability of the spatially homogeneous periodic solution to (E3), Yoshida [7] has proved that the bifurcating periodic solution near the bifurcation point, $a=\pi/2$, is stable. However, it has not been made clear in [7] how the stability region of the bifurcation parameter, a, depends on other factors such as the diffusion constant, d, and the shape of the domain, Ω.

In the case where the space dimension is $N=1$, Lin and Kahn [8] have shown by the two-timing method that the bifurcating spatially homogeneous periodic solution loses its stability fairly near the bifurcation point when d is sufficiently small; they suggest that such a phenomenon of destabilization may lead to a chaotic behavior as d continues to decrease.

In this paper we shall discuss the destabilization of the spatially homogeneous periodic solution in quite a general framework. Applying our main results (to be given in §4) to the equation (E3), we see that for any $\Omega\subset\boldsymbol{R}^N$ the spatially homogeneous periodic solution becomes unstable near the bifurcation point if the diffusion coefficient, d, is taken sufficiently small. Moreover, in the case of several space dimensions (i.e., $N\geqq 2$), such destabilization also occurs when the shape of the domain, Ω, is varied. More precisely, this occurs when the second eigenvalue of the Laplacian on Ω with homogeneous Neumann boundary condition becomes sufficiently small. An illustrative example of such domains is given in Fig. A in §4.

We will also give a more general sufficient condition on the diffusion coefficients and the domain, Ω, for such destabilization of the spatially homogeneous periodic solution. The same argument applies to periodic solutions occurring in reaction-diffusion systems without time delay provided that we replace function spaces, inner products and other notations by appropriate ones (cf. Example B in §5; for further details, see [16]).

In §1 we will formulate the differential-delay equation (E2) in a fairly general form of functional differential equation (see. (1.1)). And we will give the Hopf bifurcation theorem for this equation without proof (Theorem H).

In §2 we will expand the periodic solution given by Theorem H in terms of a certain parameter, ε, and determine the coefficients of lower order terms. Using the result of § 2, we shall discuss, in § 3 and § 4, the linearized stability of the spatially homogeneous periodic solution to the diffusion equation with time delay; we thereby obtain theorems on the destabilization of the periodic solution mentioned above (Theorems B and C).

In the last section we shall apply our theorems to some specific equations. For example, Theorems B and C apply to the equation (E3), so, it is easily checked that the spatially homogeneous periodic solution loses its stability very near the bifurcation point if the diffusion constant is sufficiently small, or if the domain, Ω, is of a certain shape, as illustrated in Fig. A.

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1. The Hopf Bifurcation of Functional Differential Equations

For any finite interval $[a, b]$, we let $C[a, b]$ denote the space of all $\boldsymbol{R}^n$-valued continuous functions defined on $[a, b]$ with the usual supremum norm, $\|\cdot\|$. Let $\tau \in R^1$, $r>0$ and $a>0$. For any $X \in C[\tau-r, \tau+a]$ and $t \in [\tau, \tau+a]$, the symbol, X_t, will denote the element in $C[-r, 0]$ defined by the relation,

$$X_t(\theta)=X(t+\theta), \qquad -r \leqq \theta \leqq 0 .$$

From the definition it is clear that $X_t(0)$ is equal to $X(t)$.

In this section we shall consider the following functional differential equation:

$$\dot{X}(t)=L(\mu)X_t+G(\mu, X_t) \equiv F(\mu, X_t), \qquad t>0, \tag{1.1}$$

$$X(\theta)=\phi_0(\theta), \qquad -r \leqq \theta \leqq 0, \qquad \phi_0 \in C[-r, 0], \tag{1.2}$$

where $\cdot$ denotes d/dt. We assume that the mapping,

$$F: I_0 \times C[-r, 0] \to \boldsymbol{R}^n,$$

is of class C^4, where $I_0 \subset \boldsymbol{R}^1$ is an open interval containing $0 \in \boldsymbol{R}^1$. Furthermore, we assume that,

$$F(\mu, 0)=0 \qquad \text{for every} \quad \mu \in I_0 . \tag{1.3}$$

$L(\mu)$, $G(\mu, \cdot)$ are the linear part and the higher order nonlinear part of $F(\mu, \cdot)$ respectively. It is known that the initial value problem, (1.1), (1.2) admits a unique smooth solution (see Hale [5]).

The linear equation associated with (1.1) is:

$$\dot{Y}(t)=L(\mu)Y_t . \tag{1.4}$$

As $L(\mu)$ is a continuous linear mapping of $C[-r, 0]$ into $\boldsymbol{R}^n$, there is an $n \times n$ matrix function, $\eta(\theta; \mu)$, $-r \leqq \theta \leqq 0$, whose elements have bounded variation in θ on $[-r, 0]$, such that,

$$(1.5) \qquad L(\mu)\phi = \int_{-r}^{0} [d\eta(\theta; \mu)]\phi(\theta) \qquad \text{for} \quad \phi \in C[-r, 0] .$$

When $\mu = 0$, we simply write,

$$(1.6) \qquad \eta(\theta) = \eta(\theta; 0) \qquad \text{for} \quad \theta \in [-r, 0] .$$

The domain of $L(\mu)$ is naturally extended into $C([-r, 0]; \boldsymbol{C}^n)$ and (1.5) holds for $\phi \in C([-r, 0]; \boldsymbol{C}^n)$, where $\boldsymbol{C}^n$ is the n-dimensional complex space. Hereafter, the notation, $C[-r, 0]$, also denotes $C([-r, 0]; \boldsymbol{C}^n)$ as long as there is no confusion.

Now we define the characteristic equation associated with (1.4):

$$(1.7) \qquad \det\left(\lambda I - \int_{-r}^{0} e^{\lambda\theta}[d\eta(\theta; \mu)]\right) = 0 ,$$

where I is the $n \times n$ identity matrix. There are countably many roots of (1.7), each of them being at most finitely degenerated. It is known that the set of the roots of (1.7) coincides with the set of the eigenvalues of the linear system, (1.4) (see [5]). For example, let $A(\mu)$ be the infinitesimal generator of the semigroup associated with (1.4); namely $A(\mu)$ is defined as,

$$(1.8) \qquad A(\mu)\phi = \begin{cases} \dfrac{d}{d\theta}\phi(\theta) , & -r \leqq \theta < 0 , \\ L(\mu)\phi , & \theta = 0 , \end{cases} \qquad \text{for} \quad \phi \in \mathfrak{D}(A(\mu)) ,$$

where $\mathfrak{D}(A(\mu))$ denotes the domain of the operator $A(\mu)$. Then the spectrum of $A(\mu)$ consists only of eigenvalues, each of which is a root of (1.7) with corresponding multiplicity. In particular, the generalized eigenspace in $C[-r, 0]$ subject to each eigenvalue of $A(\mu)$ is finite dimensional.

In terms of the above operator, the equation, (1.1), can be written as,

$$(1.9) \qquad \frac{d}{dt}v(t) = A(\mu)v(t) + N(\mu, v(t)) \equiv f(\mu, v(t)) ,$$

where

$$[v(t)](\theta) = X_t(\theta) , \qquad -r \leqq \theta \leqq 0 ,$$

$$(1.10) \qquad [N(\mu, \phi)](\theta) = \begin{cases} 0 , & -r \leqq \theta < 0 , \\ G(\mu, \phi) , & \theta = 0 , \end{cases} \qquad \text{for} \quad \phi \in C[-r, 0] .$$

By the definition, the equation (1.9) implies,

$$\frac{d}{dt} X(t+\theta) = \begin{cases} \dfrac{d}{d\theta} X(t+\theta) , & -r \leqq \theta < 0 , \\ L(\mu)X_t + G(\mu, X_t) , & \theta = 0 . \end{cases}$$

If $X(t)$ is a solution to (1.1), then $[v(t)](\theta)=X(t+\theta)$ is a solution to (1.9) for $t>r$.

Here we assume the following two assumptions:

(A1) $A(\mu)$ has a pair of simple complex conjugate eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ such that

$$\lambda(\mu)=\sigma(\mu)+i\omega(\mu)\,,$$
$$\sigma(0)=0\,, \qquad \omega(0)=\omega_0\neq 0\,,$$

and that

$$\text{(1.11)} \qquad \operatorname{Re}\frac{d\lambda}{d\mu}(0)=\frac{d\sigma}{d\mu}(0)\neq 0\,;$$

(A2) the remaining eigenvalues of $A(0)$ have strictly negative real parts.

Then the following Hopf bifurcation theorem holds:

THEOREM H. *Let* (A1), (A2) *be true. Then the equation* (1.1) *has a family of periodic solutions: there is the positive constant,* ε_H, *and there are the* C^1*-functions,* $\mu(\varepsilon)$, $\omega(\varepsilon)$, $\beta(\varepsilon)$,

$$\text{(1.12)} \qquad \begin{cases}\mu=\mu(\varepsilon)=\mu_2\varepsilon^2+O(\varepsilon^3)\\ \omega=\omega(\varepsilon)=\omega_0+\omega_2\varepsilon^2+O(\varepsilon^3)\end{cases} \qquad \textit{for} \quad \varepsilon\in(0,\varepsilon_H)\,,$$

$$\text{(1.13)} \qquad \begin{cases}\beta=\beta(\varepsilon)=\beta_2\varepsilon^2+O(\varepsilon^3)\\ \beta_2=-2\dfrac{d\sigma}{d\mu}(0)\mu_2\,,\end{cases} \qquad \textit{for} \quad \varepsilon\in(0,\varepsilon_H)\,,$$

such that for each $\varepsilon\in(0,\varepsilon_H)$ *and* $\mu=\mu(\varepsilon)$ *there exists a periodic solution,* $p(t;\varepsilon)$, *with period* $2\pi/\omega(\varepsilon)$. *This periodic solution,* $p(\cdot;\varepsilon)$, *has Floquet exponents* 0 *and* $\beta=\beta(\varepsilon)$. *Except for the family of periodic solutions,* $p(\cdot;\varepsilon)$, $\varepsilon\in(0,\varepsilon_H)$, *there is no non-trivial periodic solution in a sufficiently small neighborhood of* $(0,0)\in I_0\times C[-r,0]$.

COROLLARY H. *Assume the assumptions in Theorem H and* $\beta_2<0$ *in* (1.13). *Then there is a constant,* ε_0, $0<\varepsilon_0<\varepsilon_H$, *such that for each* $\varepsilon\in(0,\varepsilon_0)$ *the bifurcating periodic solution,* $p(\cdot;\varepsilon)$, *to* (1.1) *is asymptotically stable* (*with asymptotic phase*).

We omit the proof of Theorem H (see [11]). For the sake of later arguments, however, we shall calculate the coefficients, μ_2, ω_2, appearing in (1.12). The calculation will be carried out in the next section.

REMARK 1. In the above discussion the equation, (1.9), is regarded as a differential equation on the Banach space $C[-r,0]$. Actually it is more appropriate to consider the equation, (1.9), on a slightly wider space, $\hat{C}[-r,0]=C[-r,0]\oplus\tilde{X}$, where $\tilde{X}=\{\phi\mid\phi(\theta)\in\boldsymbol{R}^n$ for each $\theta\in[-r,0]$, $\phi(\theta)=0$, $-r\leqq\theta<0\}$ (see [4]). With this notation, the domain of the operator, $A(\mu)$, can naturally be extended so that,

$$A(\mu):\ C^1[-r,0]\to\hat{C}[-r,0]$$

be a well-defined closed operator.

In this paper we shall simply use the notation $C[-r, 0]$ to represent $\hat{C}[-r, 0]$ as long as there is no confusion.

2. Construction of a Bifurcating Periodic Solution

In this section we determine the coefficients of lower order terms in the ε-expansion of $\mu(\varepsilon)$, $\omega(\varepsilon)$ and $p(t; \varepsilon)$ in Theorem H. The results will be used in the next section.

Iooss and Joseph's work [9] gives a formal calculation of the bifurcating periodic solution to (1.9). To show the validity of their calculation, however, we need to prove that the Fredholm alternative holds for the operator appearing in (1.9). (It is not trivial.) To avoid a lengthy argument, we turn back to the equation (1.1) and discuss the bifurcation problem in (1.1) instead of its generalized version (1.9). Using the result in [5; Chap. 9] (Fredholm alternative), we can solve the bifurcation equation occurring in (1.1) and determine the coefficients of lower order terms in ε. One also sees that the equation (obtained in (2.37)) which determines the coefficients μ_2 and ω_2 coincides with the one derived by the formal calculation in [9].

To simplify our notation for derivatives of $F(\mu, u)$ and $f(\mu, u)$, we write

$$F_u(\mu) \underset{\text{def}}{=} \frac{\partial}{\partial u} F(\mu, 0)\ (=L(\mu)), \qquad f_u(\mu) \underset{\text{def}}{=} \frac{\partial}{\partial u} f(\mu, 0)\ (=A(\mu)),$$

$$F_{\mu u}(\mu) \underset{\text{def}}{=} \frac{\partial^2}{\partial \mu \partial u} F(\mu, 0), \qquad f_{\mu u}(\mu) \underset{\text{def}}{=} \frac{\partial^2}{\partial \mu \partial u} f(\mu, 0),$$

$$F_{uu}(\mu)(\cdot, \cdot) \underset{\text{def}}{=} \frac{\partial^2}{\partial u^2} F(\mu, 0)(\cdot, \cdot), \qquad f_{uu}(\mu)(\cdot, \cdot) \underset{\text{def}}{=} \frac{\partial^2}{\partial u^2} f(\mu, 0)(\cdot, \cdot),$$

$$F_{uuu}(\mu)(\cdot, \cdot, \cdot) \underset{\text{def}}{=} \frac{\partial^3}{\partial u^3} F(\mu, 0)(\cdot, \cdot, \cdot), \quad f_{uuu}(\mu)(\cdot, \cdot, \cdot) \underset{\text{def}}{=} \frac{\partial^3}{\partial u^3} f(\mu, 0)(\cdot, \cdot, \cdot),$$

and so forth. By the definition of f, $[f(\mu, \phi)](0) = F(\mu, \phi)$ for $\phi \in C[-r, 0]$; hence $F_u(\mu)\phi = [f_u(\mu)\phi](0)$, $F_{\mu u}(\mu)\phi = [f_{\mu u}(\mu)\phi](0)$, $F_{uu}(\mu)(\phi, \phi) = [f_{uu}(\mu)(\phi, \phi)](0)$, $\cdots$ for $\phi \in C[-r, 0]$.

In order to study the bifurcation problem, we need to introduce the adjoint operator of $A(0)$ with respect to the (formal) duality product defined by,

$$\langle \phi, \psi \rangle \underset{\text{def}}{=} (\phi(0), \psi(0)) - \int_{-r}^{0} \int_{0}^{\theta} \overline{{}^t\psi(\xi - \theta)[d\eta(\theta)]}\phi(\xi) d\xi \tag{2.1}$$

$$\text{for} \quad \phi \in C[-r, 0] \quad \text{and} \quad \psi \in C[0, r],$$

where $\eta(\theta)$ is as in (1.6), ${}^t\psi$ denotes the transpose of the n-vector, ψ, and $(\cdot, \cdot)$ stands for the Hermite inner product in $\boldsymbol{C}^n$, i.e.,

$$(a, b) \underset{\text{def}}{=} {}^t\bar{b}a = \sum_{i=1}^{n} a_i \bar{b}_i\,, \qquad a = {}^t(a_1, \cdots, a_n),\ b = {}^t(b_1, \cdots, b_n)\,. \tag{2.2}$$

The adjoint operator, $A^*(0)$, of $A(0) = f_u(0)$ with respect to the product (2.1) is given by,

$$[A^*(0)\psi](\theta) = \begin{cases} -\dfrac{d\psi}{d\theta}\,, & 0 < \theta \leqq r\,, \\ L^*(0)\psi \underset{\text{def}}{=} \displaystyle\int_{-r}^{0} {}^t[d\eta(\tau)]\psi(-\tau)\,, & \theta = 0\,, \end{cases} \tag{2.3}$$

$$\text{for} \quad \psi \in \mathfrak{D}(A^*(0)) \subset C[0, r]\,,$$

where $\mathfrak{D}(A^*(0)) = \{\psi \mid \psi \in C[0, r],\ A^*(0)\psi \in C[0, r]\}$. It is known that the set of all eigenvalues of $A(0)$ coincides with that of $A^*(0)$. Let $\zeta_1(\theta)$, $-r \leqq \theta \leqq 0$, be an eigenfunction of $f_u(0) = A(0)$ corresponding to the simple eigenvalue, $i\omega_0$, that is,

$$A(0)\zeta_1 = i\omega_0\zeta_1\,, \tag{2.4}$$

and let $\zeta_1^*(\theta)$, $0 \leqq \theta \leqq r$, be an eigenfunction of $f_u^*(0) = A^*(0)$ corresponding to the simple eigenvalue, $-i\omega_0$,

$$A^*(0)\zeta_1^* = -i\omega_0\zeta_1^*\,. \tag{2.5}$$

ζ_1 and ζ_1^* are also written as

$$\zeta_1(\theta) = \zeta_0 e^{i\omega_0\theta} \quad (-r \leqq \theta \leqq 0)\,, \qquad \zeta_1^*(\theta) = \zeta_0^* e^{i\omega_0\theta} \quad (0 \leqq \theta \leqq r)\,, \tag{2.6}$$

where ζ_0 and ζ_0^* are n-vectors satisfying

$$\left(i\omega_0 - \int_{-r}^{0} e^{i\omega_0\theta}[d\eta(\theta)]\right)\zeta_0 = 0\,, \tag{2.7a}$$

$$\left(-i\omega_0 - \int_{-r}^{0} e^{-i\omega_0\theta}\,{}^t[d\eta(\theta)]\right)\zeta_0^* = 0\,, \tag{2.7b}$$

respectively.

It is shown in [5; Chapter 7] that a function, $\phi \in C[-r, 0]$, belongs to the range of the operator, $i\omega_0 - A(0)$, if and only if ϕ satisfies $\langle \phi, \zeta_1^* \rangle = 0$. Thus the space $C[-r, 0]$ is decomposed as,

$$\begin{aligned} C[-r, 0] &= \mathfrak{N}(i\omega_0 - A(0)) \oplus \mathfrak{R}(i\omega_0 - A(0))\,, \\ \mathfrak{N}(i\omega_0 - A(0)) &= \{\phi \mid (i\omega_0 - A(0))\phi = 0\}\,, \\ \mathfrak{R}(i\omega_0 - A(0)) &= \{\phi \mid \langle \phi, \zeta_1^* \rangle = 0\}\,. \end{aligned} \tag{2.8}$$

By normalization, we may assume,

$$\langle \zeta_1, \zeta_1^* \rangle = 1\,. \tag{2.9}$$

And it is clear from (2.8) that

(2.10) $$\langle \zeta_1, \overline{\zeta_1^*} \rangle = \langle \overline{\zeta_1}, \zeta_1^* \rangle = 0$$

holds. It is also checked that

(2.11) $$\langle \zeta_1, \zeta_1^* \rangle = (M\zeta_0, \zeta_0^*),$$

$$M \underset{\text{def}}{=} I - \int_{-r}^{0} \theta e^{i\omega_0\theta}[d\eta(\theta)] = I - L(0)(\cdot e^{i\omega_0 \cdot}).$$

Let $\zeta(\mu)$ be an eigenfunction of $A(\mu) = f_u(\mu)$ corresponding to $\lambda(\mu)$. Differentiating the equality,

$$f_u(\mu)\zeta(\mu) = \lambda(\mu)\zeta(\mu) \qquad (\lambda(0) = i\omega_0, \ \zeta(0) = \zeta_1),$$

with respect to μ and using (2.5) and (2.9), we get the relation,

(2.12) $$\frac{d\lambda}{d\mu}(0) = \langle f_{\mu u}(0)\zeta_1, \zeta_1^* \rangle.$$

By the definition of f and $\langle \cdot, \cdot \rangle$, (2.12) implies,

(2.13) $$\frac{d\lambda}{d\mu}(0) = (F_{\mu u}(0)\zeta_1, \zeta_0^*).$$

Next we define ζ_2 and $\hat{\zeta}_2 \in C[-r, 0]$ by

(2.14a) $$(2i\omega_0 - A(0))\zeta_2 = \frac{1}{2} f_{uu}(0)(\zeta_1, \zeta_1),$$

(2.14b) $$-A(0)\hat{\zeta}_2 = f_{uu}(0)(\zeta_1, \overline{\zeta}_1).$$

By the assumptions on the operator $A(0)$, both ζ_2 and $\hat{\zeta}_2$ are uniquely determined by (2.14): specifically, ζ_2 has the form,

(2.15a) $$\zeta_2(\theta) = \zeta_2(0)e^{2i\omega_0\theta} \qquad (-r \leqq \theta \leqq 0),$$

where $\zeta_2(0)$ satisfies,

(2.15b) $$(2i\omega_0 I - L(0)(e^{2i\omega_0 \cdot}))\zeta_2(0) = \frac{1}{2} F_{uu}(0)(\zeta_1, \zeta_1),$$

and $\hat{\zeta}_2$ is a constant n-vector satisfying,

(2.15c) $$-L(0)\hat{\zeta}_2 = F_{uu}(0)(\zeta_1, \overline{\zeta}_1).$$

Now we adopt new variables,

$$s = \omega t, \qquad y(s) = X(s/\omega) = X(t).$$

Then the equations (1.1) and (1.9) are transformed into,

(2.16) $$\omega \frac{d}{ds} y(s) = F(\mu, y_{s,\omega}),$$

(2.17) $$\omega\frac{d}{ds}u(s)=f(\mu, u(s)),$$

respectively, where

$$y_{s,\omega}(\theta)\underset{\text{def}}{=}y(s+\omega\theta), \qquad [u(s)](\theta)\underset{\text{def}}{=}[v(t)](\omega\theta), \qquad (-r\leqq\theta\leqq 0).$$

Define the spaces, $P_{2\pi}$ and $P_{2\pi}^{(1)}\subset P_{2\pi}$, as

(2.18a) $$P_{2\pi}=\{g\,|\,g(s+2\pi)=g(s),\ g\in C([0, 2\pi];\ C^n)\},$$

(2.18b) $$P_{2\pi}^{(1)}=\left\{g\,|\,g\in P_{2\pi},\ \frac{dg}{ds}\in P_{2\pi}\right\}.$$

$P_{2\pi}$ and $P_{2\pi}^{(1)}$ are Banach spaces with norms

$$\|g\|=\sup_{0\leqq s\leqq 2\pi}|g(s)| \qquad (g\in P_{2\pi})$$

and

$$|||g|||\underset{\text{def}}{=}\max\left\{\|g\|, \left\|\frac{dg}{ds}\right\|\right\} \qquad (g\in P_{2\pi}^{(1)})$$

respectively. In the following we seek a 2π-periodic solution to (2.16) in the space $P_{2\pi}^{(1)}$.

Let us define an operator, J, acting in $P_{2\pi}$ as,

(2.19) $$Jy(s)\underset{\text{def}}{=}\omega_0\frac{d}{ds}y(s)-L(0)y_{s,\omega_0} \qquad \text{for} \quad y\in\mathfrak{D}(J),$$

$$y_{s,\omega_0}(\theta)\underset{\text{def}}{=}y(s+\omega_0\theta), \qquad -r\leqq\theta\leqq 0,$$

where the domain, $\mathfrak{D}(J)$, of J is $P_{2\pi}^{(1)}$. From the definition it follows that the null-space, $\mathfrak{N}(J)$, of J is spanned by the elements,

(2.20) $$z\underset{\text{def}}{=}\zeta_0e^{is}, \qquad \bar{z}=\bar{\zeta}_0e^{-is},$$

where ζ_0 is as in (2.7a). Let us introduce the formal adjoint operator, J^*, of J,

(2.21) $$J^*y(s)\underset{\text{def}}{=}-\omega_0\frac{d}{ds}y(s)-L^*(0)y^{s,\omega_0} \qquad \text{for} \quad y\in P_{2\pi}^{(1)},$$

$$y^{s,\omega_0}(\tau)\underset{\text{def}}{=}y(s+\omega_0\tau), \qquad 0\leqq\tau\leqq r,$$

$$\left(L^*(0)y^{s,\omega_0}=\int_{-r}^{0}{}^t[d\eta(\theta)]\,y^{s,\omega_0}(-\theta)=\int_{-r}^{0}{}^t[d\eta(\theta)]\,y(s-\omega_0\theta)\right).$$

By (2.7b)

$$J^*z^*=J^*\overline{z^*}=0,$$

where

(2.22) $$z^*\underset{\text{def}}{=}\zeta_0^*e^{is} \qquad (\zeta_0^* \text{ is as in (2.7b)}).$$

Define the product in $P_{2\pi}$ as

$$(u, v)_{2\pi}=\frac{1}{2\pi}\int_0^{2\pi}(u(s), v(s))ds\,, \qquad u, v\in P_{2\pi}\,. \tag{2.23}$$

The linear operator, J, has the following property (Fredholm alternative) in the space $P_{2\pi}$:

LEMMA F (Hale [5; Chapter 9]). *Consider the equation,*

$$Ju=h\,, \qquad h\in P_{2\pi}\,. \tag{2.24}$$

The equation (2.24) *is solvable for* $u\in\mathfrak{D}(J)(=P_{2\pi}^{(1)})$ *if and only if*

$$(h, z^*)_{2\pi}=(h, \overline{z^*})_{2\pi}=0\,. \tag{2.25}$$

Furthermore, there is a continuous projection, P_N: $P_{2\pi}\to P_{2\pi}$, *such that* $\mathfrak{R}(J)=(I-P_N)P_{2\pi}$ *and there is a continuous linear operator* K: $(I-P_N)P_{2\pi}\to(I-\Pi)P_{2\pi}\cap\mathfrak{D}(J)$, *such that* Kh *is a solution of* (2.24) *for each* $h\in(I-P_N)P_{2\pi}$, *where* $\mathfrak{R}(J)$ *denotes the range of* J *and* Π *is a continuous projection of* $P_{2\pi}$ *onto* $\mathfrak{N}(J)$. *If, in the equation* (2.24), h *is real valued then the condition* (2.25) *is equivalent to* $(h, z^*)_{2\pi}=0$.

Now we seek a real valued 2π-periodic solution to (2.16) in the form,

$$\begin{cases} y(s;\varepsilon)=y_1(s)\varepsilon+y_2(s)\varepsilon^2+\hat{y}(s;\varepsilon)\varepsilon^2\,, & \hat{y}(s;0)\equiv 0\,, \\ \mu=\mu(\varepsilon)=\mu_2\varepsilon^2+\hat{\mu}(\varepsilon)\varepsilon^2\,, & \hat{\mu}(0)=0\,, \\ \omega=\omega(\varepsilon)=\omega_0+\omega_2\varepsilon^2+\hat{\omega}(\varepsilon)\varepsilon^2\,, & \hat{\omega}(0)=0\,, \end{cases} \tag{2.26}$$

under the condition

$$[y_{s,\omega_0}, z^{*s,\omega_0}]\underset{\text{def}}{=}\frac{1}{2\pi}\int_0^{2\pi}\langle y_{s,\omega_0}, z^{*s,\omega_0}\rangle ds=\varepsilon\,, \tag{2.27}$$

where $z^{*s,\omega_0}(\theta)=\zeta_0^*e^{i(s+\omega_0\theta)}=\zeta_1^*(\theta)e^{is}$, $0\leqq\theta\leqq r$. The condition (2.27) determines the solution uniquely for each $\varepsilon>0$.

Inserting (2.26) into (2.16) and (2.27) yields the following equations from which we obtain the coefficients $y_1(s)$, $y_2(s)$ and $\hat{y}(s;\varepsilon)$ in (2.26) iteratively:

$$Jy_1=0\,, \qquad [y_{1,s,\omega_0}, z^{*s,\omega_0}]=1\,, \tag{2.28}$$

$$Jy_2=\frac{1}{2}F_{uu}(0)(y_{1,s,\omega_0}, y_{1,s,\omega_0})\,, \qquad [y_{2,s,\omega_0}, z^{*s,\omega_0}]=0\,, \tag{2.29}$$

$$J\hat{y}=R(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y})\,, \qquad [\hat{y}_{s,\omega_0}, z^{*s,\omega_0}]=0 \quad (\hat{y}\in P_{2\pi}^{(1)})\,, \tag{2.30}$$

where the remaining term, $R(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y})$, is C^1 in $(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y})$ and $R(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y})\in P_{2\pi}$ for $\hat{y}\in P_{2\pi}^{(1)}$; it can be expressed in the form

$$(2.31)\quad R(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y})(s)=\frac{1}{\varepsilon}L(0)(y_{1,s,\omega(\varepsilon)}-y_{1,s,\omega_0})$$

$$+L(0)(y_{2,s,\omega(\varepsilon)}-y_{2,s,\omega_0})+L(0)(\hat{y}_{s,\omega(\varepsilon)}-\hat{y}_{s,\omega_0})$$

$$+\frac{1}{2}(F_{uu}(0)(y_{1,s,\omega(\varepsilon)}, y_{1,s,\omega(\varepsilon)})-F_{uu}(0)(y_{1,s,\omega_0}, y_{1,s,\omega_0}))$$

$$+\varepsilon\Big[(\mu_2+\hat{\mu})F_{\mu u}(0)y_{1,s,\omega(\varepsilon)}-(\omega_2+\hat{\omega})\frac{d}{ds}y_1(s)$$

$$-\varepsilon(\omega_2+\hat{\omega})\frac{d}{ds}(y_2(s)+\hat{y}(s))+F_{uu}(0)(y_{1,s,\omega(\varepsilon)}, y_{2,s,\omega(\varepsilon)}+\hat{y}_{s,\omega(\varepsilon)})$$

$$+\frac{1}{6}F_{uuu}(0)(y_{1,s,\omega(\varepsilon)}, y_{1,s,\omega(\varepsilon)}, y_{1,s,\omega(\varepsilon)})\Big]+O(\varepsilon^2)\,.$$

In what follows, $y_1(s)$ and $y_2(s)$ are given as smooth functions in s. Hence we see that $R(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y})$ is C^1 in $(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y})$.

Solving (2.28) and (2.29) in y_1, y_2 gives,

$$(2.32)\qquad y_1(s)=\zeta_0e^{is}+\bar{\zeta}_0e^{-is}\quad(=z+\bar{z})\,,$$

$$y_{1,s,\omega_0}=\zeta_1e^{is}+\bar{\zeta}_1e^{-is}\,,$$

$$(2.33)\qquad y_2(s)=\zeta_2(0)e^{2is}+\overline{\zeta_2(0)}e^{-2is}+\hat{\zeta}_2\,,$$

$$y_{2,s,\omega_0}=\zeta_2e^{2is}+\bar{\zeta}_2e^{-2is}+\hat{\zeta}_2\,,$$

where ζ_0, ζ_1 are as in (2.6), (2.7a) and ζ_2, $\hat{\zeta}_2$ are as defined in (2.14), (2.15).

It remains to solve the equation (2.30). As $\mathfrak{N}(J)\subset P_{2\pi}$ is spanned by the elements z and $\bar{z}$, $[a_{s,\omega_0}, z^{*s,\omega_0}]\neq 0$ for $a\in\mathfrak{N}(J)$ $(a\neq 0)$ by (2.9). Hence, $[\hat{y}_{s,\omega_0}, z^{*s,\omega_0}]=0$ implies $\hat{y}\in(I-\Pi)P_{2\pi}^{(1)}$, where Π is the projection in Lemma F. Using the operators, K and P_N, in Lemma F, we obtain the following equations which are equivalent to (2.30):

$$(2.34a)\qquad \hat{y}=K(I-P_N)R(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y})\qquad(\hat{y}\in P_{2\pi}^{(1)})\,,$$

$$(2.34b)\qquad P_NR(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y})=0\,.$$

By the Implicit Function Theorem, the equation, (2.34a), is solvable, that is, there exists a C^1-function $\hat{y}=\hat{y}(\varepsilon, \hat{\mu}, \hat{\omega})$ $(\in P_{2\pi}^{(1)})$ in $(\varepsilon, \hat{\mu}, \hat{\omega})$ satisfying (2.34a) and

$$\hat{y}(0,\cdot,\cdot)=\frac{\partial\hat{y}}{\partial\hat{\mu}}(0,\cdot,\cdot)=\frac{\partial\hat{y}}{\partial\hat{\omega}}(0,\cdot,\cdot)=0\,.$$

(Note that $R(0,\cdot,\cdot,\cdot)=0$ uniformly in $\hat{\mu}, \hat{\omega}, \hat{y}$.) Thus, (2.34a) and (2.34b) are reduced to the equation,

$$P_NR(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y}(\varepsilon, \hat{\mu}, \hat{\omega}))=0\,,$$

or equivalently, $(R(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y}(\varepsilon, \hat{\mu}, \hat{\omega})), z^*)_{2\pi}=0$. Put

$$H(\varepsilon, \hat{\mu}, \hat{\omega}) \equiv \frac{1}{\varepsilon} (R(\varepsilon, \hat{\mu}, \hat{\omega}, \hat{y}(\varepsilon, \hat{\mu}, \hat{\omega})), z^*)_{2\pi}=0\,. \tag{2.35}$$

We solve (2.35).

$$H(0, 0, 0)=0$$

holds if and only if,

$$\begin{aligned} 0=&-i\omega_2(\zeta_0, \zeta_0^*)+i\omega_2(L(0)(\cdot e^{i\omega_0 \cdot}\zeta_0), \zeta_0^*) \\ &+\mu_2(F_{\mu u}(0)\zeta_1, \zeta_0^*)+(F_{uu}(0)(\zeta_1, \hat{\zeta}_2), \zeta_0^*) \\ &+(F_{uu}(0)(\bar{\zeta}_1, \zeta_2), \zeta_0^*)+\frac{1}{2}(F_{uuu}(0)(\zeta_1, \zeta_1, \bar{\zeta}_1), \zeta_0^*)\,, \end{aligned}$$

which implies,

$$\begin{aligned} &-i\omega_2+\mu_2\frac{d\lambda}{d\mu}(0)+(F_{uu}(0)(\zeta_1, \hat{\zeta}_2), \zeta_0^*) \\ &\qquad +(F_{uu}(0)(\bar{\zeta}_1, \zeta_2), \zeta_0^*)+\frac{1}{2}(F_{uuu}(0)(\zeta_1, \zeta_1, \bar{\zeta}_1), \zeta_0^*)=0\,, \end{aligned} \tag{2.36}$$

(by (2.11) and (2.13)). The coefficients, μ_2 and ω_2, are determined by the relation (2.36). Differentiating (2.35) yields,

$$\frac{\partial H}{\partial \hat{\mu}}(0, 0, 0)=(F_{\mu u}(0)\zeta_1, \zeta_0^*)=\frac{d\lambda}{d\mu}(0) \qquad \text{(from (2.11))}\,,$$

$$\begin{aligned} \frac{\partial H}{\partial \hat{\omega}}(0, 0, 0)&=-i(\zeta_0, \zeta_0^*)+i(L(0)(\cdot e^{i\omega_0 \cdot}\zeta_0), \zeta_0^*) \\ &=-i \qquad \text{(from (2.13))}\,. \end{aligned}$$

This implies that one can apply the Implicit Function Theorem to obtain $\hat{\mu}$, $\hat{\omega}$, as functions of ε. Hence, $\hat{y}=\hat{y}(\varepsilon, \hat{\mu}(\varepsilon), \hat{\omega}(\varepsilon))$ as a function of ε. (Note (1.11).)

Thus, we get a periodic solution to (2.16) having the form (2.26).

Finally we note that by the definition of $f(\mu, u)$ in (1.9) the equation, (2.36), can be written as,

$$\begin{aligned} &-i\omega_2+\mu_2\frac{d\lambda}{d\mu}(0)+\langle f_{uu}(0)(\zeta_1, \hat{\zeta}_2), \zeta_1^*\rangle \\ &\qquad +\langle f_{uu}(0)(\bar{\zeta}_1, \zeta_2), \zeta_1^*\rangle+\frac{1}{2}\langle f_{uuu}(0)(\zeta_1, \zeta_1, \bar{\zeta}_1), \zeta_1^*\rangle=0\,, \end{aligned} \tag{2.37}$$

where ζ_1^* is as in (2.5) (and (2.6)). The above equation coincides with the one obtained by the formal calculation in Iooss and Joseph's work [9].

3. Destabilization of Spatially Homogeneous Periodic Solutions

Let us introduce some function spaces. We denote by $W^{2,p}(\Omega)$ the Sobolev space of all real valued $L^p(\Omega)$ functions whose derivatives up to order 2 belong to $L^p(\Omega)$, where Ω is a bounded domain in $\boldsymbol{R}^N$ with a smooth boundary $\partial\Omega$.

Given a Banach space, X, let $C([-r, 0]; X)$ be a Banach space of X-valued continuous function defined on the interval $[-r, 0]$. For simplicity, as in the preceding section, $C[-r, 0]$ stands for $C([-r, 0]; \boldsymbol{R}^n)$ (or $C([-r, 0]; \boldsymbol{C}^n)$). $(X)^n$ stands for the n-th product of X.

Let us put,

$$W_N^{2,p}(\Omega)=\{u\in W^{2,p}(\Omega);\ \partial u/\partial n=0 \text{ on } \partial\Omega\},$$

where $\partial/\partial n$ denotes the outer normal derivative to $\partial\Omega$. In what follows we shall understand that p is sufficiently large, for instance $p>N/2$ so that the correspondence,

$$(\mu, \Phi)\longmapsto F(\mu, \Phi),$$

defines a mapping,

$$F\colon I_0\times(W^{2,p}(\Omega))^n\to(W^{2,p}(\Omega))^n,$$

of C^4 class, where $F(\mu, \cdot)$ is as in (1.1) (satisfying (1.3), (A1), (A2)), and I_0 is an interval containing $0\in\boldsymbol{R}^1$.

Now we shall consider the following equation:

$$\text{(3.1)}\qquad \begin{cases} \dfrac{\partial}{\partial t}V(t, x)=D\Delta V(t, x)+F(\mu, V_t(\cdot, x)), & (t, x)\in(0, \infty)\times\Omega, \\ \dfrac{\partial V}{\partial n}(t, x)=0, & (t, x)\in(0, \infty)\times\partial\Omega, \\ V(\theta, x)=[\Phi_0(\theta)](x), & (\theta, x)\in[-r, 0]\times\Omega, \\ \qquad\Phi_0\in C([-r, 0]; (W_N^{2,p}(\Omega))^n), & \end{cases}$$

where

$$V_t(\theta, x)=V(t+\theta, x), \qquad -r\leqq\theta\leqq 0,$$

$$\Delta=\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2},$$

$$D=\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix},$$

$$d_i\geqq 0\ (i=1, \cdots, n), \qquad d_1+\cdots+d_n>0.$$

To avoid a lengthy argument on the well-posedness of (3.1), which is not the subject of the present paper, we assume that for any $\Phi_0\in C([-r, 0]; (W_N^{2,p}(\Omega))^n)$ there exists a unique solution $V(t, \cdot)\in C([-r, 0]; (W_N^{2,p}(\Omega))^n)$ to (3.1) such that

$(\partial/\partial t)V(t,\cdot)\in C([0,\infty);(L^p(\Omega))^n)$. See [12] for such existence theorems.

Let $\Delta_N(\Omega)$ be a closed operator in $L^p(\Omega)$, with dense domain $\mathfrak{D}(\Delta_N(\Omega))=W_N^{2,p}(\Omega)$, defined by $\Delta_N(\Omega)v=\Delta v$ for $v\in\mathfrak{D}(\Delta_N(\Omega))$. For simplicity Δ_N denotes $\Delta_N(\Omega)$ hereafter.

Put

$$[(U(t))](x)=V(t,x), \qquad (t,x)\in[-r,\infty)\times\Omega.$$

The equation (3.1) is written as,

$$\frac{d}{dt}U(t)=D\Delta_N U(t)+F(\mu, U_t(\cdot)), \qquad t>0, \tag{3.2}$$

$$U(t)=\Phi_0(t), \qquad -r\leqq t\leqq 0.$$

For any matrix, D, and any domain, Ω, it is clear from Theorem H that for each $\varepsilon\in(0,\varepsilon_H)$ the equation (3.2) has a spatially homogeneous periodic solution, $U(t)=p(t;\varepsilon)$, with period $2\pi/\omega(\varepsilon)$ occurring for $\mu=\mu(\varepsilon)$, where $\mu(\varepsilon)$, $\omega(\varepsilon)$, ε_H are as in Theorem H. Here we understand that $y(s;\varepsilon)=p(s/\omega(\varepsilon);\varepsilon)$ satisfies the condition (2.27) so that the coefficients in (1.12), (1.13) are uniquely determined. If, in addition, the conditions in Corollary H are satisfied, then $p(t;\varepsilon)$ is stable with respect to spatially homogeneous perturbation. Note that the stability in the above sense does not necessarily imply the stability required in this section. Since our problem in this section involves space variables, we have to consider the stability with respect to all possible perturbations (either spatially homogeneous or inhomogeneous).

As mentioned in the Introduction, Yoshida [7] has shown for some specific equation that the spatially homogeneous periodic solution $p(\cdot;\varepsilon)$ is stable in the right above sense near the bifurcation point. That is, the stability region for $p(\cdot;\varepsilon)$ (the set of all ε for which $p(\cdot;\varepsilon)$ is stable) is not empty for any diffusion coefficient and any domain Ω (in the case of $n=1$).

It is clear that the spatially homogeneous periodic solution, $p(\cdot;\varepsilon)$, to (3.1) is virtually independent of the matrix, D, and the domain Ω; hence, it is defined on some fixed ε-interval that does not depend on D and Ω. However, the stability region for $p(\cdot;\varepsilon)$ as mentioned above may vary according to D and Ω, even if it continues to be non-empty. This fact suggests the possibility of the occurrence of destabilization that might be observed when we vary D or Ω. More precisely, it will be shown that the stability region shrinks when the diffusion coefficients, d_i $(i=1,\cdots,n)$, become very small or when the shape of Ω becomes far from being convex; hence, the bifurcating periodic solution loses its stability very near the bifurcation point. We shall discuss this in the present and next sections.

Hereafter we shall always assume the conditions in Corollary H to be true. To see how the destabilization of the spatially homogeneous periodic solution occurs, let us consider the following linearized equation of (3.2) around the periodic solution, $p(t;\varepsilon)$:

$$\frac{d}{dt}z(t)=D\Delta_N z(t)+F_u(\mu(\varepsilon),\ p_t(\cdot\,;\varepsilon))z_t\,. \tag{3.3}$$

For any $\varepsilon\in(0,\varepsilon_0)$, (3.3) is a periodic system with period $T(\varepsilon)=2\pi/\omega(\varepsilon)$, where ε_0 is given in Corollary H.

We assume that $T=T(\varepsilon)>r$ for $0<\varepsilon<\varepsilon_0$. Then a Floquet exponent of (3.3) is defined as follows:

For each initial function $\Phi_0\in C([-r,0];\ (W_N^{2,p})^n)$, let $z(t;\Phi_0)$ $(t\geqq -r)$ be a solution satisfying $z(t;\Phi_0)=\Phi_0(t)$, $-r\leqq t\leqq 0$. The correspondence,

$$\Phi_0\longmapsto z_T(\cdot\,;\Phi_0)\qquad (z_T(\theta;\Phi_0)=z(T+\theta;\Phi_0),\ -r\leqq\theta\leqq 0)\,,$$

defines a completely continuous linear operator from $C([-r,0];\ (W_N^{2,p})^n)$ into itself (see [12]). We call a complex number, γ, a Floquet exponent of (3.3) if $\nu=e^{\gamma T}$ is an eigenvalue of the above operator.

Using arguments analogous to those found in [5; Chapter 8], we see that determining the Floquet exponent, γ, of (3.3) is reduced to seeking a solution to (3.3) in the form,

$$z(t)=e^{\gamma t}\chi(t)\,,\qquad \chi(t+T)=\chi(t)\qquad (\chi(t)\not\equiv 0)\,, \tag{3.4}$$
$$\chi(t)\in(W_N^{2,p})^n\qquad \text{for each}\quad t\geqq -r\,.$$

One can easily verify that if (3.3) has a Floquet exponent with strictly positive real part, then the periodic solution is unstable.

Now we adopt the new variables,

$$s=\omega(\varepsilon)t\,,\qquad w(s)=z(s/\omega(\varepsilon))=z(t)\,.$$

Then (3.3) is transformed into,

$$\omega(\varepsilon)\frac{d}{ds}w(s)=D\Delta_N w(s)+F_u(\mu(\varepsilon),\ y_{s,\omega(\varepsilon)}(\cdot\,;\varepsilon))w_{s,\omega(\varepsilon)}\,, \tag{3.5}$$

where

$$y(s;\varepsilon)=p(s/\omega(\varepsilon);\varepsilon)\,,$$
$$w_{s,\omega(\varepsilon)}(\theta)\underset{\text{def}}{=}w(s+\omega(\varepsilon)\theta)\qquad(-r\leqq\theta\leqq 0)\,.$$

Let λ_m be the m-th eigenvalue of the operator $-\Delta_N$, and ψ_m be the eigenfunction corresponding to λ_m, i.e.,

$$\Delta_N\psi_m=-\lambda_m\psi_m\,,\qquad m=1,2,3,\cdots\,,\qquad (0=\lambda_1<\lambda_2\leqq\lambda_3\cdots)\,. \tag{3.6}$$

Considering that $W_N^{2,p}(\Omega)$ is spanned by $\{\psi_m\}_{m=1,2,3,\cdots}$, we see that γ is a Floquet exponent of the linear 2π-periodic system (3.5) if and only if there exists a function, $q(s)\not\equiv 0$, in $P_{2\pi}$ and a positive integer, m, such that,

$$[w(s)](x)=e^{\gamma s/\omega(\varepsilon)}q(s)\psi_m(x) \tag{3.7}$$

satisfies the equation (3.5), where $P_{2\pi}$ is defined by (2.18). Thus, calculating the Floquet exponents is reduced to finding solutions, $y(s)$, to (3.5) in the form (3.7).

Substituting (3.7) into (3.5), and comparing the coefficients of ψ_m on the both sides of (3.5), we obtain,

$$\omega(\varepsilon)\frac{d}{ds}q(s) = -(\gamma+\lambda_m D)q(s) + F_u(\mu(\varepsilon),\ y_{s,\omega(\varepsilon)}(\cdot\,;\,\varepsilon))(q_{s,\omega(\varepsilon)}(\cdot)e^{\gamma\cdot}), \qquad q\in P_{2\pi}. \tag{3.8}$$

The equation (3.8) is independent of the spatial variable x.

When $m=1$, the equation (3.8) coincides with the one induced from the linearized equation of the diffusion-free equation (2.16), that is,

$$\omega(\varepsilon)\frac{d}{ds}q(s) = -\gamma q(s) + F_u(\mu(\varepsilon),\ y_{s,\omega(\varepsilon)}(\cdot\,;\,\varepsilon))(q_{s,\omega(\varepsilon)}(\cdot)e^{\gamma\cdot}). \tag{3.9}$$

The equation (3.9) has Floquet exponents 0 and $\beta(\varepsilon)<0$ for $\varepsilon\in(0,\varepsilon_0)$, where $\beta(\varepsilon)$ is defined by (1.13). Moreover, we see from the stability assumption that all the remaining Floquet exponents have strictly negative real parts.

Next we consider the case $m\neq 1$ in (3.8). Take any positive integer m, greater than 1 and fix it. Let

$$E=\lambda_m D.$$

Then the equation (3.8) is written as,

$$\omega(\varepsilon)\frac{d}{ds}q(s) = -(\gamma+E)q(s) + F_u(\mu(\varepsilon),\ y_{s,\omega(\varepsilon)}(\cdot\,;\,\varepsilon))(q_{s,\omega(\varepsilon)}(\cdot)e^{\gamma\cdot}), \qquad q\in P_{2\pi}. \tag{3.10}$$

We can regard the equation (3.10) as a perturbed equation of (3.9) with perturbation term, $-Eq$.

In the rest of this section we shall show that if E is a sufficiently small perturbation matrix then we can find a pair of Floquet exponents of (3.10) (explicitly in terms of ε) corresponding to the Floquet exponents 0, $\beta(\varepsilon)$ of the unperturbed equation (3.9). The results will be used in the next section for the stability analysis.

Let us parametrize the deformation of a domain and diffusion coefficients in terms of the same parameter, ε, as above. Then D and λ_m (and hence $E\equiv\lambda_m D$) become functions of ε. In what follows we assume the parametrization is taken in such a way that,

$$E\equiv\varepsilon^2 E_2 \tag{3.11}$$

for some matrix E_2. (ε will not be a bifurcation parameter.) Actually, if E_2 is sufficiently small in a certain sense (to be clarified later), then the perturbed Floquet Exponents, γ, corresponding to $\beta(\varepsilon)$ and 0 are real numbers of order $O(\varepsilon^2)$.

Thus, in the equation (3.10), we are to find a pair of a real valued function, $q\in P_{2\pi}$, and a real number, γ, in the form,

$$(3.12)\quad \begin{cases} \gamma=\gamma_2\varepsilon^2+\hat{\gamma}(\varepsilon)\varepsilon^2\,, & \hat{\gamma}(0)=0\,,\\ q(s;\,\varepsilon)=q_0(s;\,\varepsilon)+q_1(s;\,\varepsilon)\varepsilon+\hat{q}(s;\,\varepsilon)\varepsilon\,,\\ \hat{q},\ q_i\in P^{(1)}_{2\pi}\ (i=0,\,1)\,,\quad q_0(s;\,0)\not\equiv 0,\quad q_1(s;\,0)\not\equiv 0,\quad \hat{q}(s;\,0)\equiv 0\,, \end{cases}$$

$$(3.13)\qquad [q_{s,\omega_0},\, z^{*s,\omega_0}]=c+\eta(\varepsilon)\,,\qquad \eta(0)=0\,,$$

under the condition $E\equiv E_2\varepsilon^2$, where $[\cdot,\,\cdot]$ and z^* are defined in (2.27) and (2.22). We substitute (3.11) and (3.12) into (3.10) and (3.13). Then we get the following equations that determine q_0, q_1, $\hat{q}$:

$$(3.14)\qquad Jq_0=0\,,\qquad [q_{0,s,\omega_0},\, z^{*s,\omega_0}]=c+\eta\,,$$

$$(3.15)\qquad Jq_1=F_{uu}(0)(y_{1,s,\omega_0},\, q_{0,s,\omega_0})\,,\qquad [q_{1,s,\omega_0},\, z^{*s,\omega_0}]=0\,,$$

$$(3.16a)\qquad J\hat{q}=r(\varepsilon,\,\hat{\gamma},\,\eta,\,\hat{q})\,,\qquad [\hat{q}_{s,\omega_0},\, z^{*s,\omega_0}]=0\quad (\hat{q}\in P^{(1)}_{2\pi})\,,$$

$$(3.16b)\qquad r(\varepsilon,\,\hat{\gamma},\,\eta,\,\hat{q})=r_1(\varepsilon,\,\hat{\gamma},\,\eta,\,\hat{q})+\varepsilon r_2(\varepsilon,\,\hat{\gamma},\,\eta,\,\hat{q})\qquad (r(0,\,\cdot,\,\cdot,\,\cdot)=0)\,,$$

$$\begin{aligned} r_1(\varepsilon,\,\hat{\gamma},\,\eta,\,\hat{q})(s)=L(0)(\hat{q}_{s,\omega(\varepsilon)}-\hat{q}_{s,\omega_0})+\varepsilon\Big[&-\omega^2\frac{d}{ds}q_0(s)\\ &+\omega_2L(0)\Big(\cdot\frac{d}{ds}q_{0,s,\omega_0}(\cdot)\Big)-(\gamma_2+\hat{\gamma}+E_2)q_0(s)+\mu_2F_{\mu u}(0)q_{0,s,\omega_0}\\ &+(\gamma_2+\hat{\gamma})L(0)(\cdot q_{0,s,\omega_0}(\cdot))+F_{uu}(0)(y_{1,s,\omega_0},\, q_{1,s,\omega_0}+\hat{q}_{s,\omega(\varepsilon)})\\ &+F_{uu}(0)(y_{2,s,\omega_0},\, q_{0,s,\omega_0})+\frac{1}{2}F_{uuu}(0)(y_{1,s,\omega_0},\, y_{1,s,\omega_0},\, q_{0,s,\omega_0})\Big]\,, \end{aligned}$$

$$r_2(0,\,\cdot,\,\cdot,\,\cdot)(s)\equiv 0\,,$$

where J is defined by (2.19) and y_1, y_2 are as in (2.32), (2.33).

Considering that q_0 and q_1 are real valued, we obtain the solutions to (3.14) and (3.15) in the form,

$$(3.17)\qquad q_0=(c+\eta)z+(\bar{c}+\bar{\eta})\bar{z}\,,\qquad q_{0,s,\omega_0}=(c+\eta)\zeta_1e^{is}+(\bar{c}+\bar{\eta})\bar{\zeta}_1e^{-is}\,,$$

$$\begin{aligned} (3.18)\qquad q_1(s)&=2(c+\eta)\zeta_2(0)e^{2is}+2(\bar{c}+\bar{\eta})\overline{\zeta_2(0)}e^{-2is}+(c+\bar{c}+\eta+\bar{\eta})\hat{\zeta}_2\,,\\ q_{1,s,\omega_0}&=2(c+\eta)\zeta_2e^{2is}+2(\bar{c}+\bar{\eta})\bar{\zeta}_2e^{-2is}+(c+\bar{c}+\eta+\bar{\eta})\hat{\zeta}_2\,, \end{aligned}$$

respectively, where ζ_2 and $\hat{\zeta}_2$ are as defined in (2.14). (Compare (3.14) and (3.15) with (2.28) and (2.29).)

We solve the equation (3.16). Using operators K and P_N, defined in Lemma F in §2, we get the following equations which are equivalent to (3.16):

$$(3.19a)\qquad \hat{q}=K(I-P_N)r(\varepsilon,\,\hat{\gamma},\,\eta,\,\hat{q})\qquad (\hat{q}\in P^{(1)}_{2\pi})\,,$$

$$(3.19b)\qquad P_Nr(\varepsilon,\,\hat{\gamma},\,\eta,\,\hat{q})=0\,.$$

(3.19a) is solved by the Implicit Function Theorem; we get a C^1-function $\hat{q}=\hat{q}(\varepsilon, \hat{\gamma}, \eta)(\in P_{2\pi}^{(1)})$ of $(\varepsilon, \hat{\gamma}, \eta)$ satisfying (3.19a) and

$$\hat{q}(0, \cdot, \cdot)=\frac{\partial \hat{q}}{\partial \hat{\gamma}}(0, \cdot, \cdot)=\frac{\partial \hat{q}}{\partial \eta}(0, \cdot, \cdot)=0 .$$

Thus, the equations (3.19a) and (3.19b) are reduced to,

$$P_N r(\varepsilon, \hat{\gamma}, \eta, \hat{q}(\varepsilon, \hat{\gamma}, \eta))=0 ,$$

which is equivalent to $(r(\varepsilon, \hat{\gamma}, \eta, \hat{q}(\varepsilon, \hat{\gamma}, \eta)), z^*)_{2\pi}=0$. Let

$$W(\varepsilon, \hat{\gamma}, \eta)\underset{\text{def}}{\equiv}\frac{1}{\varepsilon}(r(\varepsilon, \hat{\gamma}, \eta, \hat{q}(\varepsilon, \hat{\gamma}, \eta), z^*)_{2\pi}=0 . \tag{3.20}$$

We solve the equation (3.20) around $(\varepsilon, \hat{\gamma}, \eta)=(0, 0, 0)$.

After a slightly lengthy but simple calculation similar to that employed in getting (2.36), we obtain,

$$\begin{aligned} W(0, 0, 0)=&\left(-\omega_2 i-\gamma_2-(E_2\zeta_0, \zeta_0^*)+\mu_2\frac{d\lambda}{d\mu}(0)\right)c \\ &+(2c+\bar{c})(F_{uu}(0)(\zeta_1, \hat{\zeta}_2), \zeta_0^*)+(2c+\bar{c})(F_{uu}(0)(\bar{\zeta}_1, \zeta_2), \zeta_0^*) \\ &+\left(c+\frac{\bar{c}}{2}\right)(F_{uuu}(0)(\zeta_1, \zeta_1, \bar{\zeta}_1), \zeta_0^*), \qquad \text{(by (2.11), (2.13))}. \end{aligned}$$

Hence $W(0, 0, 0)=0$ implies

$$-(\gamma_2+(E_2\zeta_0, \zeta_0^*)-B_1)c+B_1\bar{c}=0 \qquad (c\neq 0) , \tag{3.21}$$

$$\begin{aligned} B_1 \underset{\text{def}}{=} &(F_{uu}(0)(\zeta_1, \hat{\zeta}_2), \zeta_0^*)+(F_{uu}(0)(\bar{\zeta}_1, \zeta_2), \zeta_0^*) \\ &+\frac{1}{2}(F_{uuu}(0)(\zeta_1, \zeta_1, \bar{\zeta}_1), \zeta_0^*) \\ =&\,\omega_2 i-\mu_2\frac{d\lambda}{d\mu}(0) \qquad \text{(see (2.36))}. \end{aligned} \tag{3.22}$$

(3.21) holds for some $c\neq 0$ if and only if

$$\gamma_2^2+2\mathrm{Re}((E_2\zeta_0, \zeta_0^*)-B_1)\gamma_2+|(E_2\zeta_0, \zeta_0^*)|^2-2\mathrm{Re}(\bar{B}_1(E_2\zeta_0, \zeta_0^*))=0 . \tag{3.23}$$

(3.23) determines the coefficient γ_2. If (3.23) has distinct roots, then (3.20) can be solved for c and γ_2 satisfying (3.21). The detail is left to the Appendix.

Combining the observations above, we obtain

LEMMA A. *Consider the linearized equation* (3.3) *of* (3.2) (*or* (3.1)) *around the spatially homogeneous periodic solution*, $p(\cdot; \varepsilon)$, *constructed in* §2. *If the matrix*, $E=\lambda_m D$, *is scaled such as* $E=E_2\varepsilon^2$ *and if* (3.23) *has real distinct roots, then there exists a positive constant*, ε_p, *such that for each* $\varepsilon\in(0, \varepsilon_p)$ *the linear periodic system*, (3.3), *has a*

Floquet exponent, $\gamma=\gamma(\varepsilon)$, *having the form in* (3.12), *where* λ_m *is defined by* (3.6) *and* γ_2 *is one of the roots of* (3.23).

4. Main Theorems

Let us consider the equation (3.1) (or (3.2)) and discuss the stability of the spatially homogeneous periodic solution by using Lemma A. Recall that the matrix, E, in (3.10) has the form,

$$E=\lambda_m D\,, \tag{4.1}$$

where λ_m ($m>1$) is the m-th eigenvalue of the operator $-\Delta_N(\Omega)$. The matrix E_2 is defined as,

$$E=E_2\varepsilon^2\,. \tag{4.2}$$

We assume the hypotheses in Corollary H on the diffusion-free equation (1.1) (or (1.9)). It therefore holds that the spatially homogeneous periodic solution, $p(\cdot;\varepsilon)$, to (3.1) is stable with respect to spatially homogeneous perturbation in an ε-interval independent of the matrix, D, and the domain, Ω.

First we consider the case where the diffusion coefficients vary while the domain is fixed. Let ε_p be as in Lemma A and let $\varepsilon\in(0, \varepsilon_p)$ be fixed. Let

$$D=D_2\varepsilon^2\,. \tag{4.3}$$

In order to study the destabilization caused by change of diffusion coefficients, we shall let D_2 vary. The matrix, E_2, in (4.2) is, in this case, expressed as,

$$E_2=\lambda_m D_2\,. \tag{4.4}$$

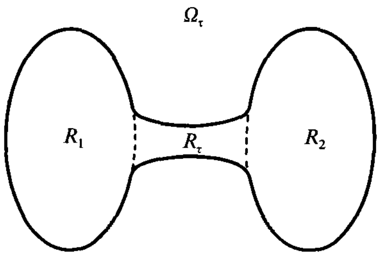
By Lemma A, the existence of a positive root γ_2 of the equation (3.23) implies the instability of the spatially homogeneous periodic solution to (3.1) (or (3.2)) for $\mu=\mu(\varepsilon)$ and $D=D_2\varepsilon^2$ with ε sufficiently small. Thus we obtain the following theorem:

THEOREM B. *Consider the equation* (3.1) (*or* (3.2)) *under the assumptions in Corollary H. Let* D_2 *be an* $n\times n$ *matrix and let* E_2 *be defined by* (4.4). *If* (3.23) *has a positive root,* γ_2, *then there exists a constant,* $\tilde{\varepsilon}>0$, *such that the spatially homogeneous periodic solution,* $p(\cdot;\varepsilon)$, *is unstable* (*with respect to some spatially inhomogeneous perturbation*) *for each* $\varepsilon\in(0,\tilde{\varepsilon})$ *and* $D=D_2\varepsilon^2$, $\mu=\mu(\varepsilon)$, *where* $\mu(\varepsilon)$, $p(\cdot;\varepsilon)$ *are as in Theorem H.*

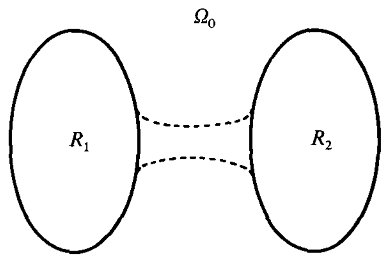
Next, let the matrix, D, be fixed and let us vary the domain, Ω; in this case, the eigenvalues, λ_m ($m=2, 3, \cdots$), of the operator, $\Delta_N(\Omega)$, vary accordingly. It suffices to consider the case where the space dimension is $N\geqq 2$, since, in the case $N=1$, varying the domain, Ω, can be reduced to varying the diffusion coefficients after a suitable change of the space variable.

Let us consider a one-parameter family of bounded domains, Ω_τ ($\tau>0$), with smooth boundaries, such that $\lambda_2(\Omega_\tau)\to 0$ as $\tau\to 0$.

Examples of such a family of domains are well-known. For instance, let $\Omega_\tau = R_1 \cup R_2 \cup R_\tau$ $(\tau>0)\subset \boldsymbol{R}^2$ be dumbell-shaped (Fig. A) and suppose Ω_τ approaches a set Ω_0 (Fig. B) consisting of two connected components R_1 and R_2 as τ tends to 0. In this example, one easily sees that $\lambda_2(\Omega_\tau)\to 0$ as $\tau\to 0$. (Hale and Vegas [17] has further shown that, under suitable assumptions, $\lambda_3(\Omega_\tau)$ is bounded away from zero by a positive constant as $\tau\to 0$.)



(Fig. A)



(Fig. B)

In what follows we shall simply write as $\lambda_2(\tau)$ instead of $\lambda_2(\Omega_\tau)$, where Ω_τ is a family of domains such that $\lambda_2(\tau)\to 0$ as $\tau\to 0$.

Let $\tilde{\lambda}$ be a positive number and let D be fixed. We now give the matrix, E_2, in (4.2) by,

$$E_2 = \tilde{\lambda} D\,. \tag{4.5}$$

Then, by virtue of Lemma A, we have the following theorem:

THEOREM C. *Consider the equation* (3.1) (*or* (3.2)) *under the assumptions in Corollary H. Suppose that the space dimension is* $N \geqq 2$, *and let the family of domains* Ω_τ $(\tau>0)$ *be as above. Let the matrix* E_2 *be defined by* (4.5), *and assume that the equation* (3.23) *has a positive root* γ_2. *Then there exists an* $\bar{\varepsilon}>0$ *such that for each* $\varepsilon\in(0,\bar{\varepsilon})$, $\mu=\mu(\varepsilon)$ *and sufficiently small* $\tau>0$ (*i.e.*, $\lambda(\tau)<\tilde{\lambda}\varepsilon^2$) *the spatially homogeneous periodic solution*, $p(\cdot\,;\varepsilon)$, *is unstable* (*with respect to some spatially inhomogeneous perturbation*).

REMARK 2. As mentioned in §3, it is often the case that the bifurcating solution, $p(\cdot\,;\varepsilon)$, is stable (with respect to either spatially homogeneous or inhomogeneous perturbation) at least near the bifurcation point. However, even in such a case, we see from Theorems B and C that if we fix ε and change D and Ω appropriately then the spatially homogeneous periodic solution, $p(\cdot\,;\varepsilon)$, may eventually lose its stability. This shows that the stability region of μ (for which $p(\cdot\,;\varepsilon)$ is stable) may become smaller and smaller when D and Ω are changed in a certain manner.

In the next section, we shall give an example in which the condition in Theorems B and C (that is, the positivity of γ_2) is satisfied. One easily sees that this condition is satisfied in various equations.

5. Applications

In this section we shall apply the results obtained in the preceding section to some specific equations.

Example A.

Let us consider the following scalar-valued equation:

$$(5.1)\quad \begin{cases} \dfrac{\partial v(t,x)}{\partial t} = d\Delta v(t,x) - \left(\dfrac{\pi}{2}+\mu\right)(1+v(t,x))v(t-1,x)\,, & (t,x)\in(0,\infty)\times\Omega\,,\\ \dfrac{\partial v}{\partial n}=0\,, & (t,x)\in(0,\infty)\times\partial\Omega\,,\\ v(\theta,x)=[\Phi_0(\theta)](x)\,, & (\theta,x)\in[-1,0]\times\Omega\,,\\ \qquad \Phi_0\in C([-1,0]\,;\,W_N^{2,p}(\Omega))\,, & \end{cases}$$

where Ω denotes a bounded domain in $\boldsymbol{R}^N$ with a smooth boundary $\partial\Omega$. As mentioned in the Introduction, for any fixed positive diffusion constant, d, and any domain, Ω, a spatially homogeneous periodic solution, $p(\cdot\,;\varepsilon)$, bifurcates at $\mu=0$ from the steady state, $v(t,x)\equiv 0$; and Yoshida has shown in [7] that the bifurcating solution is stable near the bifurcation point. By applying Theorems B and C, however, we shall see that the destabilization of the periodic solution occurs in the sense of Remark 2.

To check that Theorems B and C apply to the present case, let us calculate the roots, γ_2, of the equation (3.23). The operators, $A(\mu)$ and $G(\mu,\cdot)$, in (1.8) and (1.1) are, in the present case, written as

$$(5.2)\quad [A(\mu)\phi](\theta)=\begin{cases} \dfrac{d}{d\theta}\phi(\theta) & -1\leqq\theta<0\\ -\left(\dfrac{\pi}{2}+\mu\right)\phi(-1)\,, & \theta=0 \end{cases} \qquad \text{for}\quad \phi\in\mathfrak{D}(A(\mu))\subset C[-1,0]\,,$$

$$G(\mu,\phi)=-\left(\frac{\pi}{2}+\mu\right)\phi(0)\phi(-1)\,, \qquad \text{for}\quad \phi\in C[-1,0]\,.$$

It is easily seen that $A(0)$ has a pair of conjugate eigenvalues, $\pm(\pi/2)i$, and that the eigenfunction, ζ_1, corresponding to eigenvalue $(\pi/2)i$ is given by,

$$(5.3)\qquad \zeta_1(\theta)=e^{i\pi\theta/2}\,,\qquad -1\leqq\theta\leqq 0\ (\zeta_0=\zeta_1(0)=1)\,.$$

The formal product, (2.1), and the adjoint operator, $A^*(0)$, of $A(0)$ with respect to this product are expressed as,

$$\langle\phi, \psi\rangle=\overline{\psi(0)}\phi(0)-\frac{\pi}{2}\int_{-1}^{0}\overline{\psi(\xi+1)}\phi(\xi)d\xi \tag{5.4}$$

$$\text{for} \quad \phi\in C[-1,0], \qquad \psi\in C[0,1],$$

$$A^*(0)\psi=\begin{cases}-\dfrac{d\psi}{d\theta}(\theta), & 0<\theta\leqq 1\\ -\dfrac{\pi}{2}\psi(1), & \theta=0\end{cases} \quad \text{for} \quad \psi\in\mathfrak{D}(A^*(0)) \ (\subset C[0,1]), \tag{5.5}$$

respectively. Due to the normalizing condition, (2.9), the eigenfunction, ζ_1^*, of $A^*(0)$ corresponding to eigenvalue, $-(\pi/2)i$, takes the form,

$$\zeta_1^*(\theta)=\kappa e^{i\pi\theta/2}, \qquad 0\leqq\theta\leqq 1 \ (\zeta_0^*=\zeta_1^*(0)=\kappa), \tag{5.6}$$

$$\kappa=\frac{1}{1-\dfrac{\pi}{2}i}.$$

Moreover, ζ_2 and $\hat{\zeta}_2$ defined in (2.14) are easily calculated as,

$$\begin{cases}\zeta_2(\theta)=\dfrac{1}{5}(2-i)e^{i\pi\theta}, \\ \hat{\zeta}_2(\theta)=0,\end{cases} \qquad -1\leqq\theta\leqq 0. \tag{5.7}$$

We scale $\lambda_2 d$ in terms of ε such as,

$$\lambda_2 d=d_2\varepsilon^2. \tag{5.8}$$

In this case the matrix, E_2, in (3.11) is the 1×1 matrix (d_2). From (5.3), (5.6), (5.7) and (5.8) it follows that,

$$(E_2\zeta_0, \zeta_0^*)=d_2\bar{\kappa}, \qquad B_1=\frac{\pi}{10}(1-3i)\bar{\kappa}.$$

Thus the equation (3.23) is written as,

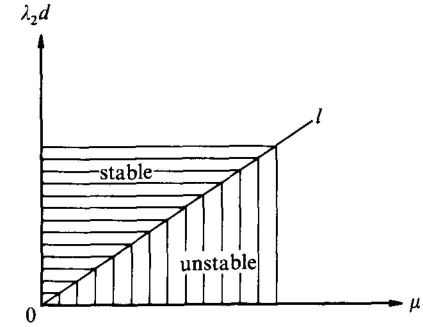
$$\left(1+\left(\frac{\pi}{2}\right)^2\right)\gamma_2^2+2\left(d_2-\frac{\pi}{10}+\frac{3\pi^2}{20}\right)\gamma_2+d_2^2-\frac{\pi}{5}d_2=0. \tag{5.9}$$

The equation (5.9) has a positive root, γ_2, for $0<d_2<\pi/5$, which ensures the occurrence of destabilization of the spatially homogeneous periodic solution in the sense of Remark 2.

In Fig. C we illustrate the stability and instability regions in the $(\mu, \lambda_2 d)$-parameter space. (Note that this figure is valid only in a sufficiently small neighborhood of $(\mu, \lambda_2 d)=(0, 0)$.) Since $\mu=\mu(\varepsilon)$ is expanded as,

$$\mu=\mu(\varepsilon)=\frac{3\pi-2}{10}\varepsilon^2+O(\varepsilon^3),$$

the curve, l, in Fig. C has slope $2\pi/(3\pi-2)$ at the origin.



(Fig. C)

REMARK 3. Let d and λ_2 (resp. μ) be fixed. When the parameter μ (resp. $\lambda_2 d$) passes across the curve, l, a secondary bifurcation will occur from the spatially homogeneous periodic solution. In this paper, however, we shall not discuss this problem.

Example B.

Our method can be applied to reaction-diffusion system without time delay. Consider the equation,

$$(5.10)\quad \begin{cases} \dfrac{\partial}{\partial t} U(t, x) = D\Delta U(t, x) + f(\mu, U(t, x)) & (t, x) \in (0, \infty) \times \Omega, \\ \dfrac{\partial U}{\partial n} = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \\ U(0, x) = \Phi_0(x), \quad x \in \Omega, \quad \Phi_0 \in (W_N^{2,p}(\Omega))^n, \end{cases}$$

where

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}, \quad d_i \geqq 0 \ (i=1, \cdots, n), \quad d_1 + \cdots + d_n > 0,$$

$$U = {}^t(u_1, \cdots, u_n).$$

$$f: I_0 \times \boldsymbol{R}^n \to \boldsymbol{R}^n$$

is of class C^4 and $f(\mu, 0) = 0$ for $\mu \in I_0$, (I_0 is an interval containing $0 \in \boldsymbol{R}^1$). We assume that $(\partial f/\partial U)(0, 0)$ has simple eigenvalues $\pm i\omega_0$ and a spatially homogeneous periodic solution bifurcates from the zero solution at $\mu = 0$. Moreover, assume that this bifurcating periodic solution is stable with respect to homogeneous perturbation. By using similar arguments, we can obtain destabilization results analogous to that in Example A. However, as it is easily seen, such destabilization does not occur if $d_1 = d_2 = \cdots = d_n$; so we must exclude this case. More detailed discussion together with interesting examples will be given in the forthcoming paper [16]; in this case, the equation corresponding to (3.23) is obtained, provided that we replace ζ_0 and ζ_0^* by the eigenvectors of $(\partial/\partial U)f(0, 0)$ and ${}^t((\partial/\partial U)f(0, 0))$ respectively and determine B_1 by

$$B_1 = i\omega_2 - \mu_2 \frac{\partial\lambda}{d\mu}(0),$$

where ω_2 and μ_2 are coefficients of ε^2 in $\omega(\varepsilon)$ and $\mu(\varepsilon)$.

Example C.

We shall consider the following time-lag equation with an integral kernel:

$$\text{(5.11)} \qquad \frac{d}{dt} v(t) = a\left(1 - \int_{-\infty}^{t} k(t-s)v(s)ds\right)v(t) \qquad (a>0).$$

For simplicity, we consider the case,

$$k(t) = \nu(e^{-bt} - e^{-ct}), \qquad \nu = \frac{bc}{c-b},$$

where b and c are positive constants. Note that $k(t)$ satisfies $\int_0^\infty k(t)dt = 1$. The equation (5.11) is transformed into,

$$\text{(5.12)} \qquad \begin{cases} \dfrac{d}{dt} u_1(t) = -a(1+u_1(t))u_3(t), \\ \dfrac{d}{dt} u_2(t) = c(u_1(t) - u_2(t)), \\ \dfrac{d}{dt} u_3(t) = b(u_2(t) - u_3(t)), \end{cases}$$

by the following transformation:

$$\text{(5.13)} \qquad \begin{cases} u_1(t) = v(t) - 1, \\ u_2(t) = c\displaystyle\int_{-\infty}^{t} e^{-c(t-s)}v(s)ds - 1, \\ u_3(t) = \displaystyle\int_{-\infty}^{t} k(t-s)v(s)ds - 1. \end{cases}$$

Let a be a bifurcation parameter. It is easily verified that the Hopf bifurcation occurs at $a = b + c$ and that the bifurcating periodic solution is stable.

Now we couple the equation (5.11) with a diffusion term and study this time-lag diffusion equation. Again by the transformation (5.13), this time-lag diffusion equation is converted into a diffusion system without time lag. In this case, the matrix, D, that determines the diffusion coefficients is written as,

$$D=\begin{pmatrix} d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad d>0\,.$$

As described in Example B, our arguments can apply to this equation (without time lag). After a little tedious calculation, one can check that, in the present case, the destabilization of the spatially homogeneous periodic solution occurs as in Example A.

Appendix

We shall solve the equation (3.20) for γ_2 and c satisfying (3.21), under the assumption that (3.23) has distinct roots. It is easily checked that (3.20) is reduced to the equation,

$$(\beta 1) \qquad W(\varepsilon, \hat{\gamma}, \eta) = -(\gamma_2 + (E_2\zeta_0, \zeta_0^*) - B_1)\eta + B_1\bar{\eta} - \hat{\gamma}(c+\eta) + O(\varepsilon) = 0\,.$$

(Use (3.17) and (3.18).) We define a vector function,

$$(\beta 2) \qquad \tilde{W}(\varepsilon, \hat{\gamma}, \eta_1, \eta_2) = {}^t(\mathrm{Re}\, W(\varepsilon, \hat{\gamma}, \eta_1 + i\eta_2), \mathrm{Im}\, W(\varepsilon, \hat{\gamma}, \eta_1 + i\eta_2))$$

where $\eta_1 = \mathrm{Re}\,\eta$, $\eta_2 = \mathrm{Im}\,\eta$. It is clear that,

$$\tilde{W}(0, 0, 0, 0) = 0\,,$$

and

$$(\beta 3) \qquad \frac{\partial \tilde{W}}{\partial \hat{\gamma}}(0, 0, 0, 0) = {}^t(-\mathrm{Re}\, c, -\mathrm{Im}\, c)\,,$$

$$(\beta 4) \qquad \frac{\partial \tilde{W}}{\partial \eta_1}(0, 0, 0, 0) = {}^t(-\gamma_2 - \mathrm{Re}\,(E_2\zeta_0, \zeta_0^*) + 2\,\mathrm{Re}\, B_1, -\mathrm{Im}\,(E_2\zeta_0, \zeta_0^*) + 2\,\mathrm{Im}\, B_1)\,,$$

$$(\beta 5) \qquad \frac{\partial \tilde{W}}{\partial \eta_2}(0, 0, 0, 0) = {}^t(\mathrm{Im}\,(E_2\zeta_0, \zeta_0^*), -\gamma_2 - \mathrm{Re}\,(E_2\zeta_0, \zeta_0^*))\,.$$

First we suppose that $\mathrm{Im}\,(E_2\zeta_0, \zeta_0^*) = 0$. Then the equation (3.23) is written as,

$$\gamma_2^2 + 2(e - b_1)\gamma_2 + e^2 - 2b_1 e = 0\,,$$

which implies

$$\gamma_2 = -e \quad \text{or} \quad \gamma_2 = -e + 2b_1\,,$$

where $e = (E_2\zeta_0, \zeta_0^*)$, $b_1 = \mathrm{Re}\, B_1 \neq 0$. (Note that (1.13) and (3.21) imply $2\mathrm{Re}\, B_1 = \beta_2$.) From the equation (3.21) it follows that,

$$(\beta 6) \qquad \begin{cases} (\gamma_2 + e)c_1 = 2b_1 c_1\,, \\ (\gamma_2 + e)c_2 = 2b_2 c_1\,, \end{cases}$$

$$c_1 = \mathrm{Re}\, c\,, \qquad c_2 = \mathrm{Im}\, c\,, \qquad b_2 = \mathrm{Im}\, B_1\,.$$

In the case, $\gamma_2 = -e$, we have $c_1 = 0$ and $c_2 \neq 0$. Thus,

$$\det\left(\frac{\partial \tilde{W}(0,0,0,0)}{\partial(\hat{\gamma}, \eta_1)}\right) = \begin{pmatrix} 0 & , 2b_1 \\ -c_2 & , 2b_2 \end{pmatrix} = 2b_1 c_2 \neq 0\,.$$

By the Implicit Function Theorem we obtain a pair of C^1-functions, $(\hat{\gamma}(\varepsilon, \eta_2),$ $\eta_1(\varepsilon, \eta_2))$, satisfying $\tilde{W}(\varepsilon, \hat{\gamma}(\varepsilon, \eta_2), \eta_1(\varepsilon, \eta_2), \eta_2) = 0$ in a sufficiently small neighborhood of $(\varepsilon, \eta_2) = (0, 0)$, and $\hat{\gamma}(0, 0) = \eta_1(0, 0) = 0$.

Next, in the case $\gamma_2 = -e_1 + 2b_1$, it follows from (β6) that, $c_2 = b_2 c_1 / b_1$, $c_1 \neq 0$. By (β3) and (β5) we obtain,

$$\det\left(\frac{\partial \tilde{W}(0,0,0,0)}{\partial(\hat{\gamma}, \eta_2)}\right) = \begin{pmatrix} c_1 & , & 0 \\ b_2 c_1 / b_1 & , & -\gamma_2 - e_1 \end{pmatrix} = -2c_1 b_1 \neq 0\,.$$

This shows that we can apply the Implicit Function Theorem to obtain a pair of C^1-functions $(\hat{\gamma}(\varepsilon, \eta_1), \eta_2(\varepsilon, \eta_1))$ satisfying, $\tilde{W}(\varepsilon, \hat{\gamma}(\varepsilon, \eta_1), \eta_1, \eta_2(\varepsilon, \eta_1)) = 0$ in a neighborhood of $(\hat{\gamma}, \eta_1) = (0, 0)$, and $\hat{\gamma}(0, 0) = \eta_2(0, 0) = 0$. Thus, in either case, we have solved the equation, (β1), under the assumption $\mathrm{Im}\,(E_2\zeta_0, \zeta_0^*) = 0$.

Next we assume $\mathrm{Im}\,(E_2\zeta_0, \zeta_0^*) \neq 0$. Let $e_1 = \mathrm{Re}\,(E_2\zeta_0, \zeta_0^*)$, $e_2 = \mathrm{Im}\,(E_2\zeta_1, \zeta_1^*)$. The equation (3.21) is reduced to,

$$(\beta 7) \qquad \begin{cases} (\gamma_2 + e_1)c_1 - c_2 e_2 = 2b_1 c_1\,, \\ (\gamma_2 + e_1)c_2 + c_1 e_2 = 2b_2 c_1\,. \end{cases}$$

(β3), (β5) and (β7) imply,

$$(\beta 8) \qquad \det\left(\frac{\partial \tilde{W}(0,0,0,0)}{\partial(\hat{\gamma}, \eta_2)}\right) = \begin{pmatrix} -c_1\,, & e_2 \\ -c_2\,, & -\gamma_2 - e_1 \end{pmatrix} = c_1(\gamma_2 + e_1) + c_2 e_2$$
$$= 2(b_1 c_1 + c_2 e_2)\,.$$

If $c_1 = 0$, the right hand side of (β8) is nonzero, thus, showing that (β1) is solvable.

Now consider the case $c_1 \neq 0$. We may assume $c_1 = 1$ without loss of generality. Then (β7) is written as,

$$(\beta 9) \qquad \begin{cases} (\gamma_2 + e_1) - c_2 e_2 = 2b_1\,, \\ (\gamma_2 + e_1)c_2 + e_2 = 2b_2\,. \end{cases}$$

It follows from (β9) that,

$$(\beta 10) \qquad (c_2 e_2 + 2b_1)c_2 + e_2 = 2b_2\,.$$

Under the condition (β10),

$$b_1 c_1 + c_2 e_2 = 0$$

holds if and only if

$$e_2^2 - 2b_2 e_2 - b_1^2 = 0\,.$$

In view of this and (β8), we see that except for the case $e_2 = b_2 \pm \sqrt{b_1^2 + b_2^2}$ we can solve the equation, (β1), by the Implicit Function Theorem, to obtain the solution pair, $\hat{\gamma}(\varepsilon, \eta_1)$, $\eta_2(\varepsilon, \eta_1)$. The case of $e_2 = b_2 \pm \sqrt{b_1^2 + b_2^2}$ means that the equation (3.23) has a double root. By the assumption we have excluded this case.

Thus, the equation, (β1), is solvable and the solution is given by the form $(\hat{\gamma}(\varepsilon, \eta_1), \eta_2(\varepsilon, \eta_1))$ or $(\hat{\gamma}(\varepsilon, \eta_2), \eta_1(\varepsilon, \eta_2))$ in a sufficiently small neighborhood of (0, 0). If the solution is given by the form $(\hat{\gamma}(\varepsilon, \eta_1), \eta_2(\varepsilon, \eta_1))$ (resp. $(\hat{\gamma}(\varepsilon, \eta_2), \eta_1(\varepsilon, \eta_2))$), then we put $\eta_1 = 0$ (resp. $\eta_2 = 0$) and set $\hat{\gamma}(\varepsilon) = \hat{\gamma}(\varepsilon, 0)$, $\eta(\varepsilon) = \eta_2(\varepsilon, 0)$ (resp. $\hat{\gamma}(\varepsilon) = \hat{\gamma}(\varepsilon, 0)$, $\eta(\varepsilon) = \eta_1(\varepsilon, 0)$). This gives a solution to (3.10) having the form (3.12). Hence, the completion of the proof.

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