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On the Incompressible Limit of the Compressible Euler Equation

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The solution of the initial value problem for the compressible Euler equation tends to the solution of the corresponding incompressible Euler equation with the corresponding initial data, as the Mach number (which is proportional to a parameter $1/\lambda$) tends to zero. Under suitable conditions, we also obtain the asymptotic expansion theorem for those solutions, when λ is large.

Key words: compressible Euler equation, incompressible limit, asymptotic expansion

1. Introduction

This is a completion of the work made by Klainerman and Majda [10], [11]. Consider the initial value problem of the compressible or incompressible Euler equation (C.E.Eq. or I.E.Eq.) which describes the state of flow of the compressible or incompressible ideal fluid, respectively:

$$\text{(1.1)} \qquad \begin{aligned} &\frac{\partial}{\partial t} p + v\cdot\nabla p + \gamma p \nabla\cdot v = 0\,, \qquad t\geq 0\,, \quad x\in R^3\,, \\ &\frac{\partial}{\partial t} v + v\nabla v + \frac{\lambda^2}{\rho}\nabla p = 0\,, \end{aligned}$$

$$\text{(1.1)}_0 \qquad (p, v)\big|_{t=0} = (p_0(x), v_0(x))\,,$$

$$\text{(1.2)} \qquad \begin{aligned} &\frac{\partial}{\partial t} v + v\cdot\nabla v + \frac{1}{\bar\rho}\nabla q^\infty = 0\,, \qquad t\geq 0\,, \quad x\in R^3\,, \\ &\nabla\cdot v = 0\,, \end{aligned}$$

$$\text{(1.2)}_0 \qquad v\big|_{t=0} = v_0(x) \quad (\text{and } \nabla\cdot v_0(x) = 0, \text{ usually})\,.$$

Here $p(t, x)$, $v(t, x)$ and $\rho(t, x)$ denote the pressure, velocity and density of the fluid at time $t\geq 0$ and point $x\in R^3$. The notation $\cdot$ or $\langle\ ,\ \rangle$ (resp. $\times$) means the scalar product in R^3 (resp. the vector product in R^3), or sometimes in R^n. The parameter $\gamma\geq 1$ is constant and λ varies in $[1, \infty)$. The density ρ is calculated by means of the ideal gas condition

$$\text{(1.3)} \qquad p = \rho^\gamma s\,,$$

where s is the entropy of the fluid and assumed to be constant, $s(t, x) \equiv \bar{s} > 0$ on the whole space. The fluid is assumed to be isentropic.

The sound speed calculated from (1.1) is $\lambda(\gamma p/\rho)^{1/2} = \sigma$. Hence roughly speaking, λ is (proportional to) the reciprocal of the Mach number $M = |v_m|/\sigma_m = |v_m|\lambda^{-1}(\gamma p_m/\rho_m)^{-1/2}$.

Besides the existence theorem and others, Agemi [1], Ebin [7] and Klainerman-Majda [10], [11] proved (under some restrictive conditions on the initial data (p_0, v_0)) that the solution (p^λ, v^λ) of the equation (1.1)–$(1.1)_0$ approaches the solution (p^∞, v^∞) of (1.2)–$(1.2)_0$, as λ tends to ∞. The aim of this paper is to weaken the assumptions, especially to remove the condition $\nabla \cdot v_0 = 0$. Hence the initial layer, i.e., the discontinuity of $v^\lambda(t, x)$ at $\lambda = \infty$ (or at $t = 0$) may appear in our limiting process. Our main results are the following (see (1.22)–(1.26) for the definition of function spaces):

THEOREM 1.1. *Assume that the initial data* (p_0, v_0) *satisfies*

$$(1.4)_0 \qquad p_0 = p_0(\lambda, x) = \bar{p}(1 + \lambda^{-1} q_0(x)), \qquad \bar{p} = constant > 0,$$

$$(1.5)_0 \qquad \text{(i)} \quad (q_0, v_0) = u_0 \in H^l \quad with \quad l \geq 3,$$

$$\text{(ii)} \quad |u_0|_2 \leq a \quad for\ a\ sufficiently\ small\ constant \quad a \in (0, 1/5].$$

Then: (i) *there exists a solution* $(p(\lambda, t, x), v(\lambda, t, x))$ *of the C.E.Eq.* (1.1)–$(1.1)_0$ *satisfying the following properties in a time interval* $[0, T]$:

$$(1.4) \qquad p(\lambda, t, x) = \bar{p}(1 + \lambda^{-1} q(\lambda, t, x)),$$

$$(1.5) \qquad \text{(i)} \quad (q, v) \in M^j([1, \infty); B^0([0, T]; H^{l-j})), \qquad 0 \leq j \leq l,$$

$$\text{(ii)} \quad |(q, v)|_2 \leq 2a \quad for \quad (\lambda, t) \in [1, \infty) \times [0, T].$$

Here $T > 0$ *depends only on* $|u_0|_3$, $\bar{p}$, $\bar{s}$ *and* a, *but not on* $\lambda \in [1, \infty)$. *The solution* (q, v) *is unique in* $B^0([0, T]; H^3)$, *and depends continuously on the initial data* (q_0, v_0) *as the mapping from* H^l *to* $M^j([1, \infty); B^0([0, T]; \tilde{H}^{l-j}))$, $0 \leq j \leq l$.

(ii) *The solution* (q, v) *also satisfies the following*

$$(1.6) \qquad \text{(i)} \quad (q, v) \in B^0([1, \infty] \times [0, T] \backslash (\infty, 0); B^{l-2+\delta}), \qquad 0 \leq \delta < 1/2,$$

$$\text{(ii)} \quad \|q(\lambda, t, \cdot)\|_{l-2+\delta} + \|\nabla \cdot v(\lambda, t, \cdot)\|_{l-3+\delta} \to 0 \qquad as \quad \lambda \to \infty \quad (t > 0),$$

$$\text{(iii)} \quad (q(\lambda, t, \cdot), v(\lambda, t, \cdot)) \to (0, v(\infty, t, \cdot)) \quad weakly\ in\ H^l \quad (\lambda \to \infty),$$

where the convergence is uniform on $[t_0, T]$ *for any* $t_0 > 0$.

(iii) *There exists a unique solution* $(q^\infty(t, x), v^\infty(t, x))$ *of the I.E.Eq.* (1.2), *satisfying on* $[0, T]$

$$(1.7) \qquad v^\infty(0, x) = P v_0(x),$$

$$(1.8) \qquad (q^\infty, v^\infty) \in B^j([0, T]; \tilde{H}^{l+1-j} \times H^{l-j}), \qquad 0 \leq j \leq l-1,$$

(1.9) $$v^{\infty}(t, x)=v(\infty, t, x), \qquad (t, x)\in(0, T]\times R^3 .$$

Here P denotes the projection of $L^2(R^3)^3$ into the divergence free subspace.

THEOREM 1.2. *In addition to the condition of Theorem 1.1 we assume further $u_0=(q_0, v_0)\in H_2^{k+2}$, $1\le k\le l-2$. Then we have the asymptotic expansion in $\lambda\in[\lambda_k, \infty)$*

(1.10)
$$q(\lambda, t, \cdot)=q^0(\lambda, t)+\lambda^{-1}q^{\infty}(t)+\sum_{i=1}^{k}\lambda^{-i}q^i(\lambda, t),$$
$$v(\lambda, t, \cdot)=v^0(\lambda, t)+v^{\infty}(t)+\sum_{i=1}^{k}\lambda^{-i}v^i(\lambda, t),$$

where (q^i, v^i), $0\le i\le k$, satisfy

(1.11)
$$\frac{\partial}{\partial t}q^0+\lambda\gamma\bar{p}\nabla\cdot v^0=0,$$
$$\frac{\partial}{\partial t}v^0+\lambda/\bar{\rho}\nabla q^0=0,$$

$(1.11)_0$ $$^t(q^0, v^0)|_{t=0}=P_1{}^t(q_0, v_0) \qquad \text{(for the definition of } P_1 \text{ see (3.13))}.$$

$(1.12)^1$
$$\frac{\partial}{\partial t}(q^{\infty}+q^1)+(v^0+v^{\infty})\cdot\nabla(q^{\infty}+q^1)+\lambda\gamma p_1\nabla\cdot v^1$$
$$+\gamma\bar{p}(q^{\infty}+q^1)\nabla\cdot v^0+v^1\cdot\nabla q^0=-\lambda\gamma\bar{p}q^0\nabla\cdot v^0-\lambda(v^0+v^{\infty})\cdot\nabla q^0,$$
$$\frac{\partial}{\partial t}v^1+(v^0+v^{\infty})\cdot\nabla v^1+\lambda/\rho_1\nabla q^1+1/(\gamma\bar{\rho})(q^{\infty}+q^1)\nabla q^0+v^1\cdot\nabla(v^0+v^{\infty})$$
$$=-\lambda v^0\cdot\nabla(v^0+v^{\infty})-v^{\infty}\cdot\nabla(\lambda v^0)+1/(\gamma\rho_1)\lambda q^0\nabla q^0$$

$(1.12)^1_0$ $$(q^{\infty}+q^1, v^1)|_{t=0}=0,$$

$(1.13)^1$ (i) $$p_1=p_1(\lambda, t, x)=\bar{p}(1+\lambda^{-1}q^0),$$
(ii) $$\rho_1=\rho_1(\lambda, t, x)=\bar{\rho}\{1-1/(\lambda\gamma)q^0\},$$

$(1.12)^j$
$$\frac{\partial}{\partial t}q^j+(v^0+v^{\infty})\cdot\nabla q^j+\lambda\gamma p_j\nabla\cdot v^j$$
$$+\gamma\bar{p}q^j\nabla\cdot v^0+v^j\cdot\nabla q^0=g_j((q, v)_{j-1}, \nabla(q, v)_{j-1}),$$
$$\frac{\partial}{\partial t}v^j+(v^0+v^{\infty})\cdot\nabla v^j+\lambda/\rho_j\nabla q^j$$
$$-1/(\gamma\bar{\rho})q^j\nabla q^0+v^j\cdot\nabla(v^0+v^{\infty})=h_j((q, v)_{j-1}, \nabla(q, v)_{j-1}),$$

$(1.12)^j_0$ $$(q^j, v^j)|_{t=0}=0, \qquad 2\le j<k,$$

$(1.13)^j$ (i) $p_j = p_j(\lambda, t, x) = \bar{p}\left\{1 + \lambda^{-1}q^0 + \lambda^{-2}q^\infty + \sum_{i=1}^{j-1} \lambda^{-i-1}q^i\right\}$,

(ii) ρ_j = *the sum of the terms up to order* λ^{-j} *whivh appear in the expansion of* $\rho = \left\{1 + \lambda^{-1}q^\infty + \sum_{i=0}^{j-1} \lambda^{-i-1}q^i\right\}^{1/\gamma}$ (*see* (1.21)),

$$(1.12)^k \quad \begin{aligned} &\frac{\partial}{\partial t} q^k + v\cdot\nabla q^k + \lambda\gamma p \nabla\cdot v^k = g_k(\lambda^{-1}, (q, v)_{k-1}), \\ &\frac{\partial}{\partial t} v_k + v\cdot\nabla v^k + \lambda/\rho \nabla q^k - 1/(\gamma\bar{p}) q^k \nabla q = h_k(\lambda^{-1}, (q, v)_{k-1}), \end{aligned}$$

$$(1.12)^k_0 \quad (q^k, v^k)|_{t=0} = 0,$$

$(1.13)^k$ (i) (p, v), $p = \bar{p}(1 + \lambda^{-1}q)$, *is the solution of* (1.1)–(1.2),

(ii) ρ *is the density calculated by* (1.3),

$(1.13)^j$ (iii) g_i *and* h_j, $2 \le j \le k$, *are bilinear functions of* $(q, v)_{j-1} = (q^0, q^\infty, \cdots, q^{j-1}, v^0, v^\infty, \cdots, v^{j-1})$ *and its first derivatives* $\nabla(q, v)_{j-1} = (\nabla q^0, \cdots, \nabla v^{j-1})$. *The coefficients appearing in* g_k *and* h_k *depend smoothly on* $\lambda^{-1} \in [0, 1]$.

$$(1.14)^0 \quad (q^0, v^0) \in B^0([1, \infty); B^0([0, T]; H^l)) \cap B^0([1, \infty] \times [0, T] \backslash (\infty, 0); B^{l-2+\delta})), \qquad 0 \le \delta < 1/2,$$

$$(1.15)^0 \quad \|(q^0, v^0)\|_k \le C_0(1 + \lambda t)^{-1} |u_0|_{k+2,2},$$

$$(1.14)^1 \quad (q^\infty + q^1, v^1) \in C^0([1, \infty); B^0([0, T]; H^{l-1})),$$

$$(1.15)^1 \quad |(q^\infty + q^1, v^1)|_{k-1} \le C_1 \log(1 + \lambda) |u_0|_{k+2,2},$$

$$(1.14)^j \quad (q^j, v^j) \in C^0([\lambda_j, \infty); B^0([0, T]; H^{l-j})), \qquad 1 < j \le k,$$

$$(1.15)^j \quad |(q^j, v^j)|_{k-j} \le C_j \{\log(1 + \lambda)\}^j |u_0|_{k+2,2}, \qquad 1 < j \le k,$$

where $1 = \lambda_1 \le \cdots \le \lambda_k < \infty$. *In particular we have*

$$(1.16) \quad \begin{aligned} &|(q(\lambda, t) - q^0(\lambda, t), v(\lambda, t) - v^0(\lambda, t) - v^\infty(t))|_{k-1} \\ &\le C\lambda^{-1} \log(1 + \lambda) |u_0|_{k+2,2}, \qquad 0 \le t \le T. \end{aligned}$$

REMARK. Ukai [15] proved the main part of Theorem 1.1 simultaneously with (exactly a week ahead of) the author. However, his method is different from ours except the use of the evolution operator for the linear C.E.Eq. with constant coefficients. Klainerman-Majda [11] showed the corresponding results of Theorem 1.2 under the condition

$$(1.17)\qquad \begin{aligned} &p_0(\lambda, x)=\bar{p}+\lambda^{-2}q_0^1(x)\,,\\ &v_0(\lambda, x)=v_0^0(x)+\lambda^{-1}v_0^1(x)\,,\qquad \nabla\cdot v_0^0(x)=0\,,\\ &(q_0^1, v_0^1)\in H^l\,.\end{aligned}$$

In our study the assumption on the initial data are weakened extensively. We note the similarity of the asymptotic expansion (1.10) with the one given in [4]. We also note that our method can be applied to the study of the relation between the compressible and incompressible Navier-Stokes equations. This theme will be discussed in the forthcoming paper which will succeed [3].

Using the description of Klainerman-Majda (i.e., that of Theorem 1.1) we write the variable p as in (1.4). Then the C.E.Eq. (1.1) reduces to

$$(1.18)\qquad \begin{aligned} &\frac{\partial}{\partial t}q+v\cdot\nabla q+\lambda\gamma p\nabla\cdot v=0\,,\\ &\frac{\partial}{\partial t}v+v\cdot\nabla v+\lambda/\rho\nabla q=0\,,\end{aligned}$$

$$(1.18)_0\qquad (q, v)\big|_{t=0}=(q_0.\, v_0)\,,$$

with the conditions

$$(1.19)\qquad p(\lambda, t, x)=\bar{p}\{1+\lambda^{-1}q(\lambda, t, x)\}\,,\qquad \bar{p}>0\,,$$

$$(1.20)\qquad \rho(\lambda, t, x)=\{\bar{p}(1+\lambda^{-1}q(\lambda, t, x))/\bar{s}\}^{1/\gamma}\,,\qquad \bar{s}>0\,.$$

If q is sufficiently small, $|q|\le a$ for some $a\in(0, 1/5]$ as in Theorem 1.1, we can rewrite (1.20) as below:

$$(1.21)\quad \text{(i)}\quad \begin{aligned} &\rho=\bar{\rho}\{1+\psi_0(\lambda^{-1}q)\}\,,\qquad \bar{\rho}=(\bar{p}/\bar{s})^{1/\gamma}>0\,,\\ &\psi_0(z)=\{1+z\}^{1/\gamma}-1=\sum_{j=1}^{k-1}a_jz^j+z^k\psi_k(z)\,,\\ &\psi_k\in C^\infty([-2a, 2a])\,,\quad \psi_0(0)=0\,,\quad 0\le\psi'_0\le 5/4\quad\text{and}\\ &|\psi_0|\le 1/2\qquad\text{on}\quad[-2a, 2a]\,.\end{aligned}$$

$$\text{(ii)}\quad \{1+\psi_0(z)\}^{-1}-1=\phi_0(z)=\sum_{j=1}^{k-1}a_j^1z^j+z^k\phi_k(z)\,,\qquad \phi_j\in C^\infty([-2a, 2a])\,.$$

Under the assumption (1.5) (ii) the equation (1.18) is Friedrichs' symmetric hyperbolic system ([8], [13]), and the well-established theory can be applied to solve (1.18) ([9]). This is the method of Klainerman-Majda [10], [11]. We proceed in the same way.

We introduce the function spaces H^l, $\tilde{H}^l$ and H^l_β of scalar or vector valued measurable functions for $l\in R$ and $\beta\in R$:

(1.22) $H^l \ni f(x) \Leftrightarrow (1+|\xi|^2)^{l/2}\hat{f}(\xi) \in L^2(R^3)$, where

$$\hat{f}(\xi)=(2\pi)^{-3/2}\int e^{-ix\cdot\xi}f(x)dx=(Ff)(\xi).$$

The norm $|f|_l$ is defined by $|f|_l^2=\int(1+|\xi|^2)^l|\hat{f}(\xi)|^2d\xi$.

(1.23) $\tilde{H}^l=\bigcap_{\delta>0} H^{l-\delta}$ is a complete metric space,

(1.24) $H^l_\beta \ni f(x) \Leftrightarrow (1+|x|^2)^{\beta/2}\nabla^j_x f(x) \in L^2(R^3)=H^0$ for $0\le j\le l$. Here $l\ge 0$ is an integer. The norm $|f|_{l,\beta}$ is defined by

$$|f|^2_{l,\beta}=\sum_{j=0}^{l}\int(1+|x|^2)^\beta|\nabla^j f(x)|^2dx.$$

Only once or twice we use $\tilde{H}^l_\beta=\cap H^{l-\delta}_{\beta-\delta}$ and $\bar{H}^l=\{f\in B^0(R^3);\ \nabla f\in H^{l-1}\}$.

For a (closed) domain $\Omega\subset R^n$ and a Banach space (more generally, a linear topological space) Y we denote by $C^k(\Omega;Y)$ the space of Y-valued continuous functions which are k times continuously differentiable on Ω in the topology of Y. By $B^k(\Omega;Y)$ (resp. $B^{k+\theta}(\Omega;Y)$, $0<\theta<1$) we denote the subspace of $f(x)\in C^k(\Omega;Y)$ whose derivatives $(\partial/\partial x)^\alpha f(x)$, $|\alpha|\le k$, are bounded on Ω (resp. bounded and Hölder continous with exponent θ). If Y is a Banach space with the norm $|\ \ |_Y$, $B^{k+\theta}(\Omega;Y)$, $0\le\theta<1$, is a Banach space with the norm $\|f\|_{Y,k+\theta}$:

(1.25) (i) $\|f\|_{Y,k}=\|f\|_k=\sum_{|\alpha|\le k}\sup_{x\in\Omega}|(\partial/\partial x)^\alpha f(x)|_Y$,

(ii) $\|f\|_{Y,k+\theta}=\|f\|_{k+\theta}=\|f\|_{k-1}+\|\nabla^k f\|_\theta$, $\quad 0<\theta<1$,

$$\|g\|_\theta=\|g\|_0+\sup_{x\neq y}|g(x)-g(y)|_Y/|x-y|^\theta.$$

Similarly by $M^k(\Omega;Y)$ we denote the space of Y-valued strongly measurable functions $f(x)$ on Ω whose derivatives $(\partial/\partial x)^\alpha f(x)$ (in the distribution sense), $|\alpha|\le k$, are essentially bounded on Ω. $M^k(\Omega;Y)$ is also a Banach space with the norm defined by (1.25) (i), if Y is a Banach space. We use the notation $C^k=C^k(R^3;C^n)$ and $B^{k+\theta}=B^{k+\theta}(R^3;C^n)$.

For a function $f(t,x)\in B^0([0,T]^i;H^l)$ or $M^0([0,T]^i;H^l)$ $(i=1,2)$ we put

$$|f|_{l,T}=\sup_{0\le t\le T}|f(t,\cdot)|_l \qquad (\text{or } =\sup_{t,s}|f(t,s,\cdot)|_l).$$

The diagonal set of $[0,T]^2$ is denoted by Δ, $\Delta=\{(t,t);\ t\in[0,T]\}$.

We use b_i for the constant associated with the Sobolev inequality, $b(l)$ (and K, L in section 4) associated with the equation and C or $C(M)$ in general. $C(M)$ means that the constant $C(M)$ depends (mainly) on the quantity M. With a Banach space Y, $B(Y)$ denotes the space of the bounded linear operators in Y.

2. The Linear C.E.Eq. (I)

We consider the initial value problem of the linearized C.E.Eq. corresponding to (1.18):

$$(2.1)\qquad \begin{aligned}&\frac{\partial}{\partial t}q+\tilde{v}\cdot\nabla q+\lambda\gamma\tilde{p}\nabla\cdot v=g\,,\\ &\frac{\partial}{\partial t}v+\tilde{v}\cdot\nabla v+(\lambda/\tilde{\rho})\nabla q=h\,,\end{aligned}$$

$$(2.1)_s\qquad (q,v)\big|_{t=s}=(q_0,v_0)\,,$$

$$(2.2)\qquad \begin{aligned}&\tilde{p}(\lambda,t,x)=\bar{p}\{1+\lambda^{-1}\tilde{q}(\lambda,t,x)\}\,,\\ &\tilde{\rho}(\lambda,t,x)=\bar{\rho}\{1+\psi_0(\lambda^{-1}\tilde{q})\}=\bar{\rho}\{1+\lambda^{-1}\tilde{q}\psi_1(\lambda^{-1}\tilde{q})\}\,.\end{aligned}$$

To solve the above equation we assume

[A.0]$_l$ (i) $(q_0,v_0)=u_0\in H^l$ with an integer $l\geq 3$,

(ii) $|u_0|_2\leq a$ with a constant a of Theorem 1.1,

[A.1]$_l$ (i) $\tilde{u}=(\tilde{q},\tilde{v})\in M^j([1,\infty);B^0([0,\tilde{T}];H^{l-j}))$, $l\geq 3$, $0\leq j\leq l$,

(i)′ $\tilde{u}\in M^j([1,\infty);M^0([0,\tilde{T}];H^{l-j}))\cap B^j([1,\infty);B^0([0,\tilde{T}];\tilde{H}^{l-j}))$.

(ii) $|\tilde{u}|_2\leq 8|u_0|_2$ for $(\lambda,t)\in[1,\infty)\times[0,\tilde{T}]$,

(iii) $(\gamma\bar{p})^{-1}|\tilde{q}|_j^2+\bar{\rho}|\tilde{v}|_j^2\leq M_3^2$ $(j=3)$, $\leq\tilde{M}^2$ $(4\leq j\leq l)$,

(iv) $(\lambda\bar{p})^{-1}\left\|\left(\frac{\partial}{\partial t}+\tilde{v}\cdot\nabla\right)\tilde{q}\right\|_0+\frac{1}{2}\|\nabla\tilde{v}\|_0\leq b_0\{(1+\lambda^{-1}M_3)+1/2\}M_3\leq M_0$,

(Note that (i)′ follows from (i).)

[A.2]$_k$ $\quad f=(g,h)\in M^j([1,\infty);\;M^0([0,T];H^{k-j})\,,\quad 0\leq j\leq k\,.$

To estimate the solution $u=(q,v)$ of (2.1)–(2.1)$_0$, we define an equivalent norm $|u|_j'$ of H^j by

$$(2.3)\qquad |u|_j'^2=\sum_{|\alpha|\leq j}\left((\gamma\tilde{p})^{-1}\left(\frac{\partial}{\partial x}\right)^{\alpha}q,\left(\frac{\partial}{\partial x}\right)^{\alpha}q\right)+\sum_{|\alpha|\leq j}\left(\tilde{\rho}\left(\frac{\partial}{\partial x}\right)^{\alpha}v,\left(\frac{\partial}{\partial x}\right)^{\alpha}v\right),$$

where (,) denotes the usual inner product in $L^2(R^3)$. If we put

$$(2.4)\qquad L=L(\tilde{v})=\frac{\partial}{\partial t}+\tilde{v}\cdot\nabla\,,$$

and take a real valued function $\phi(t,x)$, an easy calculation proves

$$(2.5)\qquad \frac{d}{dt}(\phi w,w)=((L\phi)w,w)+(\phi(\nabla\cdot\tilde{v})w,w)+2\mathrm{Re}(\phi Lw,w).$$

Using (2.5), [A.1]$_l$ (ii)–(iv) and the equation (2.1), we obtain

$$(2.6)\qquad \frac{d}{dt}|u|_0'^2=\left(\left\{\left(\frac{\partial}{\partial t}+\tilde{v}\cdot\nabla\right)(\gamma\tilde{p})^{-1}\right\}q, q\right)+\left(\left\{\left(\frac{\partial}{\partial t}+\tilde{v}\cdot\nabla\right)\tilde{\rho}\right\}v, v\right)$$

$$+((\gamma\tilde{p})^{-1}(\nabla\cdot\tilde{v})q, q)+(\tilde{\rho}(\nabla\cdot\tilde{v})v, v)+2\mathrm{Re}(g, q)'+2\mathrm{Re}(h, v)'\,,\quad \text{i.e.,}$$

$$(2.6)'\qquad \left|\frac{d}{dt}|u|_0'\right|\le M_0|u|_0'+|f|_0'\,.$$

This gives

$$(2.7)\qquad |u|_0'\le e^{M_0|t-s|}(|u_0|_0'+|t-s||f|_{0,T}')\,.$$

Using (2.5) and (2.7) to estimate $w=((\partial/\partial x)^\alpha q, (\partial/\partial x)^\alpha v)$, we have the following

LEMMA 2.1. *Assume* $[\mathrm{A.0}]_l$, $[\mathrm{A.1}]_l$ *and* $[\mathrm{A.2}]_l$ *with* $l\ge 3$. *Then*:

(i) *the linear C.E.Eq.* (2.1)–$(2.1)_s$ *has a unique solution* $u=(q, v)$ *satisfying* $[\mathrm{A.1}]_l$ (i)$'$ *and the estimates*

$$(2.8)\quad \text{(i)}\quad |u|_j'\le e^{|t-s|b(l)M_3}\{|u_0|_j'+|t-s|(b(l)|\tilde{u}|_{j,t}'|u|_{j-3,t}'+c|f|_{j,t}')\}\,,\quad 0\le j\le l\,,$$

$$\text{(ii)}\quad |u|_3'\le e^{|t-s|b(3)M_3}\{|u_0|_3'+c|f|_{3,t}'\}\,,$$

where $|u|_{j-3,t}'=0$ *for* $0\le j\le 3$. *The solution is unique in* $B^0([0, T]; H^1)\cap B^1([0, T]; H^0)$.

(ii) *Assume further* $f=0$ *and*

$$(2.9)\qquad |u_0|_3'\le M_3/m \qquad \text{with some}\quad m>4\,.$$

Choose $T\in(0, \tilde{T}]$ *so that there holds*

$$(2.10)\qquad \text{(i)}\quad M_3\ge 4e^{Tb(3)M_3}|u_0|_3'\,,$$

$$\text{(ii)}\quad e^{Tb(3)M_3}\le 2\sqrt{2}\,.$$

Then the solution $u=(q, v)$ *of* (2.1)–$(2.1)_s$ *satisfies the conditions* $[\mathrm{A.1}]_l$ (ii)–(iv) *with the same* M_3 *and with* $\tilde{T}$ *and* $\tilde{M}$ *replaced by* T *and* $M=C(\tilde{M}, |u_0|_l')$.

(iii) *If we assume further* (*except* (2.9))

$[\mathrm{A.3}]_l$ $\quad \tilde{u}=(\tilde{q}, \tilde{v})\in M^j([1, \infty]; B^0([0, T]; B^{l+1}))\,,$

then $u=(q, v)$ *satisfies the condition* $[\mathrm{A.1}]_l$ (i).

As the proof is quite standard, we omit it except a brief comment on the condition [A.1] (iv). (For example, see Kato [9], Klainerman-Majda [11] or Mizohata [13].) If (q, v) satisfies (2.1) (with $f=0$), it follows

$$(2.11)\qquad \left(\frac{\partial}{\partial t}+v\cdot\nabla\right)q=-\lambda\gamma\bar{p}(1+\lambda^{-1}\tilde{q})\nabla\cdot v+(v-\tilde{v})\cdot\nabla v\,.$$

Combining (2.11), $[\mathrm{A.1}]_l$ (ii) and (2.8) (ii), we can show that $[\mathrm{A.1}]_l$ (iv) holds for (q, v) under the condition (2.10). (Note $b_0\le b(l)$.) We also note that under the condition $[\mathrm{A.1}]_l$ (ii) and (2.10) there hold

$$\begin{aligned}(2.12)\qquad &\bar{p}/2\le\tilde{p}=\bar{p}(1+\lambda^{-1}q)\le 3\bar{p}/2\,,\\ &\bar{\rho}/2\le\tilde{\rho}=\bar{\rho}\{1+\psi(\lambda^{-1}\tilde{q})\}\le 3\bar{\rho}/2\,,\end{aligned}$$

$$\begin{aligned}(2.13)\qquad \|q\|_0\le|q|_2/4\le|u|_2/4\le(2\gamma\bar{p})^{1/2}|u|_2'/4\\ \le(2\gamma\bar{p})^{1/2}|u_0|_2/\sqrt{2}\le 2|u_0|_2\le 2a\,.\end{aligned}$$

Now we give an improved version of Lemma 2.1.

LEMMA 2.2. *Assume* $[A.0]_l$, $[A.1]_l$ *and* $[A.2]_k$ *with* $l\ge 3$ *and* $0\le k\le l$. *Then the solution* $u=(q,v)$ *of* (2.1)–$(2.1)_s$ *satisfies the condition* $[A.1]_k$ (i). *In particular, the claims* (ii) *and* (iii) *of Lemma* 2.1 *hold with the same constants T and M of Lemma* 2.1 *without assuming* $[A.3]_l$.

Proof. First we consider the case $f=0$. By Lemma 2.1 $u=(q,v)$ belongs to $M^0([0,\tilde{T}]^2;H^l)\cap B^0([0,\tilde{T}]^2;H^{l-1})$, and then by the interpolation theorem to $B^0([0,\tilde{T}]^2;H^{l-1+\theta})$, $0<\theta<1$. By the interpolation, we can define $|u|_{j+\theta}'$ for $0<\theta<1$ and $0<j\le l-1$. Then (2.8) (ii) implies

$$(2.14)\qquad |u(t')|_{l-1+\theta}'\le e^{|t'-t|c}|u(t)|_{l-1+\theta}'\,,\qquad t',t\in[0,\tilde{T}]\,,$$

where the constant c is independent of $\theta\in(0,1)$. Since $|u|_{l-1+\theta}$ is continuous in $\theta\in[0,1]$ for $u\in H^l$ and $|u|_{l-1+\theta}$ is uniformly equivalent with $|u|_{l-1+\theta}'$ for $\theta\in[0,1]$, $|u|_{l-1+\theta}'$ is continuous in $\theta\in[0,1]$ for $u\in H^l$. Since $u(t')$ and $u(t)$ belong to H^l, (2.14) gives

$$(2.15)\qquad |u(t')|_l'\le e^{|t'-t|c}|u(t)|_l'\,,\qquad t,t'\in[0,\tilde{T}]\,.$$

This implies $|u(t)|_l'$ is continuous in $[0,\tilde{T}]$. Since $u(t)$ is weakly continuous in H^l, $u(t)$ is strongly continuous in the topology of H^l. If $u_0\in H^j$, $0\le j\le l$, $u(t)$ is strongly continuous in $t\in[0,\tilde{T}]$ in the topology of H^j. The strong continuity in the variable s is proved similarly.

Denote the solution $u(t)=u(t,s)$ of (2.1)–$(2.1)_s$ with $f=0$ as

$$(2.16)\qquad u(t)=U(t,s;\lambda)u_0=U(t,s;\lambda,\tilde{u})u_0\,,$$

and define by (2.16) the evolution operator $U(t,s;\lambda,\tilde{u})$ associated with the initial value problem (2.1)–$(2.1)_s$. Then the solution $u(t)$ of (2.1)–$(2.1)_s$ with f satisfying $[A.2]_k$ is described as

$$(2.17)\qquad u(\lambda,t,s)=U(t,s;\lambda,\tilde{u})u_0+\int_s^t U(t,r;\lambda,\tilde{u})f(r)dr\,.$$

This gives the conclusion.

REMARK. Define differential operators $A(\tilde{v},\lambda\tilde{p},\lambda/\tilde{\rho},\nabla)$ and $B(\lambda\tilde{p},\lambda/\tilde{\rho},\nabla)$ by

$$(2.18)\qquad A(\tilde{v},\lambda\tilde{p},\lambda/\tilde{\rho},\nabla)=\tilde{v}\cdot\nabla+B(\lambda\tilde{p},\lambda/\tilde{\rho},\nabla)\,,$$

$$(2.19)\qquad B(\lambda\tilde{p}, \lambda/\tilde{\rho}, \nabla)=\begin{pmatrix} 0 & \lambda\gamma\tilde{p}{}^t\nabla \\ (\lambda/\tilde{\rho})\nabla & 0 \end{pmatrix}\equiv B(\lambda\tilde{p}, \lambda/\tilde{\rho})\cdot\nabla_x .$$

The followings are trivial properties of $U(t, s; \lambda, \tilde{q}, \tilde{v})=U(t, s; \lambda, \tilde{u})$:

$$(2.20)\qquad U(t, r; \lambda, \tilde{u})U(r, s; \lambda, \tilde{u})=U(t, s; \lambda, \tilde{u}) ,$$

$$(2.21)\qquad \frac{d}{dt}U(t, s; \lambda, \tilde{u})=-A(\tilde{v}, \lambda\tilde{p}, \lambda/\tilde{\rho}, \nabla)U(t, s; \lambda, \tilde{u}) ,$$

$$(2.22)\qquad \frac{d}{ds}U(t, s; \lambda, \tilde{u})=U(t, s; \lambda, \tilde{u})A(\tilde{v}, \lambda\tilde{p}, \lambda/\tilde{\rho}, \nabla) .$$

Here $\tilde{p}$ and $\tilde{\rho}$ are calculated by the formula (1.19)–(1.20) from $\tilde{q}$.

Finally we state the continuous dependence of $U(t, s; \lambda, \tilde{u})$ on $\tilde{u}=(\tilde{q}, \tilde{v})$.

Lemma 2.3. *Let $\tilde{u}=(\tilde{q}, \tilde{v})$ and $\tilde{u}_1=(\tilde{q}_1, \tilde{v}_1)$ satisfy* [A.1]$_l$ *and $u_0\in H^k$, $3\le k\le l$. Then we have*

$$(2.23)\qquad \begin{aligned} &|U(t, s; \lambda, \tilde{u})u_0-U(t, s; \lambda, \tilde{u}_1)u_0|_{k-1} \\ &\le C|t-s|e^{|t-s|b(l)C(M)}|\tilde{u}-\tilde{u}_1|_{k-1, T}|u_0|_k . \end{aligned}$$

Proof. Using (2.21) and (2.22), we obtain

$$(2.24)\qquad \begin{aligned} &\frac{d}{dr}U(t, r; \lambda, \tilde{u})U(r, s; \lambda, \tilde{u}_1) \\ &=U(t, r; \lambda, \tilde{u})\{(\tilde{v}_1-\tilde{v})\cdot\nabla+B((\tilde{q}_1-\tilde{q})\tilde{\psi}_1(\lambda^{-1}\tilde{q}_1, \lambda^{-1}\tilde{q}) \\ &\quad+(\tilde{q}_1-\tilde{q})\tilde{\phi}_1(\lambda^{-1}\tilde{q}_1, \lambda^{-1}\tilde{q}), \nabla)\}U(r, s; \lambda, \tilde{u}_1) , \end{aligned}$$

where $\tilde{\psi}_1$, $\tilde{\phi}_1\in C^\infty([-2a, 2a]^2)$. Integrating (2.24) and using (2.8), we obtain the desired result (2.23).

The following Lemma 2.4 is a version of Lemma 2.2 and needed to solve the linear C.E.Eq. $(1.12)^j$–$(1.12)^j_0$, $1\le j\le k$. The proof is similar to that of Lemma 2.2.

Lemma 2.4. *Let $(\tilde{q}, \tilde{\mu}, \tilde{v})(\lambda, t, x)$ satisfy the condition*

$$\begin{aligned} [\text{A.1}']_{l, \tilde{\Lambda}}\quad &\text{(i)}\quad (\tilde{q}, \tilde{\mu}, \tilde{v})\in B^0([\tilde{\Lambda}, \infty); B^0([0, T]; H^l)) \\ &\qquad\cap C^0([\tilde{\Lambda}, \infty); B^1([0, T]; H^{l-1})) , \qquad l\ge 3 \\ &\text{(ii)}\quad \|(\partial/\partial t+\tilde{v}\cdot\nabla)(\tilde{q}/\tilde{p}, \tilde{\mu}/\tilde{\rho})\|_{0, T}\le b_0\lambda . \end{aligned}$$

Consider the initial value problem of the linear C.E.Eq. (2.1)–(2.1)$_s$ *with the initial data $u_0={}^t(q_0, v_0)\in H^l$, the non-homogeneous term $(g, h)=0$ and the coefficients defined by*

$$(2.25)\qquad \begin{aligned} &\tilde{p}(\lambda, t, x)=\bar{p}(1+\lambda^{-1}\tilde{q}) , \\ &\tilde{\rho}(\lambda, t, x)=\bar{\rho}(1+\lambda^{-1}\tilde{\mu})^{1/\gamma} . \end{aligned}$$

Then there exists a unique solution $u(\lambda, t, x)=(q, v)$ *of the above equation* $((2.1)$–$(2.1)_s$ *with* $(g, h)=(0, 0))$ *satisfying* $[\mathrm{A}.1']_{l,\Lambda}$ *with some* $\Lambda \geq \tilde{\Lambda}$. *Moreover u satisfies the same estimates stated in Lemma* 2.1 *for* $\lambda \in [\Lambda, \infty)$.

REMARK. We describe the above solution u as $u=U(\lambda, t, s; \tilde{q}, \tilde{\mu}, \tilde{v})u_0$. This is the definition of the evolution operator $U(\lambda, t, s,; \tilde{q}, \tilde{\mu}, \tilde{v})$ associated with the above linear C.E.Eq. Then $U(\lambda, t, s; \tilde{q}, \tilde{\mu}, \tilde{v})-U(\lambda, t, s; \tilde{q}_1, \tilde{\mu}_1, \tilde{v}_1)$ satisfies the similar estimate as (2.23) for $\lambda \in [\Lambda, \infty)$, where both of $(\tilde{q}, \tilde{\mu}, \tilde{v})$ and $(\tilde{q}_1, \tilde{\mu}_1, \tilde{v}_1)$ are assumed to satisfy $[\mathrm{A}.1']_{l,\Lambda}$.

3. The Linear C.E.Eq. (II)

We study the evolution operator $U(t, s; \lambda, \tilde{q}, \tilde{v})$ in detail. First we consider the linear C.E.Eq. corresponding to the case $(\tilde{q}, \tilde{v})=(0, 0)$:

(3.1)
$$\frac{\partial}{\partial t} q+\lambda\gamma\bar{p}\nabla\cdot v=0\,,$$
$$\frac{\partial}{\partial t} v+(\lambda/\bar{\rho})\nabla q=0\,,$$

$(3.1)_s$
$$(q, v)\big|_{t=s}=(q_0, v_0)\,.$$

Putting $u={}^t(q, v)$, we rewrite the above equation as below:

(3.2)
$$\frac{\partial}{\partial t} u=-B(\lambda\bar{p}, \lambda/\bar{\rho}, \nabla)u=-\lambda B(\bar{p}, 1/\bar{\rho}, \nabla)u\,,$$

$(3.2)_0$
$$u\big|_{t=s}=u_0={}^t(q_0, v_0)\,.$$

The symbol $B(\xi)=B(\bar{p}, 1/\bar{\rho}, \xi)$ of $B(\bar{p}, 1/\bar{\rho}, \nabla)$ is expressed as

(3.3)
$$B(\xi)=\begin{pmatrix} 0 & \gamma\bar{p}\xi_1 & \gamma\bar{p}\xi_2 & \gamma\bar{p}\xi_3 \\ \xi_1/\bar{\rho} & 0 & 0 & 0 \\ \xi_2/\bar{\rho} & 0 & 0 & 0 \\ \xi_3/\bar{\rho} & 0 & 0 & 0 \end{pmatrix}.$$

We put

(3.4)
$$\mu=(\gamma\bar{p}/\bar{\rho})^{1/2} \quad \text{and} \quad \nu=(\gamma\bar{p}\bar{\rho})^{1/2}\,,$$
$$e_\pm(\xi)={}^t(\pm\nu, {}^t\tilde{\xi})\,, \qquad \tilde{\xi}=\xi/|\xi|=e_1(\xi)\,,$$
$$e_\pm(\xi)={}^t(\pm 1, \nu\,{}^t\tilde{\xi})$$

(3.5)
$$P_1(\xi)=(2\nu)^{-1}\sum\langle\cdot, e^\pm(\xi)\rangle e_\pm(\xi)=\sum P_{1,\pm}(\xi)\,,$$

$$P_{1,\pm}(\xi)=\frac{1}{2}\begin{pmatrix} 1 & \pm v\tilde{\xi}_1 & \pm v\tilde{\xi}_2 & \pm v\tilde{\xi}_3 \\ \pm\tilde{\xi}_1/v & & & \\ \pm\tilde{\xi}_2/v & & \tilde{\xi}_i\tilde{\xi}_j & \\ \pm\tilde{\xi}_3/v & & & \end{pmatrix}$$

(3.6) $$P_0(\xi)=1-\mathrm{P}_1(\xi)\,.$$

Then $P_1(\xi)$ and $P_2(\xi)$ are projections in C^4 and satisfy

(3.7) $$B(\xi)P_{1,\pm}(\xi)=\pm\mu|\xi|P_{1,\pm}(\xi)\,,\qquad B(\xi)P_0(\xi)=0\,.$$

If we take an orthonormal set $\{e_1(\xi)=\xi/|\xi|,\ e_2(\xi),\ e_3(\xi)\}$ of R^3, we have

(3.8) $$P_0(\xi)=\sum_{j=2}^{3}\langle\cdot,e_{0,j}(\xi)\rangle e_{0,j}(\xi)\,,\qquad e_{0,j}(\xi)={}^t(0,{}^te_j(\xi))\,.$$

We define the Fourier transform $\hat{u}(\xi)=Fu(\xi)$ by

(3.9) $$\hat{u}(\xi)=(Fu)(\xi)=(2\pi)^{-3/2}\int e^{-ix\cdot\xi}u(x)dx\,.$$

Noting that $iB(\xi)$ generates a unitary group in C^4:

(3.10) $$e^{-itB(\xi)}=\sum e^{\pm it\mu|\xi|}P_{1,\pm}(\xi)+P_0(\xi)\,,$$

we define an evolution operator $U_0(t)$ acting in H^l by

(3.11) $$U_0(t)=F^{-1}e^{-itB(\xi)}F \equiv e^{-tB(\bar{p},1/\bar{\rho},\nabla)}=U_0(t;\bar{p},\bar{\rho})\,.$$

Clearly we have

(3.12) $$|U_0(t)u_0|_l\le a(v)|u_0|_l \qquad \text{for}\quad u_0\in H^l\,,$$

and the unique solution $u(\lambda,t)$ of (3.1)–$(3.1)_s$ is described by

$$u(\lambda,t)=U_0(\lambda(t-s))u_0\,.$$

We define projections P_0 and P_1 by (3.13). Then they have the property (3.14):

(3.13) $$P_j=F^{-1}P_j(\xi)F\,,\qquad j=0,1\,,$$

(3.14) (i) $|P_j u|_l \leq a(v)|u|_l$, $j=0, 1$, $P_0+P_1=1$,

(ii) $P_0 u=0 \Leftrightarrow P_1 u=u \Leftrightarrow \nabla\times v=0$,

(iii) $P_1 u=0 \Leftrightarrow P_0 u=u \Leftrightarrow q=0$ and $\nabla\cdot v=0$.

(iv) $U_0(t)P_0=P_0 U_0(t)=P_0$,

(v) $U_0(t)P_1=P_1 U_0(t)=F^{-1}(\sum e^{\pm it\mu|\xi|}P_{1,\pm}(\xi))F$.

The following Lemma 3.1 will be proved in the Appendix (I).

LEMMA 3.1. *Let* $u\in H^l_\beta$, $\beta>1$, *or* $u\in H^l$ $(l\geq 2)$. *Then there hold*

$$\|U_0(t)P_1 u_0\|_{l-2}\leq C(1+|t|)^{-1}|u_0|_{l,\beta}, \tag{3.15}$$

$$\|U_0(t)P_1 u_0\|_{l-2+\delta}\to 0 \quad \text{as} \quad t\to\pm\infty, \quad 0\leq\delta<1/2. \tag{3.16}$$

Second we consider the linear C.E.Eq. corresponding to the case $\lambda=0$, i.e., the linear transport equation with a vector field $\tilde{v}=\tilde{v}(t)$:

$$\frac{\partial}{\partial t}w+\tilde{v}(t)\cdot\nabla w=0, \tag{3.17}$$

$$w\big|_{t=s}=w_0. \tag{3.17$_s$}$$

The (backward) characteristic equation associated with the equation (3.17)–(3.17)$_s$ is described as below and easily solved:

$$\frac{d}{dt}X=-\tilde{v}(t; X),\ X|_{t=s}=x\in R^3. \tag{3.18}$$

We denote the solution $X(t, s)$ of (3.18) by

$$X(t, s)=X(t, s, x; \tilde{v})=X(t, s; \tilde{v})(x). \tag{3.19}$$

The mapping $X(t, s; \tilde{v})$ is a C^{l-2}-diffeomorphism in R^3 and has the following property.

LEMMA 3.2. (i) *If* $\tilde{v}(\lambda, t)\in B^j([1, \infty); B^0([0, \tilde{T}]; \tilde{H}^{l-j}))$ *with* $l\geq 3$ *and* $0\leq j\leq l$, *then* $X(t, s, x; \tilde{v})\in C^j([1, \infty); C^0([0, \tilde{T}]^2; C^{l-2-j}))$, $0\leq j\leq l-2$.

(ii) *If we assume further* $\tilde{v}\in B^0([1, \infty]\times[0, T]\setminus(\infty, s); B^{l-2-j}))$, *then* $X(t, s, x; \tilde{v})\in C^0([1, \infty]\times[0, \tilde{T}]^2; C^{l-2-j}))$, $0\leq j\leq l-2$.

Proof. We prove the latter part. First we note

$$|X(t, s, x; \tilde{v})-x|\leq\|\tilde{v}\|_{0,\tilde{T}}|t-s|\leq b_2|\tilde{v}|_{2,\tilde{T}}|t-s|. \tag{3.20}$$

Then, by a simple calculation we obtain

$$(3.21)\qquad \begin{aligned} &|X(t, s, x; \tilde{v}(\lambda)) - X(t, s, x; \tilde{v}(\mu))| \\ &\leq |X(s+\varepsilon, s, x; \tilde{v}(\lambda)) - X(s+\varepsilon, s, x; \tilde{v}(\mu))| \\ &\quad + \int_{s+\varepsilon}^{t} |v(\lambda, r, X(r, s, x; \tilde{v}(\lambda))) - \tilde{v}(\mu, r, X(r, s, x; \tilde{v}(\lambda)))|\, dr \\ &\quad + L\int_{s+\varepsilon}^{t} |X(r, s, x; \tilde{v}(\lambda)) - X(r, s, x; \tilde{v}(\mu))|\, dr\,, \\ &L \geq b_2 \sup_{\lambda} |\tilde{v}(\lambda)|_{3,\tilde{T}} \geq \sup_{\lambda} \|v(\lambda)\|_{1,\tilde{T}} \geq \|\nabla\tilde{v}(\lambda)\|_{0,\tilde{T}}\,. \end{aligned}$$

If we take λ and μ so large that the first integrand is majorized by η, then it follows from the Gronwall inequality

$$(3.22)\qquad \begin{aligned} &|X(t, s, x; \tilde{v}(\lambda)) - X(t, s, x; \tilde{v}(\mu)| \\ &\leq e^{L|t-s|}\{2b_2 \sup_{\lambda} |\tilde{v}|_{2,\tilde{T}}\varepsilon + T\eta\}\,. \end{aligned}$$

This shows that $\{X(t, s, x; \tilde{v}(\lambda))\}$ is uniformly convergent as $\lambda\to\infty$, if $(t, s, x)\in[0, \tilde{T}]^2\times R^3$. Thus $X(t, s, x; \tilde{v}(\lambda))\in C^0([1, \infty]\times[0, T]^2;\ C^0(R^3))$. Differentiating (3.18) by x, we obtain the desired results.

The solution $w(t, x) = w(t, s, x)$ of (3.17)–$(3.17)_s$ is uniquely determined and described by

$$(3.23)\qquad w(t, s, x) = w_0(X(t, s, x; \tilde{v})) \equiv T(t, s; \tilde{v})w_0\,,$$

which is the definition of the transport operator $T(t, s; \tilde{v})$. In a similar way used to prove Lemmas 2.1 and 2.2, we can show

LEMMA 3.3. (i) *Let* $\tilde{v}\in B^0([0, \tilde{T}]; H^l)$ *with* $l\geq 3$ *and* $w_0\in H^j$ *for some* $j, 0\leq j\leq l$. *Then there exists a unique solution* $w\in B^1([0, \tilde{T}]^2; H^j)$ *of* (3.17)–$(3.17)_s$ *described by* (3.23). $w = T(t, s; \tilde{v})w_0$ *satisfies*

$$(3.24)\qquad |T(t, s; \tilde{v})w_0|_j \leq e^{|t-s|b(l)|\tilde{v}|_{l,\tilde{T}}}|w_0|_j\,, \qquad t, s\in[0, \tilde{T}]\,.$$

(ii) *If* $\tilde{v}(\lambda, t)\in M^j([1, \infty); B^0([0, \tilde{T}]; H^{l-j}))$ *with* $l\geq 3$ *and* $0\leq j\leq l$, *then* $w(\lambda, t, s, x) = T(t, s; \tilde{v}(\lambda))w_0$ *enjoys the same property as* $\tilde{v}$.

(iii) *If* $\tilde{v}\in B^0([1, \infty]\times[0, \tilde{T}]\backslash(\infty, s); B^{l-2})$ *in addition to the assumption of* (ii) *and* $w_0\in H^k$, *then* $T(t, s; \tilde{v})w_0\in B^0([1, \infty)\times[0, \tilde{T}]^2; H^k)$ *for* $0\leq k\leq l-2$, *and* $\in B^0([1, \infty]\times[0, \tilde{T}]^2; \tilde{H}^k)$ *for* $l-1\leq k\leq l$.

(iv) *Let* $w_0\in H^l_\beta$. Then $T(t, s; \tilde{v})w_0\in M^j([1, \infty); B^0([0, \tilde{T}]^2; H^{l-j}_\beta)$ *with* $0\leq j\leq l$ (resp. $B^0([1, \infty]\times[0, \tilde{T}]^2; \tilde{H}^l_\beta)$) *under the assumption of* (ii) (resp. (iii)). *The corresponding estimate of* (3.24) *holds with* $|\ \ |_j$ *replaced by* $|\ \ |_{l,\beta}$.

(v) *If* $\tilde{v}$ *and* $\tilde{v}_1$ *satisfy the assumption of* (i) (*or* (ii)), *then there holds for* $1\leq k\leq l$ *with* $m(k) = 2$, $1\leq k\leq 2$, *and* $m(k) = k$, $3\leq k\leq l$:

$$(3.25)\qquad |T(t,s;\tilde{v})w_0 - T(t,s;\tilde{v}_1)w_0|_{k-1} \le Ce^{|t-s|b(l)(|\tilde{v}|_{l,T}+|\tilde{v}_1|_{l,T})}|w_0|_{m(k)}\int_s^t |v-\tilde{v}_1|_{k-1}\,dr\,, \qquad 1\le k\le l\,.$$

The same estimate also holds with $|\tilde{v}(r)-\tilde{v}_1(r)|_{k-1}$ *replaced by* $\|\tilde{v}(r)-\tilde{v}_1(r)\|_{k-1}$.

Proof. The claims (i)–(iv) are easy consequences of Lemma 3.2 and the interpolation theorem. The claim (v) follows from (3.24) and an equality

$$(3.26)\qquad \frac{d}{dr}T(t,r;\tilde{v})T(r,s;\tilde{v}_1) = T(t,r;\tilde{v})\{\tilde{v}_1\cdot\nabla - \tilde{v}\cdot\nabla\}T(r,s;\tilde{v}_1)\,,\ \text{i.e.,}$$

$$T(t,s;\tilde{v}) - T(t,s;\tilde{v}_1) = -\int_s^t T(t,r;\tilde{v})\{\tilde{v}_1\cdot\nabla - \tilde{v}\cdot\nabla\}T(r,s;\tilde{v}_1)dr\,.$$

Third we define an operator $V(t,s;\lambda,\tilde{v}) = V(t,s:\lambda,\bar{p},\bar{\rho},\tilde{v})$ which approximates $U(t,s;\lambda,\tilde{q},\tilde{v})$ *when* $\lambda\to\infty$:

$$(3.27)\qquad V(t,s;\lambda,\tilde{v}) = U_0(\lambda(t-s);\bar{p},\bar{\rho})P_1 + P_0T(t,s;\tilde{v})\,.$$

Remembering (2.22) and noting the equality (which follows from (3.14) (iii))

$$(3.28)\qquad B(\lambda\bar{p},\lambda/\bar{\rho},\nabla)P_0 = 0\,,$$

we have the following equality (on H^1)

$$(3.29)\qquad \frac{d}{dr}\{U(t,r;\lambda,\tilde{q},\tilde{v})V(r,s;\lambda,\tilde{v})\}$$

$$= U(t,r;\lambda,\tilde{q},\tilde{v})\{\tilde{v}\cdot\nabla + B(\tilde{q},\tilde{\rho}_1,\nabla)\}U_0(\lambda(r-s))P_1 + U(t,r;\lambda,\tilde{q},\tilde{v})\{\tilde{v}\cdot\nabla P_0 - P_0(\tilde{v}\cdot\nabla)\}T(r,s;\tilde{v})\,,$$

$$\tilde{\rho}_1 = \lambda/\tilde{\rho} - \lambda/\bar{\rho} = \tilde{q}\phi_1(\lambda^{-1}\tilde{q})/\bar{\rho}\,,$$

where the function $\phi_1\in C^\infty([-2a,2a])$ is defined in (1.21) (ii) with $k=1$, $0\le\phi_1\le 5/3$. From (3.29) it follows

$$(3.30)\qquad U(t,s;\lambda,\tilde{q},\tilde{v}) - V(t,s;\lambda,\tilde{v})$$

$$= -\int_s^t U(t,r;\lambda,\tilde{q},\tilde{v})A(\tilde{v},\tilde{q},\tilde{\rho}_1,\nabla)U_0(\lambda(r-s))P_1\,dr + \int_s^t U(t,r;\lambda,\tilde{q},\tilde{v})\{P_0(\tilde{v}\cdot\nabla) - \tilde{v}\cdot\nabla P_0\}T(r,s;\tilde{v})\,dr\,.$$

Putting

$$(3.31)\qquad Q(r,s;\tilde{v}) = \{P_0(\tilde{v}(r)\cdot\nabla) - (\tilde{v}(r)\cdot\nabla)P_0\}T(r,s;\tilde{v})\,,$$

we set the following Volterra equation for $S(t,r;\tilde{v})$:

$$S(t, s; \tilde{v}) - \int_s^t Q(t, r; \tilde{v}) S(r, s; \tilde{v}) dr = Q(t, s; \tilde{v}) . \tag{3.32}$$

To solve the equation (3.32) we need the following Lemmas 3.4 and 3.5. Lemma 3.4 is proved by using Lemmas 3.3 and 6.1 (Appendix (II)).

LEMMA 3.4. (i) *Let* $\tilde{v} \in B^0([0, \tilde{T}]; H^l)$ *with* $l \geq 3$ *and* $w_0 \in H^k$ *for some* k, $0 \leq k \leq l$. *Then* $Q(t, s; \tilde{v}) w_0 \in B^0([0, \tilde{T}]^2; H^k)$ *and satisfies*

$$| Q(t, s; \tilde{v}) w_0 |_j \leq d(l) | \tilde{v}(t) |_l e^{|t-s| b(l) |\tilde{v}|_{l, \tilde{T}}} | w_0 |_j . \tag{3.33}$$

(ii) *If* $\tilde{v}(\lambda, t) \in M^j([1, \infty); B^0([0, \tilde{T}]; H^{l-j})$ *for* $0 \leq j \leq l$ *and* $w_0 \in H^k$, *then* $Q(t, s; \tilde{v}(\lambda)) w_0$ *enjoys the same property as* $\tilde{v}$ *with* $[0, \tilde{T}]$ *and* l *replaced by* $[0, \tilde{T}]^2$ *and* k $(0 \leq k \leq l)$.

(iii) *If* $\tilde{v} \in B^0([1, \infty] \times [0, \tilde{T}] \backslash (\infty, s); B^{l-2+\delta})$ *with* $\delta > 0$ *in addition to the assumption of* (ii), *then* $Q(t, s; \tilde{v}(\lambda)) w_0 \in B^0([1, \infty] \times [0, \tilde{T}]^2 \backslash (\infty, \Delta); H^k)$ *for* $0 \leq k \leq l-2$. *The same holds for* $l-1 \leq k < l$ *with* H^k *replaced by* $\tilde{H}^k$.

(iv) *If* $w_0 \in H^k_\beta$ $(1 < \beta < 3/2)$, *then the above claims* (i)–(iii) *hold with* H^k (resp. $\tilde{H}^k$) *replaced by* H^k_β (resp. $\tilde{H}^k_\beta$).

(v) *If* $\tilde{v}$ *and* $\tilde{v}_1$ *satisfy the assumption of* (i) (*or* (ii)), *then we have*

$$\begin{aligned} &| Q(t, s; \tilde{v}) w_0 - Q(t, s; \tilde{v}_1) w_0 |_{k-1} \\ &\quad \leq C(| t-s | (| \tilde{v} |_{l, \tilde{T}} + | \tilde{v}_1 |_{l, \tilde{T}})) | w_0 |_{m(k)} \int_s^t | \tilde{v}(r) - \tilde{v}_1(r) |_{k-1} dr \end{aligned} \tag{3.34}$$

with $m(k)$ *defined in Lemma* 3.3 (v). *The same estimate holds with* $| \tilde{v} - \tilde{v}_1 |_{k-1}$ *replaced by* $\| \tilde{v} - \tilde{v}_1 \|_{k-1}$.

LEMMA 3.5. *Let* $\tilde{v} \in M^j([1, \infty); B^0([0, \tilde{T}]; H^{l-j}))$ *with* $l \geq 3$ *and for* $0 \leq j \leq l$. *Then, the equation* (3.32) *has a unique solution* $S(t, s; \tilde{v})$ *enjoying the same properties* (ii)–(v) *stated in Lemma* 3.4 *for* $Q(t, s; \tilde{v})$.

Proof. Associating with the operator $Q(t, r; \tilde{v})$ in H^k, we define an operator Q_s acting in $B^0([0, \tilde{T}]; H^k)$ $(0 \leq k \leq l)$ by

$$(Q_s f)(t) = \int_s^t Q(t, r; \tilde{v}) f(r) dr . \tag{3.35}$$

Clearly $Q_s \in B^0([0, \tilde{T}]; B(B^0([0, \tilde{T}]; H^k)))$ and satisfies

$$| Q_s f(t) |_k \leq | t-s | c(l) | \tilde{v} |_{l, \tilde{T}} e^{|t-s| b(l) | \tilde{v} |_{l, \tilde{T}}} | f |_{k, \tilde{T}} , \qquad 0 \leq k \leq l . \tag{3.36}$$

An easy calculation shows

$$| Q_s{}^n f(t) |_k \leq \frac{| t-s |^n}{(n-1)!} \{ c(l) | \tilde{v} |_{l, \tilde{T}} \}^n e^{|t-s| b(l) | \tilde{v} |_{l, \tilde{T}}} | f |_{k, \tilde{T}} . \tag{3.37}$$

Thus we can obtain the solution $S(t, s; \tilde{v})$ by means of the convergent Neumann series:

(3.38) $$S(t, s; \tilde{v}) = Q(t, s; \tilde{v}) + \int_s^t Q(t, r; \tilde{v})(1-Q_s)^{-1}Q(\cdot, s; \tilde{v})(r)dr = Q(t, s; \tilde{v}) + S_1(t, s; \tilde{v}).$$

$S(t, s; \tilde{v})w_0$ (resp. $S_1(t, s; \tilde{v})$) is continuous in $(t, s) \in [0, \tilde{T}]^2$ in H^k for $w_0 \in H^k$ (resp. in $B(H^k)$), $0 \le k \le l$, and satisfies

(3.39) $$|S(t, s; \tilde{v})w_0|_k \le Ce^{|t-s|C|\tilde{v}|_{l,\tilde{T}}}|w_0|_k, \qquad 0 \le k \le l.$$

The properties of $S(t, s; \tilde{v})$ are proved by using the expression (3.38) and Lemma 3.4.

From (3.30) and (3.32) it follows

(3.40) $$U(t, s; \lambda, \tilde{q}, \tilde{v}) = V(t, s; \lambda, \tilde{v}) + \int_s^t V(t, r; \lambda, \tilde{v})S(r, s; \tilde{v})dr + W(t, s; \lambda) + \int_s^t W(t, r; \lambda)S(r, s; \tilde{v})dr,$$

(3.41) $$W(t, s; \lambda) = -\int_s^t U(t, r; \lambda, \tilde{u})A(\tilde{v}, \tilde{q}, \tilde{\rho}_1, \nabla)U_0(\lambda(r-s))P_1 dr.$$

Noting that $A(\tilde{v}, \tilde{q}, \tilde{\rho}_1, \nabla)$ is a differential operator of 1st order with coefficients $(\tilde{v}, \tilde{q}, \tilde{\rho}_1) \in M^j([1, \infty); B^0([0, \tilde{T}]; H^{l-j}))$ for $0 \le j \le l$, we apply Lemmas 2.2 and 3.1 to investigate the operator $W(t, s; \lambda)$. Then we have

LEMMA 3.6. *Let $\tilde{u} = (\tilde{q}, \tilde{v})$ satisfy* [A.1]$_l$ *and $u_0 = (q_0, v_0) \in H^l$. Then:*

(i) *$W(t, s; \lambda, \tilde{u}(\lambda))u_0$ enjoys the same property as $\tilde{u}(\lambda, t)$ with l and* $[0, \tilde{T}]$ *replaced by $l-1$ and* $[0, T]^2$. *Moreover we have*

(3.42) (i) $|W(t, s; \lambda, \tilde{u}(\lambda))u_0|_{l-1} \le C(\tilde{M})|u_0|_l$,

(ii) $|W(t, s; \lambda, \tilde{u}(\lambda))u_0|_{l-1-\delta} \to 0$ *uniformly in* $(t, s) \in [0, T]^2$ $(\delta > 0)$, *as* $\lambda \to \infty$.

(ii) *If $u_0 \in H^k_\beta$ for some k, $3 \le k \le l$, and β, $1 < \beta < 3/2$, then there holds*

(3.43) $$|W(t, s; \lambda, \tilde{u}(\lambda))u_0|_{k-3} \le C(\tilde{M})|u_0|_{k,\beta}\lambda^{-1}\log(1+\lambda|t-s|).$$

Proof. We have only to prove the claims (3.42) (ii) and (3.43). Lemma 3.1 (3.16) implies $|A(\tilde{v}, \tilde{q}, \tilde{\rho}_1; \nabla)U_0(\lambda(r-s))P_1 u_0|_{l-3} \to 0$ as $\lambda \to \infty$ uniformly in $(r, s) \in [0, T]^2$ if $|r-s| \ge \varepsilon > 0$. Since $U(t, r; \lambda, \tilde{q}, \tilde{v})$ is uniformly bounded in H^{l-3}, the same holds for $U(t, r; \lambda, \tilde{q}, \tilde{v})A(\cdots)U_0(\lambda(r-s))P_1 u_0$. Hence $|W(t, s; \lambda)u_0|_{l-3} \to 0$ uniformly in $(t, s) \in [0, \tilde{T}]^2$ as $\lambda \to \infty$. By using the interpolation theorem, we obtain the desired result. (3.43) follows from Lemma 2.2, Lemmas 3.1 (3.15) and the definition (3.41) of $W(t, s; \lambda, \tilde{u}(\lambda))$.

Combining Lemmas 3.1 and 3.3–3.6, we obtain

LEMMA 3.7. *Assume that $\tilde{u} = (\tilde{q}, \tilde{v})$ satisfies* [A.1]$_l$ *and $u_0 \in H^l$ with $l \ge 3$. Assume* *further that* $\tilde{u} \in B^0([1, \infty] \times [0, \tilde{T}] \backslash (\infty, s); B^{l-2+\delta})$ *for* $0 \le \delta < 1/2$. *Then* $U(t, s, \lambda; \tilde{u})u_0$ *satisfies* $[\mathrm{A}.1]_l$ (i) *and belongs to* $B^0([1, \infty] \times [0, \tilde{T}]^2 \backslash (\infty, \Delta); B^{l-2+\delta})$ *for* $0 \le \delta < 1/2$. *Moreover* $u(\lambda, t) = U(t, s; \lambda, \tilde{u}(\lambda))u_0$ *converges to* $u(\infty, t) = (q(\infty), v(\infty))$ *weakly in* H^l *uniformly in* (t, s), *if* $|t-s| \ge \varepsilon > 0$. $u(\infty) = (q(\infty), v(\infty))$ *satisfies*

$$(3.44)\qquad q(\infty, t) = 0 \quad \textit{and} \quad \nabla \cdot v(\infty, t) = 0 \qquad \textit{for} \quad t \neq s\,,$$
$$\partial v^{\infty}/\partial t + \tilde{v}(\infty) \cdot \nabla v(\infty) + P'S(t, s; v(\infty))u_0 = 0\,, \qquad t \neq s\,.$$

Here P' *denotes the projection into the rotation free subspace*, $P_1 u \equiv (0, P'v)$.

Proof. From (3.27), Lemmas 3.1, 3.3 (v), 3.4 (v) and 3.5 it follows

$$(3.45)\qquad \| V(t, s; \lambda, \tilde{v}(\lambda))u_0 - P_0 T(t, s; \tilde{v}(\infty))u_0 \|_{l-3} \to 0\,,$$
$$\left\| \int_s^t V(t, r; \lambda, \tilde{v}(\lambda))S(r, s; \tilde{v}(\lambda))u_0 dr - \int_s^t P_0 T(t, r; \tilde{v}(\infty))S(r, s; \tilde{v}(\infty))u_0 dr \right\|_{l-3} \to 0$$

uniformly in (r, s) as $\lambda \to \infty$, if $|t-s| \ge \varepsilon > 0$. From Lemma 3.5 (the strong continuity of $S(r, s; \tilde{v}(\lambda))u_0$) and (3.42) (ii) it follows

$$(3.46)\qquad | W(t, s; \lambda, \tilde{u}(\lambda))u_0 |_{l-1-\delta} \to 0\,, \qquad \delta > 0\,,$$
$$\left| \int_s^t W(t, r; \lambda, \tilde{u}(\lambda))S(r, s; \tilde{v}(\lambda))u_0 dr \right|_{l-1-\delta} \to 0\,, \qquad \text{as} \quad \lambda \to \infty\,,$$

uniformly in $(t, s) \in [0, \tilde{T}]^2$. Hence by (3.40) $u(\lambda, t) = U(t, s; \lambda, \tilde{u}(\lambda))u_0$ converges to the limit $u(\infty, t)$ in B^{l-3} as $\lambda \to \infty$. Since $u(\lambda, t)$ is bounded in $B^0([0, \tilde{T}]^2; H^l) \subset B^0([0, T]^2; B^{l-3/2})$ for $\lambda \in [1, \infty)$, we see that $u(\lambda, t)$ tends to $u(\infty, t)$ in $B^{l-2+\delta}$, $0 \le \delta < 1/2$, as $\lambda \to \infty$, by virtue of the interpolation theorem. The convergence is uniform, if $|t-s| \ge \varepsilon > 0$. Clearly $u(\lambda, t)$ converges to $u(\infty, t)$ weakly in H^l uniformly in (t, s), if $|t-s| \ge \varepsilon > 0$. The expression (3.40) gives

$$(3.47)\qquad u(\infty, t) = P_0 T(t, s; \tilde{v}(\infty))u_0 + \int_s^t P_0 T(t, r; \tilde{v}(\infty))S(r, s; \tilde{v}(\infty))u_0 dr\,.$$

This proves $P_1 u(\infty, t) = 0$, which implies the first part of (3.44). Differentiating (3.47) and using (3.32), we obtain the second equality of (3.44), that is,

$$(3.48)\qquad \partial u(\infty)/\partial t + \tilde{v}(\infty) \cdot \nabla u(\infty) + (1 - P_0)S(t, s; \tilde{v}(\infty))u_0 = 0\,, \qquad t \neq s\,.$$

REMARK. We have used the operator $V(t, s; \lambda, \tilde{v}) = V(t, s; \lambda\bar{p}, \lambda/\bar{p}, \tilde{v})$ to approximate $U(t, s; \lambda, \tilde{q}, \tilde{v})$. However we can give an exact decomposition of $U(t, s; \lambda, \tilde{q}, \tilde{v})$ as the product of the pure transport operator and the pure propagation operator. First we consider the propagation equation

$$\frac{\partial}{\partial t} q + \lambda\gamma\tilde{p}_1(t, x)\mathrm{tr}\{C(t, x)\nabla v\} = 0 ,$$

(3.49)

$$\frac{\partial}{\partial t} v + \lambda/\tilde{\rho}_1 C(t, x)\nabla q = 0 ,$$

(3.49)$_s$ $$(q, v)\big|_{t=s} = (q_0, v_0) = u_0 ,$$

where $C(t, x)$ is a smooth 3×3 matrix. Since this equation is symmetric hyperbolic, we have the evolution operator $Z(t, s; C)$ such that the solution $w(t, s, x)$ of (3.49)–(3.49)$_s$ is described as

$$w(t, s) = Z(t, s; \lambda\tilde{p}_1, \lambda/\tilde{\rho}_1, C)u_0 .$$

We take the solution u of (2.1) and note the equality

(3.50) $$\nabla_x(u(t, X(s, t, x; \tilde{v})) = {}^t\left(\frac{\partial X}{\partial x}\right)(\nabla_x u)(t, X(s, t, x; \tilde{v})) , \qquad \text{i.e.,}$$

(3.50)′ $$\nabla_x T(s, t; \tilde{v})u = T^*(s, t; \tilde{v})T(s, t; \tilde{v})(\nabla_x u) \quad \text{where}$$

$$T^*(s, t; \tilde{v}) = {}^t\left(\frac{\partial X}{\partial x}\right)(s, t, x; \tilde{v}) .$$

We put

(3.51) $$u_1(t) = T(s, t; \tilde{v})u(t) .$$

Then we obtain (with the notations defined by (2.18) and (2.19))

(3.52) $$\begin{aligned} \frac{\partial}{\partial t} u_1 &= T(s, t; \tilde{v})\{-B(\lambda\tilde{p}, \lambda/\tilde{\rho}, \nabla_x)u(t)\} \\ &= -T(s, t; \tilde{v})B(\lambda\tilde{p}, \lambda/\tilde{\rho}) \cdot T^*(t, s; \tilde{v})T(t, s; \tilde{v})(\nabla_x u_1) \\ &= -\{B(\lambda T(s, t; \tilde{v})(\tilde{p}, 1/\tilde{\rho})) \cdot \{T(s, t; \tilde{v})T^*(t, s; \tilde{v})\}\nabla_x u_1 . \end{aligned}$$

Thus, if we put

(3.53) $$\begin{aligned} &\tilde{p}_1(t, s, x) = T(s, t; \tilde{v})\tilde{p}(t) , \\ &\tilde{\rho}_1(t, s, x) = T(s, t; \tilde{v})\tilde{q}(t) , \\ &C(t, s; x) = T(s, t; \tilde{v})T^*(t, s, \cdot; \tilde{v})(x) , \end{aligned}$$

then we have $w(t, s) = u_1(t)$, i.e.,

$$Z(t, s; \lambda\tilde{p}_1, \lambda/\tilde{\rho}_1, C)u_0 = T(s, t; \tilde{v})u(t) .$$

This gives

(3.54) $$U(t, s; \lambda\tilde{p}, \lambda/\tilde{\rho}, \tilde{v}) = T(t, s; \tilde{v})Z(t, s; \lambda\tilde{p}_1, \lambda/\tilde{\rho}_1, C).$$

4. Proof of Theorems

Proof of Theorem 1.1. We define the approximating sequence for the solution by

$$(4.1)\qquad 1)\ \ u_1=(q_1, v_1)=U(t, 0; \lambda, q_0, v_0)u_0\,, \qquad u_0=(q_0, v_0)\,,$$
$$n)\ \ u_n=(q_n, u_n)=U(t, 0; \lambda, q_{n-1}, v_{n-1})u_0\,, \qquad n\geq 2\,.$$

Then $u_n=(q_n, v_n)$ satisfies

$$(4.2)\qquad \begin{aligned}&\frac{\partial}{\partial t}q_n+v_{n-1}\cdot\nabla q_n+\lambda\gamma p_{n-1}\nabla\cdot v_n=0\\ &\frac{\partial}{\partial t}v_n+v_{n-1}\cdot\nabla v_n+(\lambda/\rho_{n-1})\nabla q_n=0\,,\end{aligned}$$

$$(4.2)_0\qquad u_n\big|_{t=0}=(q_0, v_0)\,,$$

$$(4.3)\qquad p_{n-1}=\bar{p}(1+\lambda^{-1}q_{n-1})\,, \qquad \rho_{n-1}=\bar{\rho}\{1+\psi_0(\lambda^{-1}q_{n-1})\}\,.$$

We define the equivalent norm $|u_n|_l'$ of $|u_n|_l$ by

$$(4.4)\qquad |u_n|_l'^2=\sum_{|\alpha|\leq l}\left((\gamma p_{n-1})^{-1}\left(\frac{\partial}{\partial x}\right)^\alpha q_n, \left(\frac{\partial}{\partial x}\right)^\alpha q_n\right)+\sum_{|\alpha|\leq l}\left(\rho_{n-1}\left(\frac{\partial}{\partial x}\right)^\alpha v_n, \left(\frac{\partial}{\partial x}\right)^\alpha v_n\right).$$

Let u_0 and u_{n-1} satisfy $[\mathrm{A}.0]_l$ and $[\mathrm{A}.1]_l$, $l\geq 3$, respectively, (and also (2.9) hold) with suitable constants M_3, $\tilde{M}$, M_0 and m. Choose $T\in(0, \tilde{T}]$ satisfying (2.10). Then, by virtue of Lemmas 2.1 and 2.2 we can show that u_n satisfies $[\mathrm{A}.1]_3$ for $(\lambda, t)\in[1, \infty)\times[0, T]$ with the same constants M_3 and M_0, and then satisfies $[\mathrm{A}.1]_l$ with a suitable constant $M=C(M_3, |u_0|_l)$. We have only to divide $[0, T]$ into subintervals $[t_{i-1}, t_i]$, $e^{Tb(l)M_3}|\,t_i-t_{i-1}\,|b(l)\,|u_n|'_{j-3,T}<1$, $1\leq i\leq m$, and apply (2.8) in $[t_{i-1}, t_i]$ for $i=1, 2, \cdots$ and $j=6, 9, \cdots$. Thus we have

$$(4.5)\qquad |u_n|_3'\leq M_3/4 \qquad \text{for}\quad (\lambda, t)\in[1, \infty)\times[0, T]\,,$$

$$(4.5)'\qquad (\gamma\bar{p})^{-1}|q_n|_{l,T}^2+\bar{\rho}|v_n|_{l,T}^2\leq M^2 \qquad \text{for}\quad \lambda\in[1, \infty)\,.$$

From (4.2) it follows

$$(4.6)\qquad \begin{aligned}&\left(\frac{\partial}{\partial t}+v_n\cdot\nabla\right)(q_{n+1}-q_n)+\lambda\gamma p_n\nabla\cdot(v_{n+1}-v_n)\\ &\quad=-(v_n-v_{n-1})\cdot\nabla q_n-\gamma(q_n-q_{n-1})\nabla\cdot v_n=g_n\,,\\ &\left(\frac{\partial}{\partial t}+v_n\cdot\nabla\right)(v_{n+1}-v_n)+(\lambda/\rho_n)\nabla(q_{n+1}-q_n)\\ &\quad=-(v_n-v_{n-1})\cdot\nabla v_n+(\rho_n\rho_{n-1})^{-1}\{q_n\psi_1(\lambda^{-1}q_n)-q_{n-1}\psi_1(\lambda^{-1}q_{n-1})\}\nabla q_n=h_n\,,\end{aligned}$$

$$(4.6)_0 \qquad (q_{n+1}-q_n, v_{n+1}-v_n)|_{t=0}=(0, 0).$$

Applying Lemma 2.1, (4.5)′ and the Sobolev theorem, we obtain

$$(4.7) \qquad |u_{n+1}-u_n|_0' \leq e^{M_0 t} t |(g_n, h_n)|_{0,T}' \leq e^{M_0 t} t K M_3 |u_n - u_{n-1}|_0', \quad \text{or}$$

$$(4.7)' \qquad |u_{n+1}-u_n|_0 \leq e^{M_0 t} t L M_3 |u_n-u_{n-1}|_0, \qquad \lambda \in [1, \infty).$$

Here K and L are constants depending only on γ, $\bar{p}$ and $\bar{\rho}$. If we take $T_0 \in (0, T]$ satisfying

$$(4.8) \qquad e^{M_0 T_0} T_0 L M_3 < 1,$$

then $\{u_n\}$ converges in $H^0 = L^2(R^3)$ uniformly in $(\lambda, t) \in [1, \infty) \times [0, T_0]$. The estimate (4.5)′ and the interpolation theorem show that $\{u_n\}$ converges in $H^{l-\delta}$, $\delta > 0$, uniformly in (λ, t):

$$(4.9) \qquad u(\lambda, t) = \text{s-lim}\, u_n(\lambda, t) \quad \text{in} \quad B^0([1, \infty); B^0([0, T_0]; H^{l-\delta})).$$

By virtue of (4.5)′ u_n also converges weakly in H^l to the same limit $u(\lambda, t)$. Hence $u(\lambda, t) \in M^0([1, \infty); M^0([0, T_0]; H^l))$. By the same argument as in the proof of Lemma 2.2, we can prove that $|u(\lambda, t)|_l$ is continuous in $t \in [0, T_0]$. Thus $u(\lambda, t) \in M^0([1, \infty); B^0([0, T_0]; H^l))$. By successsive differentiation by λ we obtain

$$(4.10) \qquad u(\lambda, t) \in M^i([1, \infty); B^0([0, T_0]; H^{l-i})), \qquad 0 \leq i \leq l.$$

Letting $n \to \infty$ in (4.2), we see that $u(\lambda, t)$ satisfies (1.18)–$(1.18)_0$, i.e., $u(\lambda, t) = U(t, 0; \lambda, u(\lambda))u_0$. The continuous dependence of the solution $u(\lambda, t)$ on (λ, u_0) follows from Lemmas 2.2, 2.3 and the Gronwall inequality.

Lemma 3.7 shows that $u_n(\lambda, t) \in B^0([1, \infty] \times [0, T] \backslash (\infty, 0); B^{l-2+\delta})$, $0 \leq \delta < 1/2$. From an easy interpolation it follows with a suitable constant $b_l > 0$

$$(4.11) \qquad \begin{aligned} &\|u_n(\lambda, t) - u_m(\lambda, t)\|_{l-2+\delta} \\ &\leq b_l |u_n(\lambda, t) - u_m(\lambda, t)|_l^{1-\theta} |u_n(\lambda, t) - u_m(\lambda, t)|_0^\theta \\ &\leq b_l (2LM)^{1-\theta} |u_n(\lambda, t) - u_m(\lambda, t)|_0^\theta, \qquad \text{with} \\ &\theta = (2-\eta)(1-2\delta)/l, \qquad 0 \leq \delta < 1/2, \quad 3/2 < \eta < 2. \end{aligned}$$

This shows $\{u_n(\lambda, t)\}$ converges in $B^{l-2+\delta}$ uniformly in $(\lambda, t) \in [1, \infty] \times [0, T_0] \backslash (\infty, 0)$. Combined with (4.10) and Lemma 3.7, this proves (ii).

The solution $u(\lambda, t) = (q(\lambda, t), v(\lambda, t))$ of the equation (1.18)–$(1.18)_0$ is described as

$$(4.12) \qquad u(\lambda, t) = U(t, 0; \lambda, q, v)u_0.$$

Letting $\lambda \to \infty$, we obtain by virtue of Lemma 3.7

$$(4.13)\qquad u(\infty, t)=P_0T(t, 0; v(\infty))P_0u_0 + \int_0^t P_0T(t, r; v(\infty))P_0S(r, 0; v(\infty))u_0\,dr\,.$$

Hence $u(\infty, t)\in B^0([0, T]; H^l)\cap B^1([0, T]; H^{l-1})$ and satisfies

$$(4.13)_0\qquad u(\infty, 0)=P_0u_0\,,$$

$$(4.14)\qquad \frac{\partial}{\partial t}u(\infty, t)+v(\infty, t)\cdot\nabla u(\infty, t)$$
$$=\{v(\infty)\cdot\nabla P_0-P_0v(\infty)\cdot\nabla\}T(t, 0; v(\infty))P_0u_0$$
$$+\int_0^t\{v(\infty)\cdot\nabla P_0-P_0v(\infty)\cdot\nabla\}T(t, r; v(\infty))P_0S(r, 0; v(\infty))u_0dr$$
$$+P_0S(t, 0; v(\infty))u_0$$
$$=-(1-P_0)S(t, 0; v(\infty))u_0\quad\text{(see (3.32))}\,.$$

In the following we use notations P and $(1-P)$ for the projections of $u={}^t(q, v)$ into the divergence and rotation free parts of the fluid velocity v, respectively. Then we have

$$(4.14)'\qquad \frac{\partial}{\partial t}v(\infty)+v(\infty)\cdot\nabla v(\infty)+(1-P)S(t, 0; v(\infty))u_0=0\,.$$

Since the last term is rotation free, it is described as

$$(4.15)\qquad (1-P)S(t, 0; v(\infty))u_0\equiv 1/\bar{\rho}\nabla q^\infty(t, x)\in B^0([0, T_0]; H^l)\,,$$

$$(4.16)\qquad q^\infty=\bar{\rho}G\nabla\cdot(1-P)S(t, 0; v(\infty))u_0 \in C^0([0, T_0]; W^{l,6}_{\mathrm{loc}}(R^3)\cap B^0)\,.$$

Here G is the inverse (fundamental solution) of $-\Delta$ (Laplacian). The last assertion (4.16) is a trivial consequence of the Sobolev embedding theorem.

REMARK. We have used the well known results about the existence and uniqueness of the solution of the I.E.Eq. to claim (1.8) and (1.9). The I.E.Eq. will be studied in a more conprehensive framework in the forthcoming paper by the author [3].

Proof of Theorem 1.2. We prove (1.16) under the assumption $u_0\in H^l\cap H_2^{k+2}$, $1\le k\le l-2$. First we note that under our assumption

$$(4.17)\qquad \nabla q^\infty=\rho(1-P)S(t, 0; v^\infty)u_0\in\bigcap_{j=0}^{k+2}B^j([0, T_0]; H_\beta^{k+2-j})$$

for any $\beta<3/2$, as well as v^∞. This is a consequence of Lemmas 3.4 and 3.5. Since $H^i_\beta\subset W^{i,6/5}(R^n)$ for $\beta>1$, we have

(4.18) $$q^{\infty} \in \bigcap_{j=0}^{k+2} B^j([0, T_0]; H^{k+2-j}),$$

by virtue of the Sobolev embedding theorem. We put

(4.19) $$\begin{aligned} q(\lambda, t) &= q^0(\lambda, t) + \lambda^{-1} q^{\infty}(t) + \lambda^{-1} q^1(\lambda, t), \\ v(\lambda, t) &= v^0(\lambda, t) + v^{\infty}(t) + \lambda^{-1} v(\lambda, t), \end{aligned}$$

where $u^0(\lambda, t) = {}^t(q^0, v^0) = U_0(\lambda t) P_1 u_0$. u^0 satisfies

(4.20) $$|u^0(\lambda, t)|_l \le b|u_0|_l \quad \text{and} \quad \|u^0(\lambda, t)\|_k \le b|u_0|_{k+2,2}.$$

Substituting (4.19) into (1.18), and using (1.11)–(4.14) and (1.19)–(1.20) we obtain an equation (cf. $(1.12)^1$):

(4.21) $$\begin{aligned} &\frac{\partial}{\partial t}(q^{\infty} + q^1) + v \cdot \nabla(q^{\infty} + q^1) + \lambda \gamma p \nabla \cdot v^1 = -\lambda v \cdot \nabla q^0 - \lambda \gamma \bar{p} q \nabla \cdot v^0, \\ &\frac{\partial}{\partial t} v^1 + v \cdot \nabla v^1 + \lambda/\rho \nabla q^1 + v^1 \cdot \nabla(v^0 + v^{\infty}) \\ &\qquad = -\lambda(v^0 + v^{\infty}) \cdot \nabla v^0 - \lambda v^0 \cdot \nabla v^{\infty} - \lambda/(\gamma \bar{p}) q \phi_1(q/\lambda) q^0, \end{aligned}$$

$(4.21)_0$ $$(q^{\infty} + q^1, v^1)\big|_{t=0} = (0, 0).$$

Here (q, v) is the original solution of (1.18). Rewrite (4.20) as the equation of $w = {}^t(q^1, v^1)$ with the initial data $w_0 = {}^t(-q^{\infty}, 0)$ and the inhomogeneous term $f = {}^t(g, h)$. Then by the argument of section 2, the solution w is expressed as

(4.22) $$\begin{aligned} w = U(t, 0, \lambda, u(\lambda)) w_0 &+ \int_0^t U(t, s, \lambda, u(\lambda)) f(\lambda, s)\, ds \\ &+ \int_0^t U(t, s, \lambda, u(\lambda)) {}^t(0, v^1 \cdot \nabla(v^0 + v^{\infty})(s))\, ds. \end{aligned}$$

By virtue of (4.18) and (4.20) f satisfies

(4.23) $$|f(\lambda, s)|_{k-1} \le c\lambda(1 + \lambda s)^{-1} |u_0|_{k+2,2}.$$

Thus we can show the desired estimate for w.

If we assume that the solution $u = {}^t(q, v)$ has the expansion (1.10) then we obtain the linear C.E.Eq. $(1.12)^j$–$(1.12)^j_0$ for $u^j = {}^t(q^j, v^j)$, $1 \le j \le k$. Applying similar arguments as above, we have the desired results $(1.14)^j$ and $(1.15)^j$, $1 \le j \le k$. Thus we have completed the proof.

5. Appendix (I). The Proof of Lemma 3.1

We have only to prove the following

LEMMA 5.1. *Let $n \ge 2$, $l \ge [n/2] + 1$ and $\beta > [n/2]$. Take ϕ satisfying*

(5.1) $$\phi(w)\in C^m(S^{n-1}), \qquad m>n/2,$$

and define $v(t, x)$ *by*

(5.2) $$v(t, x)=\int e^{ix\cdot\xi}e^{it|\xi|}\phi(\xi/|\xi|)\hat{u}(\xi)d\xi, \qquad u\in H^l_\beta(R^n).$$

Then $v(t, x)$ *satisfies*

(5.3) $$\|v(t)\|_j\le b_{l,n,\beta}\|\phi\|_m|u|_{l,\beta}(1+|t|)^{-(n-1)/2}, \qquad 0\le j<l-n/2.$$

Proof. We have only to show (5.3) for the case $j=0$. First we show two lemmas.

LEMMA 5.2. *There exists a unique continuous restriction mapping from* $H^1(R^n)$ *to* $C^0((0, \infty); L^2(S^{n-1}))$ *defined by* (5.4) *and satisfying* (5.5):

(5.4) $$H^1(R^n)\ni u(\rho\omega)\longmapsto u(\rho\cdot)\in C^0((0, \infty); L^2(S^{n-1})),$$

(5.5) (i) $$|u(\rho\cdot)|_0\le b\rho^{-(n-1)/2}(1+\rho)^{-j}|u|_{0,j}^{1/2}|\nabla u|_{0,j}^{1/2},$$

(ii) $$\int_1^\infty |u(\rho\cdot)|_0^2\rho^{n-1+2j}d\rho\le b^2|u|_{0,j}^2,$$

(iii) $$\int_0^1 |u(\rho\cdot)|_0^2\rho^{n-1-2\gamma}d\rho<b^2|u|_\gamma^2, \qquad 0\le\gamma<n/2.$$

Proof. (i) There exists a trivial inequality:

(5.6) $$|u(\rho\omega)|\le|u(t\omega)|+\int_\rho^t |\omega\cdot\nabla u(s\omega)|ds.$$

Integrating both sides on $[\rho, R]$, using the Schwartz inequality and then choosing a suitable constant R, we obtain

(5.7) $$|u(\rho\omega)|^2\le 2\left(\int_\rho^\infty |u(t\omega)|^2dt\right)^{1/2}\left(\int_\rho^\infty |\nabla u(t\omega)|^2dt\right)^{1/2}.$$

Multiplying ρ^{n-1} and integrating both sides on S^{n-1}, we can show

(5.7)' $$\rho^{(n-1)}\int|u(\rho\omega)|^2d\omega\le\varepsilon\int_{|\xi|\ge\rho}|u(\xi)|^2d\xi+\varepsilon^{-1}\int_{|\xi|\ge\rho}|\nabla u(\xi)|^2d\xi.$$

Taking $\varepsilon>0$ appropriately, we have (5.5) (i) for $j=0$. The continuity of $u(\rho\cdot)$ follows from the continuity of the Lebesgue measure. The case $j>0$ is treated similarly.

(ii) Let $\beta>0$ and $0<a\le t<1$. A simple calculation gives

(5.8)
$$\int_a^t \beta s^{\beta-1}|u(s\omega)|^2 ds$$
$$= [s^\beta |u(s\omega)|^2]_a^t - \int_a^t s^\beta \frac{d}{ds}|u(s\omega)|^2 ds$$
$$\le t^\beta |u(t\omega)|^2 + 2\left\{\int_a^t s^{\beta-1}|u(s\omega)|^2 ds\right\}^{1/2}\left\{\int_a^t s^{\beta+1}|\nabla_x u(s\omega)|^2 ds\right\}^{1/2}.$$

This implies

$$\beta\int_0^t s^{\beta-1}|u(s\omega)|^2 ds \le 2t^\beta |u(t\omega)|^2 + 4/\beta \int_0^t s^{\beta+1}|\nabla u(s\omega)|^2 ds.$$

Integrating both sides in t on $(1/2, 1)$, we have

$$\beta\int_0^{1/2} s^{\beta-1}|u(s\omega)|^2 ds \le 8\int_{1/2}^1 t^{\beta+1}|u(t\omega)|^2 dt + 4/\beta\int_0^1 s^{\beta+1}|\nabla u(s\omega)|^2 ds. \tag{5.9}$$

Putting $\beta = n-2$ and integrating (5.9) on S^{n-1}, we get (5.5) (iii) for $\gamma = 1$. In a similar way we get (5.5) (iii) for a general integer $\gamma \in [0, n/2)$.

LEMMA 5.3 (Sobolev). *Let $W_l^{k,p}(R^n)$ be the space of measurable functions $u(\xi)$ satisfying*

$$|u|_{k,p,l} \equiv \sum_{|\alpha|\le k} |(1+|\xi|^2)^{l/2}\nabla^\alpha u(\xi)|_{L^p} < \infty. \tag{5.10}$$

Assume $k - n/p \le \beta - n/2$ with $2 \le p < \infty$. Then we have the continuous inclusion $H_l^\beta(R^n) \subset W_l^{k,p}(R^n)$ with

$$|u|_{k,p,l} \le b|u|_{\beta,l}. \tag{5.11}$$

Proof of Lemma 5.1 (continued). Let $u \in H_\beta^l(R^n)$ with $l \ge [n/2]+1$ and $\beta > [n/2]$. Then, the integration in (5.2) converges absolutely and there holds

$$|v(t,x)| \le \|\phi\|_0 |\hat{u}|_{0,l}. \tag{5.12}$$

By virtue of (5.5) we can rewrite (5.2) as

$$v(t,x) = \int_{S^{n-1}}\int_0^\infty e^{it\rho}e^{i\rho x\cdot\omega}\phi(\omega)\hat{u}(\rho\omega)\rho^{n-1}d\rho d\omega. \tag{5.2$'$}$$

Noting the estimate (5.5) and the equality

$$\{-i(t+x\cdot\omega)^{-1}\partial/\partial\rho\}^k e^{i\rho(t+x\cdot\omega)} = e^{i\rho(t+x\cdot\omega)},$$

and integrating by parts in ρ $k=[n/2]$ times, we obtain

$$|v(t,x)| \le C(t-|x|)^{-k}|\phi|_0 |\hat{u}|_{k,l}, \qquad t > |x|. \tag{5.13}$$

Hence the proof of Lemma 5.1 is completed, if we prove

(5.14) $$|v(t, x)| \leq C|x|^{-(n-1)/2}\|\phi\|_m|\hat{u}|_{\beta,l}.$$

Take $x=(0, \cdots, 0, y)$, $|x|=y$, and the partition $\{\chi_{-1}, \chi_0, \chi_1\} \subset C_0^\infty(R)$ of unity on $[-1, 1]$ satisfying

$$\operatorname{supp} \chi_j \subset (j-1/\sqrt{2}, j+1/\sqrt{2}), \qquad \chi_j \geq 0, \quad \sum \chi_j \equiv 1 \qquad \text{on} \quad [-1, 1].$$

Then, split $v(t, x)$ into three parts v_j, $j=-1, 0, 1$, by

(5.15) $$v_j(t, x) = \int_0^\infty e^{it\rho}\rho^{n-1}d\rho \int_{S^{n-1}} e^{i\rho y\omega_n}\chi_j(\omega_n)\phi(\omega)\hat{u}(\rho\omega)d\omega.$$

Putting

$$\omega = (\omega'\sigma(\tau), \tau), \quad \omega' \in S^{n-2}, \quad \sigma(\tau) = (1-\tau^2)^{1/2},$$

and using the equality

$$\{(i\rho y)^{-1}\partial/\partial\tau\}^k e^{i\rho y\tau} = e^{i\rho y\tau},$$

we obtain

$$v_0(t, x) = \int_0^\infty \int_{S^{n-2}} e^{it\rho}\rho^{n-1}(-it\rho y)^{-k}d\rho d\omega'$$
$$\times \int_{-1}^1 e^{i\rho y\tau}(\partial/\partial\tau)^k\{x_0(\tau)\phi(\omega'\sigma(\tau), \tau)\hat{u}(\rho\omega)\sigma(\tau)^{n-3}\}d\tau.$$

This proves

(5.16) $$|v_0(t, x)| \leq Cy^{-k}\|\chi_0\|_k\|\phi\|_k|\hat{u}|_{k,l}.$$

In order to estimate $v_1(t, x)$ (and $v_{-1}(t, x)$), we apply the stationary phase method (see e.g. Matsumura [12] Appendix) in a simple version.

Putting

(5.17) $$\omega = (\omega's, \sigma(s)), \qquad \omega' \in S^{n-2}, \qquad s \in [0, 1],$$

we obtain

(5.18) $$v_1(t, x) = \int_0^\infty \int_{S^{n-2}} e^{it\rho}\rho^{n-1}e^{i\rho y}V_1(\rho y, \omega', \rho\omega')d\rho d\omega',$$

$$V_1(\rho y, \omega', \rho\omega') = \int_0^1 e^{i\rho y(\sigma(s)-1)}\chi_1(\sigma(s))\phi(\omega's, \sigma(s))\hat{u}(\rho\omega)s^{n-2}\sigma(s)^{-1}ds.$$

It is easy to verify the following

(5.19) $$\sigma(s) = (1-s^2)^{-1/2} = 1 + (1/2)s^2\chi(s^2), \qquad \chi \in C^\infty([0, 1/2]),$$

χ is monotone increasing on $[0, 1/2]$, $x(0)=1$, $x(1/2) \leq \sqrt{3}$.

Changing the variable s into $r = s\chi(s^2)^{1/2}$ in the definition (5.18) of V_1, we get

(5.20) $$V_1=\int_0^1 e^{-i\rho y r^2/2}\tilde{\chi}_1(r)\phi(\omega'\tilde{s}(r),\,\tilde{\sigma}(r))\hat{u}(\rho\omega'\tilde{s}(r),\,\rho\tilde{\sigma}(r))r^{n-2}\zeta(r)dr\,,$$

with appropriate C^∞-functions $\tilde{s}(r)$, $\tilde{\chi}_1(r)=\chi_1(\tilde{s}(r))$, $\tilde{\sigma}(r)$ and $\zeta(r)$. Using the equality

(5.21) $$\{-(i\rho y)^{-1}\partial/\partial r\}e^{-i\rho y r^2/2}=re^{-i\rho y r^2/2}\,,$$

and integrating by parts $j=[(n-1)/2]$ times in r, we obtain

(5.20)′ $$V_1=(i\rho y)^{-j}\int_0^1 e^{-i\rho y r^2/2}\{(\partial/\partial r)r^{-1}\}^j\{\tilde{\phi}_1(\omega',r)\hat{u}(\rho\omega'\tilde{s}(r),\,\rho\tilde{\sigma}(r))r^{n-2}\}dr\,,$$

with $\tilde{\phi}_1=\tilde{\chi}_1\phi$.

If n is odd, we have only to estimate the terms of the following type:

$$U=y^{-j}\int\int\int_0^{1/\sqrt{2}}\rho^{n-1-j+\alpha}|\nabla^\alpha\hat{u}(\rho\omega's,\,\rho\sigma(s))|s^{n-2-j-\gamma}d\rho d\omega' ds\,,$$

$$\alpha+\gamma\le j=(n-1)/2\,,\qquad 0\le\gamma\le j-1\,.$$

Applying the Hölder inequality, we have

(5.22) $$U\le cy^{-j}\left(\int\int\int\rho^{n-1}(1+\rho)^{lp}|\nabla^\alpha\hat{u}|^p s^{n-2}d\rho d\omega' ds\right)^{1/p}$$
$$\times\left(\int_0^\infty\int_0^{1/\sqrt{2}}\rho^{n-1}(1+\rho)^{-(l+j-\alpha)q}s^{n-2-(j+\gamma)q}d\rho ds\right)^{1/q}$$
$$1/p+1/q=1\,,\qquad 1<q<2<p\,.$$

If we can choose p and q satisfying

(5.23)
(i) $1-1/q=1/p\ge 1/2-(\beta-\alpha)/n\,,$

(ii) $n-1-(l+j-\alpha)q<-1\,,$

(iii) $n-2-(j+\gamma)q>-1\,,$

then the integrals on the right hand side of (5.22) converge by virtue of Lemma 5.2, and we obtain the desired result

(5.24) $$U\le cy^{-j}|\hat{u}|_{\beta,l}\,.$$

We rewrite the condition (5.23) as

(5.23)′
(i) $1/q\le 1+\varepsilon-\alpha/n\,,\qquad (\beta/n=1/2+\varepsilon,\ \varepsilon>-1/(2n))\,,$

(ii) $1/q<1-\alpha/n\,,\qquad (l+j=n)\,,$

(iii) $1/q>(j+\gamma)/(n-1)\,,\qquad \gamma\le\min(j-1,j-\alpha)\,,$ i.e.,

(iii)′ $1/q>1-\max(1,\alpha)/(n-1)\,.$

It is easy to show that there exists an exponent $q=q(\alpha,\gamma)$ satisfying (5.23)′ for each

(α, γ). Thus we have proved (5.14) for odd n.

If n is even, we split V_1 into three parts by

$$(5.24)\qquad V_1 = U_0 + U_1 + U_2,$$

$$U_0 = c_0(i\rho y)^{-j}\int_0^{1/\sqrt{2}} e^{-i\rho y r^2/2}\{\psi(\omega', \rho, 0) + r\partial_r\psi(\omega', \rho, 0)\}dr,$$

$$U_1 = c_0(i\rho y)^{-j}\int_0^{1/\sqrt{2}} e^{-i\rho y r^2/2}dr\int_0^r (r-r')\partial_r^2\psi(\omega', \rho, r')dr',$$

$$\psi(\omega', \rho, r) \equiv \tilde{\chi}_1(r)\phi(\omega'\tilde{s}(r), \tilde{\sigma}(r))\hat{u}(\rho\omega'\tilde{s}(r), \rho\tilde{\sigma}(r)),$$

$$U_2 = (i\rho y)^{-j-1}\int_0^{1/\sqrt{2}} e^{-i\rho y r^2/2}\sum_{|\alpha|+\gamma\le j+1,\,\gamma\le j}\psi_{\alpha,\gamma}(\omega', r)$$
$$\times \rho^{|\alpha|}(\nabla^\alpha\hat{u})(\rho\omega'\tilde{s}(r), \rho\tilde{\sigma}(r))r^{n-2-j-\gamma}dr$$

$$\psi_{\alpha,\gamma} \in C^\infty([0, 1/\sqrt{2}\,]).$$

By the standard calculus we can rewrite U_0 as

$$U_0 = c_0(i\rho y)^{-j}\left(\int_0^\infty - \int_{1/\sqrt{2}}^\infty\right)e^{-i\rho y r^2/2}\psi(\omega', \rho, 0)dr$$
$$+ c_1(i\rho y)^{-j-1}\partial_r\psi(\omega', \rho, 0)$$
$$= c_0(2\pi)^{1/2}(i\rho y)^{-j-1/2}\phi(0', 1)\hat{u}(0', \rho) + 0((\rho y)^{-j-1})|\hat{u}(0', \rho)|$$
$$+ c_1(i\rho y)^{-j-1}\partial_r\psi(\omega', \rho, 0).$$

By virtue of the Sobolev theorem the first two terms are majorized by

$$(\rho y)^{-(n-1)/2}(1 + (\rho y)^{-1/2})(1+\rho)^{-l}|\hat{u}|_{\beta,l}, \qquad l = n/2 + 1,$$

which is integrable over $R^n \simeq (0, \infty) \times S^{n-2} \times [0, \pi]$ with $\rho^{n-1}\sin^{n-2}\theta d\rho d\omega' d\theta$. The last term is majorized by

$$(\rho y)^{-n/2-1}(1+\rho)^{-l}(1+\rho)^l\{|\hat{u}(0', \rho)| + \rho|\nabla\hat{u}(0', \rho)|\},$$

which is also integrable over R^n with the same measure. As for the integrals containing U_1 and U_2 in the integrands we can estimate them in a similar way adapted in the proof for odd n. Thus we obtain the desired result for even n.

6. Appendix (II)

For later use we prove the following

Lemma 6.1. (i) *Let* $v \in W_p^l(R^n)$, $1 < p < \infty$ *and* $l > n/p + 1$, *and* K *be the singular integral operator with the kernel* $K(x)$:

(6.1) (i) *$K(x)$ is of homogeneous degree* $-n$, $K(x) = K(x/|x|)|x|^{-n}$,

(ii) $K(\omega)\in C^l(S^{n-1})$ *with* $\|K\|_{l,S}=\sum_{|\alpha|\le l}\|\partial_x^\alpha K\|_{B^0(S^{n-1})}$,

(iii) $\int_{S^{n-1}} K(\omega)d\omega=0$,

$$(Ku)(x)=v.p.\int_{R^n} K(x-y)u(y)dy\,. \tag{6.2}$$

We define the operator R by

$$R=[(v\partial/\partial x_j),\,K]=(v\partial/\partial x_j)K-K(v\partial/\partial x_j)\equiv R(v)\,. \tag{6.3}$$

Then R is a bounded operator in $W^j{}_p(R^n)$, $0\le j\le l$, *with the estimate*

$$|R(v)u|_{j,p}\le b\|K\|_{2,S}|v|_{l,p}|u|_{j,p}\,,\qquad 0\le j\le l\,. \tag{6.4}$$

(ii) *Let* $n\ge 3$, $l>n/2+1$ *and* $v\in H^l(R^n)$. *Then the operators K and* $R(v)$ *are bounded in* $H^j_\beta(R^n)$, *if* $0\le\beta<n/2$:

$$|Ku|_{j,\beta}\le b\|K\|_{l,S}|u|_{j,\beta}\,,\qquad 0\le j\le l\,, \tag{6.5}$$

$$|R(v)u|_{j,\beta}\le b\|K\|_{l,S}|v|_{l,0}|u|_{j,\beta}\,,\qquad 0\le j\le l\,. \tag{6.6}$$

Proof. First we state the precise definition of the space $W^l_{p,\beta}(\Omega)$:

(6.7) (i) $W^l_{p,\beta}(\Omega)\ni u(x)\Leftrightarrow(1+|x|)^\beta(\partial/\partial x)^\alpha u\in L^p(\Omega)$, $|\alpha|\le l$. The norm $|f|_{l,p,\beta}$ is defined in (5.10).

(ii) We write simply $W^l_{p,0}=W^l_p$, $|\ \ |_{l,p,0}=|\ \ |_{l,p}$, and also $W^l_{2,\beta}=H^l_\beta$, $|\ \ |_{l,2,\beta}=|\ \ |_{l,\beta}$ without confusions.

To prove (i) we apply the method of Mizohata ([13], [14]) established on the basis of the Calderón-Zygmund theory on singular integrals ([5], [6]). By an easy calculation we have (with the abbreviation $(\partial/\partial x)^\alpha=\partial^\alpha$, $\partial/\partial x_j=\partial_j$ and $u_\alpha=\partial^\alpha u$)

$$\begin{aligned}\partial^\alpha\{v\partial_jKu-Kv\partial_ju\}&=\sum_{0\le\gamma\le\alpha}\binom{\alpha}{\gamma}\{(\partial^{\alpha-\gamma}v)K(\partial_j\partial^\gamma u)-K(\partial_j\partial^{\alpha-\gamma}v)(\partial_j\partial^\gamma u)\}\\&\equiv\sum_{|\alpha-\gamma|\le l-n/p-1}\binom{\alpha}{\gamma}\{v_{\alpha-\gamma}K\partial_ju_\gamma-Kv_{\alpha-\gamma}\partial_ju_\gamma\}\\&\quad+\sum_{|\alpha-\gamma|>l-n/p-1}\binom{\alpha}{\gamma}\{v_{\alpha-\gamma}K\partial_ju_\gamma-Kv_{\alpha-\gamma}\partial_ju_\gamma\}\\&\equiv R_1(v)u+R_2(v)u\,.\end{aligned} \tag{6.8}$$

To estimate the first term, we note $v_{\alpha-\gamma}\in B^{1+\theta}(R^n)$. We have only to estimate

$$S(v)u=vK\partial_ju-Kv\partial_ju\,,\qquad v\in B^{1+\theta}(R^n)\,,\quad u\in L^p(R^n)\,, \tag{6.9}$$

or

$$S_\varepsilon(v)u(x)=\int_{|x-y|\geq\varepsilon}\{v(x)-v(y)\}K(x-y)\partial_j u(y)dy\,. \tag{6.10}$$

For a while we assume $u\in W_p^1(R^n)$. We rewrite $S_\varepsilon(v)u$ as

$$\begin{aligned} S_\varepsilon(v)u(x)=&\int_{|x-y|=\varepsilon}\{v(x)-v(y)\}K(x-y)u(y)\nu_j(x-y)dS_y \\ &+\int_{|x-y|>\varepsilon}(\partial_j v(y))K(x-y)u(y)dy \\ &+\int_{|x-y|>\varepsilon}\{v(x)-v(y)\}(\partial_j K(x-y))u(y)dy\,, \\ \equiv\,& S_{1,\varepsilon}(v)u+S_{2,\varepsilon}(v)u+S_{3,\varepsilon}(v)u\,, \end{aligned} \tag{6.11}$$

where $\nu(z)=-\tilde{z}=-z/|z|$. Clearly we have

$$|S_{1,\varepsilon}(v)u|_{0,p}\leq|S^{n-1}|\,\|\nabla v\|_0\|K\|_{0,s}|u|_{0,p}\,, \tag{6.12}$$

$$|S_{2,\varepsilon}(v)u|_{0,p}\leq b\|\nabla v\|_0\|K\|_{1,s}|u|_{0,p}\,. \tag{6.13}$$

(6.13) is a direct consequence of the Calderón-Zygmund theorem. We rewrite $S_{3,\varepsilon}(v)u$ as the sum $S_{4,\varepsilon}(v)u+S_{5,\varepsilon}(v)u$ of integrals on $\{|x-y|\geq 1\}$ and $\{\varepsilon\leq|x-y|\leq 1\}$:

$$S_{3,\varepsilon}(v)u=S_{4,\varepsilon}(v)u+S_{5,\varepsilon}(v)u\,. \tag{6.14}$$

From the Hausdorff-Young thorem (Note $\partial_j K(x)=0(|x|^{-n-1})$.) it follows

$$|S_{4,\varepsilon}(v)u|_{0,p}\leq b\|v\|_0\|K\|_{1,s}|u|_{0,p}\,. \tag{6.15}$$

The Taylor formula gives

$$v(x)-v(y)=\sum_{i=1}^{n}\partial_j v(x)(x_i-y_j)+w(x,y)\,,$$

$$|w(x,y)|\leq\|v\|_{1+\theta}|x-y|^{1+\theta}\,.$$

We rewrite $S_{5,\varepsilon}(v)u$ as

$$\begin{aligned} S_{5,\varepsilon}(v)u=&\sum\partial_i v(x)\int_{\varepsilon<|x-y|<1}(x_i-y_i)(\partial_j K(x-y))u(y)dy \\ &+\int_{\varepsilon<|x-y|<1}w(x,y)(\partial_j K(x-y))u(y)dy\,. \\ \equiv\,& S_{6,\varepsilon}(v)u+S_{7,\varepsilon}(v)u\,. \end{aligned} \tag{6.16}$$

Clearly the Hausdorff-Young inequality gives

$$|S_{7,\varepsilon}(v)u|_{0,p}\leq b\|\nabla v\|_\theta\|K\|_{1,s}|u|_{0,p}\,. \tag{6.17}$$

We note the following equality

(6.18) (i) $x_i\partial_j K(x)=\partial_j\{x_i K(x)\}$, $i \neq j$,

(ii) $x_i\partial_i K(x)=\partial_i\{x_i K(x)\}-K(x)$.

It is well-known that each 1st derivative of the function of homogeneous degree $-(n-1)$ has the mean value zero on S^{n-1}, i.e., its integral on S^{n-1} is zero. (See, e.g., Agmon [2] Lemma 11.1.) Hence we can apply the Calderón-Zygmund theory to estimate $S_{6,\varepsilon}(v)u$ and obtain

(6.19) $$|S_{6,\varepsilon}(v)u|_{0,p}\leq b\|\nabla v\|_0\|K\|_{2,S}|u|_{0,p} .$$

Summing up the above arguments, we have

(6.20) $$|S(v)u|_{0,p}\leq b\|K\|_{2,S}\|v\|_{1+\theta}|u|_{0,p} .$$

Applying (6.20) to $R_1(v)u$ and using the Sobolev theorem to estimate $\|\partial_{\alpha-\gamma}v\|_{1+\theta}$ by $|v|_{l,p}$, we have the desired result

(6.21) $$|R_1(v)u|_{0,p}\leq b\|K\|_{2,S}|v|_{l,p}|u|_{|\alpha|,p} .$$

In the second term $R_2(v)u$ of (6.8), $|\alpha|\geq l-n/p-1+|\gamma|$ and each $u_\gamma\in W^r_p(R^n)$ with $r\geq 2+n/p$ (if exists). Hence $\partial_j u_\gamma$ and $K\partial_j u_\gamma$ belong to $W^{r-1}_p(R^n)\subset B^0(R^n)$. Thus $v_{\alpha-\gamma}K\partial_j u_\gamma$ and $Kv_{\alpha-\gamma}\partial_j u_\gamma$ belong to $L^p(R^n)$ and are estimated as

(6.22) (i) $$|v_{\alpha-\gamma}K\partial_j u_\gamma|_{0,p}\leq b|v_{\alpha-\gamma}|_{0,p}\|K\partial_j u_\gamma\|_0$$
$$\leq b|v|_{l,p}|K\partial_j u_\gamma|_{r-1,p}$$
$$\leq b|v|_{l,p}\|K\|_{1,S}|u|_{|\alpha|,p} ,$$

(ii) $$|Kv_{\alpha-\gamma}\partial_j u_\gamma|_{0,p}\leq b\|K\|_{1,S}|v_{\alpha-\gamma}\partial_j u_\gamma|_{0,p}$$
$$\leq b\|K\|_{1,S}|v|_{l,p}|u|_{|\alpha|,p} .$$

Thus the proof of (i) is completed.

To prove (ii) we first note that $x^\alpha K$ is a bounded operator from $H^0{}_j$ to $H^0{}_{j-|\alpha|}$, if $|\alpha|\leq j<n/2$. This fact is shown from (5.5) (ii)–(iii) of Lemma 5.2 and the following:

(6.23) $$(x_k Ku)\hat{}(\xi)=\hat{K}(\xi)i\partial\hat{u}(\xi)/\partial\xi_k+i(\partial\hat{K}(\xi)/\partial\xi_k)\hat{u}(\xi)$$
$$=(Kx_k u)\hat{}(\xi)+i(\partial_k\hat{K}(\xi))\hat{u}(\xi) ,$$

(6.24) $$|x^\alpha Ku|_0\leq c\|K\|_{1,S}|u|_{0,|\alpha|}+c'\|\hat{K}\|_{|\alpha|,S}|\hat{u}|_{|\alpha|,0}$$
$$\leq b\|K\|_{l,S}|u|_{0,|\alpha|} , \qquad |\alpha|<n/2 .$$

Here we have used the fact that $\hat{K}(\xi)\in C^l(R^n\backslash\{0\})$ and is of homogeneous degree 0, which is proved later in Lemma 6.2. With the notations $S_{i,\varepsilon}(v)u$ defined in the proof of (i) we can see easily

$$(6.25)\quad \text{(i)}\quad |x^\alpha S_{1,\varepsilon}(v)u|_0 \le c\varepsilon^{|\alpha|}\|\nabla v\|_0\|K\|_{0,S}|u|_0 + c\|\nabla v\|_0\|K\|_{0,S}|u|_{0,|\alpha|},$$

$$\text{(ii)}\quad |x^\alpha S_{2,\varepsilon}(v)u|_0 \le b\|K\|_{l,S}|(\partial_j v)u|_{0,|\alpha|} \le b\|K\|_{l,S}\|\nabla v\|_0|u|_{0,\alpha}$$

$$\text{(iii)}\quad |x^\alpha S_{7,\varepsilon}(v)u|_0 \le c\|\nabla v\|_\theta\|K\|_{1,S}|u|_{0,|\alpha|},$$

$$\text{(iv)}\quad |x^\alpha S_{6,\varepsilon}(v)u|_0 \le b\|\nabla v\|_0\|K\|_{2,S}|u|_{0,|\alpha|}.$$

We write $S_{4,\varepsilon}(v)u$ as

$$(6.26)\qquad \begin{aligned} S_{4,\varepsilon}(v)u &= v(x)\int_{|x-y|>1}(\partial_j K(x-y))u(y)dy \\ &\quad + \int_{|x-y|>1}(\partial_j K(x-y))v(y)u(y)dy \\ &\equiv S_{8,\varepsilon}(v)u + S_{9,\varepsilon}(v)u. \end{aligned}$$

Then we have

$$(6.27)\qquad \begin{aligned} x_k S_{8,\varepsilon}(v)u &= v(x)\int_{|x-y|>1}(\partial_j K(x-y))y_k u(y)dy \\ &\quad + v(x)\int_{|x-y|>1}(x_k - y_k)(\partial_j K(x-y))u(y)dy. \end{aligned}$$

By arguments similar to those used to prove (6.19) and (6.24) the $|\quad|_{0,i-1}$ norm of the second term is majorized by $\|v\|_0\|K\|_{i,S}\,|u|_{0,i}$. Clearly the first term is majorized by $\|v\|_0\|K\|_{1,S}\,|u|_{0,1}$. Repeating this argument, we have

$$(6.28)\qquad |x^\alpha S_{8,\varepsilon}(v)u|_0 \le b\|v\|_0\|K\|_{|\alpha|,S}|u|_{0,|\alpha|}.$$

In a similar way we can show

$$(6.29)\qquad |x^\alpha S_{9,\varepsilon}(v)u|_0 \le b\|v\|_0\|K\|_{|\alpha|,S}|u|_{0,|\alpha|}.$$

Summing up the above, we have

$$(6.30)\qquad |x^\alpha S_\varepsilon(v)u|_0 \le b\|v\|_{1+\theta}\|K\|_{|\alpha|,S}|u|_{0,\alpha}.$$

The rest of the proof is easy and omitted. We have only to apply (6.24) repeatedly.

REMARK. Lemma 6.1 is valid, even if $v\in B^1(R^n)$ and $\nabla^2 v\in H^{l-2}(R^n)$.

LEMMA 6.2. *Let $K(x)$ satisfy the condition* (6.1) *with $l\ge 0$. Then the Fourier transform $\hat{K}(\xi)$ of $K(x)$ is of homogeneous degree* 0 *and belongs to $C^l(R^n\backslash\{0\})$.*

Proof. By the well known calculation we have

$$(6.31) \qquad (2\pi)^{n/2}\hat{K}(\xi)=\int_{R^n} K(x)e^{-ix\cdot\xi}dx$$

$$(6.32) \qquad =\int_{S^{n-1}} K(\omega)\left\{\log|\omega\cdot\tilde{\xi}|-\frac{\pi}{2}i\,\mathrm{sgn}(\omega\cdot\xi)\right\}d\omega \qquad (\tilde{\xi}=\xi/|\xi|)$$

$$=\hat{K}_1(\xi)+\hat{K}_2(\xi).$$

Take a neighbourhood U of $e_n=(0,\cdots,0,1)\in S^{n-1}$ and a function $R(\sigma)\in C^\infty(U; SO(n))$ such that $R(\sigma)e_n=\sigma$. Then we have

$$\hat{K}_1(\tilde{\xi})=\int K(R(\tilde{\xi})\omega)\log|\omega_n|d\omega\in C^l(U),$$

which follows from the equality

$$R(\tilde{\xi})\omega\cdot\tilde{\xi}=\omega\cdot{}^tR(\xi)\xi=\omega_n.$$

We have also

$$(6.33) \qquad \partial_\xi^\alpha K_2(\xi)=-\pi i\int K(\omega)\delta^{(|\alpha|-1)}(\omega\cdot\xi)\omega^\alpha d\omega, \qquad 1\le|\alpha|\le l,$$

which belongs to $C^{l-|\alpha|+1}(R^n\backslash\{0\})$. Here δ denotes the Dirac measure on R. Thus the proof is completed.

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