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Operator Theoretic Treatment of Linear Abel Integral Equations of First Kind

Rudolf GORENFLO[†] and Masahiro YAMAMOTO[††]

[†]*Department of Mathematics and Computer Science,*
Free University of Berlin,
Arnimallee 2-6 D-14195 Berlin, Germany
E-mail: gorenflo@math.fu-berlin.de
[††]*Department of Mathematical Sciences,*
The University of Tokyo,
3-8-1 Komaba, Meguro, Tokyo 153, Japan
E-mail: myama@ms.u-tokyo.ac.jp

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We consider a linear Abel integral operator $A_\alpha : L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$(A_\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K(t,s) y(s) ds, \qquad 0 \le t \le 1, \quad 0 < \alpha \le 1.$$

We construct a scale $\{X_\beta\}_{\beta\in\mathbb{R}}$ of Hilbert spaces of functions in $(0,1)$ and relate it with a Hilbert scale of Sobolev spaces. Under suitable assumptions on K, we prove that $\|A_\alpha u\|_{L^2(0,1)}$ gives an equivalent norm in $X_{-\alpha}$. On the basis of this equivalence, we find a lower and upper estimate for the singular values of A_α and, furthermore a Hölder estimate for $\|u\|_{L^2(0,1)}$ by $\|A_\alpha u\|_{L^2(0,1)}$ provided that $\|u\|_{X_q}$ with $q > 0$ is uniformly bounded. Finally we discuss convergence rates of regularized solutions obtained by a Tikhonov method.

Key words: Abel integral equation, Ill-posedness, singular values, conditional stability, Tikhonov's regularization

1. Introduction

In this paper, we consider a linear Abel integral operator $A_\alpha : L^2(0,1) \longrightarrow L^2(0,1)$, defined by

$$(1.1) \quad (A_\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K(t,s) y(s) ds, \quad 0 \le t \le 1, \quad 0 < \alpha \le 1.$$

The function $K = K(t,s)$ is assumed to satisfy the conditions

$$(1.2) \qquad \left\{ \begin{array}{l} K \text{ is continuous on } D = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le 1\}, \\ K(s,s) = 1 \quad \text{for} \ \ 0 \le s \le 1 \\ \text{there exists a decreasing function } k \in L^2(0,1) \text{ such that} \\ \left| \dfrac{\partial K}{\partial s}(t,s) \right| \le k(s) \quad \text{for} \ \ 0 < s \le t \le 1. \end{array} \right\}$$

In particular, the third condition of (1.2) is satisfied if $\frac{\partial K}{\partial s} \in L^\infty(D)$. Moreover, in the case $K(t,s) = 1$, $(t,s) \in D$, the operator A_α is the classical Abel integral operator $J^\alpha : L^2(0,1) \longrightarrow L^2(0,1)$, defined by

$$(J^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds, \qquad 0 \le t \le 1, \quad 0 < \alpha \le 1.$$

For treatises on theory and applications of Abel integral equations, we refer to Gorenflo and Vessella [11], Samko, Kilbas and Marichev [25].

The purpose of this paper is twofold. First we treat the linear Abel integral operator (1.1) by methods of operator theory and establish an isomorphism within a framework of a Hilbert scale which is closely related to a scale of Sobolev spaces over the interval $(0,1)$. Second, by this isomorphism, we study the asymptotic behaviour of the singular values of A_α and a method of Tikhonov regularization. For estimation of the singular values, there are many papers. For example, Dostanić [6], Faber and Wing [7], Gorenflo and Vu Kim Tuan [10], Hille and Tamarkin [17], Vu Kim Tuan and Gorenflo [27]–[29]. Our method is not by "hard analysis", but on the basis of the min-max principle for singular values (e.g. Baumeister [4], Hofmann [18]). Our estimate is not sharper than one given by Vu Kim Tuan and Gorenflo [29], but it is applicable to operators in a class which is a little wider.

For the regularization, we apply results by Natterer [23] (see also Baumeister [4]). See also [24] where the application to the Radon transform is considered and the isomorphism in the Sobolev space of order $-1/2$, is essential.

As for other methods of regularization for general linear Abel integral equations, we refer to Dang Dinh Ang, R. Gorenflo and Dang Dinh Hai [3].

This paper is composed of six sections. §2 provides preliminaries. In §3 we obtain the result concerning the isomorphism. In §4 we prove a lower and upper estimate for the singular values of A_α. Finally in §5 and §6, conditional stability and a Tikhonov type method of regularization are discussed with convergence rates.

2. Preliminaries

Throughout this paper, all functions are complex-valued. As usual, $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. $L^2(0,1)$ is a Hilbert space over $\mathbb{C}$ with the scalar product $(\cdot,\cdot)_{L^2}$ and the norm $\|\cdot\|_{L^2}$:

$$(u,v)_{L^2} = \int_0^1 u(t)\overline{v(t)}dt.$$

For convenience, we also introduce the operators D^k, $k = 0,1,2,3,\dots$ according to

$$(D^k y)(t) = \frac{d^k y(t)}{dt^k}.$$

Then D^k is left-inverse to J^k, that is, $D^k J^k = I =$ the identity operator.

In this section, we form a Hilbert scale which is adequate to the mapping properties of J^1. Let us set

$$\mu_n = \left(n - \frac{1}{2}\right)\pi, \quad \phi_n(t) = \sqrt{2}\cos\mu_n t, \qquad 0 \le t \le 1, \ n \in \mathbb{N}.$$

By $\mathbb{N}$, we denote the set $\{1, 2, 3, \ldots\}$ of natural numbers. Since $\{\mu_n^2\}_{n\in\mathbb{N}}$ is the set of all the eigenvalues of the boundary-value problem,

$$\left\{ \begin{array}{ll} (D^2u)(t) = -\lambda u(t), & 0 < t < 1 \\ \dfrac{du}{dt}(0) = u(1) = 0, & \end{array} \right\}$$

and ϕ_n is an eigenfunction for μ_n^2, $n \in \mathbb{N}$, it is known (e.g. Theorem 6.2 of Chapter 1 of Levitan and Sargsjan [20]) that $\{\phi_n\}_{n\in\mathbb{N}}$ is complete in $L^2(0,1)$. Moreover, $(\phi_n, \phi_m)_{L^2} = 0$ if $n \neq m$, $(\phi_n, \phi_n)_{L^2} = 1$. We conclude that

$$\{\phi_n\}_{n\in\mathbb{N}} \text{ is an orthonormal basis in } L^2(0,1). \tag{2.1}$$

Now for $\beta \in \mathbb{R}$, we form a Hilbert scale X_β on this basis (e.g. Chapter 5 in Baumeister [4], Louis [22]). Let $\beta \in \mathbb{R}$. In span $\{\phi_n\}_{n\in\mathbb{N}}$, we define scalar products and norms by

$$\begin{aligned} (u, v)_{X_\beta} &= \sum_{n=1}^{\infty} \mu_n^{2\beta} (u, \phi_n)_{L^2} \overline{(v, \phi_n)_{L^2}}, \qquad u, v \in \operatorname{span}\{\phi_n\}_{n\in\mathbb{N}} \\ \|u\|_{X_\beta} &= (u, u)_{X_\beta}^{1/2}, \qquad u \in \operatorname{span}\{\phi_n\}_{n\in\mathbb{N}}. \end{aligned} \tag{2.2}$$

The completion of span $\{\phi_n\}_{n\in\mathbb{N}}$ in the norm $\|\cdot\|_{X_\beta}$ is a member X_β of a Hilbert scale. By (2.1), we see that

$$X_0 = L^2(0,1).$$

We note that $\{\mu_n^{-\beta}\phi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in X_β. Further characterization of X_β, $\beta > 0$, in terms of Sobolev spaces is given at the end of this section.

For simplicity, we write J, not J^1. That is, the operator $J : L^2(0,1) \longrightarrow L^2(0,1)$ is given by

$$(Jy)(t) = \int_0^t y(s)ds, \qquad 0 \le t \le 1.$$

First we establish

LEMMA 1.

$$\|Ju\|_{L^2} = \|u\|_{X_{-1}}$$

for all $u \in L^2(0,1)$.

Proof. Let us set $\psi_n(t) = \sqrt{2}\sin \mu_n t$, $0 \le t \le 1$, $n \in \mathbb{N}$. Similarly to (2.1), we see that $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(0,1)$. Therefore, by Parseval's equality, we get

$$\|Ju\|_{L^2}^2 = \sum_{n=1}^{\infty} |(Ju, \psi_n)_{L^2}|^2.$$

On the other hand, by integration by parts and $\phi_n(1) = 0$, $Ju(0) = 0$, we have

$$\begin{aligned}(Ju, \psi_n)_{L^2} &= \int_0^1 Ju(t)\overline{\psi_n(t)}dt = \frac{1}{\mu_n}\int_0^1 u(t)\overline{\phi_n(t)}dt \\ &= \frac{1}{\mu_n}(u, \phi_n)_{L^2},\end{aligned}$$

so that

$$\|Ju\|_{L^2}^2 = \sum_{n=1}^{\infty} \mu_n^{-2}|(u, \phi_n)_{L^2}|^2 = \|u\|_{X_{-1}}^2$$

by the definition of X_{-1}. Thus the proof is complete.

Next

LEMMA 2. $\mathrm{Re}\,(Ju, u)_{L^2} \ge 0$, $u \in L^2(0,1)$. *That is, J is accretive in $L^2(0,1)$.*

Proof. With $\mathrm{Re}\, Ju(t) = \phi(t)$ and $\mathrm{Im}\, Ju(t) = \psi(t)$, we get

$$\begin{aligned}\mathrm{Re}\,(Ju, u)_{L^2} &= \mathrm{Re}\int_0^1 Ju(t)D^1(\overline{Ju(t)})dt \\ &= \int_0^1 \left(\phi(t)(D^1\phi)(t) + \psi(t)(D^1\psi)(t)\right)dt = \frac{1}{2}\left[\phi(t)^2 + \psi(t)^2\right]_{t=0}^{t=1} \\ &= \frac{1}{2}|Ju(1)|^2 \ge 0.\end{aligned}$$

This completes the proof. We refer to Tanabe [26] for the notions of accretivity.

LEMMA 3. $\mathcal{R}(I + J) = L^2(0,1)$. *Here $\mathcal{R}(I + J)$ is the range of $I + J$: $\mathcal{R}(I + J) = (I + J)L^2(0,1)$.*

Proof. By a standard iteration argument, for an arbitrarily complex $\lambda \ne 0$,

$$\begin{aligned}(2.3) \qquad &(\lambda I + J)^{-1}u(t) \\ &= \lambda^{-1}u(t) - \lambda^{-2}\int_0^t \exp\left(-\frac{t-s}{\lambda}\right)u(s)ds, \qquad 0 \le t \le 1.\end{aligned}$$

Therefore, setting $\lambda = 1$, we see that for every $u \in L^2(0,1)$ there exists an element $v \in L^2(0,1)$ such that $(I + J)v = u$. This implies that $\mathcal{R}(I + J) = L^2(0,1)$. Thus the proof is complete.

By Lemmata 2 and 3, we see that J is m-accretive (e.g. Tanabe [26]).

LEMMA 4. *$J : L^2(0,1) \longrightarrow L^2(0,1)$ is m-accretive.*

Consequently we can define the fractional power $J(\alpha)$ of J for $0 < \alpha < 1$ by

$$J(\alpha)u = \frac{\sin\pi\alpha}{\pi}\int_0^{\infty} \lambda^{\alpha-1}(\lambda I + J)^{-1}Ju\, d\lambda, \quad u \in \mathcal{D}(J) = L^2(0,1)$$

(§3 and §4 of Chapter 6 in Fattorini [8]). For fractional powers of closed operators, we further refer to Henry [16], Tanabe [26]. In the context of the Abel integral operator, we refer to §5 and "Notes to §5.7" in §9 in Samko, Kilbas and Marichev [25]. Let us prove that the definition of $J(\alpha)$ is equivalent to that of J^α, the classical Abel integral operator. That is,

LEMMA 5. $(J(\alpha)u)(t) = (J^\alpha u)(t)$, $0 \le t \le 1$, *for* $u \in L^2(0,1)$, $0 < \alpha < 1$.

Proof. By (2.3), for $0 \le t \le 1$, we get

$$\begin{aligned}(\lambda I + J)^{-1} J u(t) &= u(t) - \lambda(\lambda I + J)^{-1} u(t)\\ &= \lambda^{-1} \int_0^t \exp\left(-\frac{t-s}{\lambda}\right) u(s)ds,\end{aligned}$$

so that by the change of variables $\eta = \frac{t-s}{\lambda}$,

$$\begin{aligned}&\frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda I + J)^{-1} J u(t) d\lambda\\ &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-2}\left(\int_0^t \exp\left(-\frac{t-s}{\lambda}\right) u(s)ds\right) d\lambda\\ &= \frac{\sin \pi\alpha}{\pi} \int_0^t u(s)\left(\int_0^\infty \lambda^{\alpha-2} \exp\left(-\frac{t-s}{\lambda}\right) d\lambda\right) ds\\ &= \frac{\sin \pi\alpha}{\pi} \int_0^t u(s)\left(\int_0^\infty \eta^{-\alpha} e^{-\eta} d\eta\right)(t-s)^{\alpha-1} ds\\ &= \frac{\Gamma(1-\alpha)\sin \pi\alpha}{\pi} \int_0^t u(s)(t-s)^{\alpha-1} ds.\end{aligned}$$

Now the well-known formula $\Gamma(1-\alpha)\Gamma(\alpha) = \frac{\pi}{\sin \pi\alpha}$ yields the conclusion.

Thus the fractional power $J(\alpha)$ coincides with J^α, and henceforth we write J^α for denoting the fractional power of J on $L^2(0,1)$.

For later use, we further characterize the norm in $X_{-\beta}$, $\beta > 0$. Let us define an operator $S : L^2(0,1) \longrightarrow L^2(0,1)$ by

$$(Su)(t) = \sum_{n=1}^\infty \mu_n^{-1}(u, \phi_n)_{L^2} \phi_n(t), \qquad u \in L^2(0,1).$$

Obviously $\|u\|_{X_{-1}} = \|Su\|_{L^2}$ and $\mathcal{D}(S) = L^2(0,1)$. Then

LEMMA 6. *S is m-accretive in* $L^2(0,1)$.

Proof. Since $(Su, u)_{L^2} = \sum_{n=1}^\infty \mu_n^{-1}|(u, \phi_n)_{L^2}|^2 \ge 0$ for all $u \in L^2(0,1)$, by the orthonormality of $\{\phi_n\}_{n\in\mathbb{N}}$, it follows that S is accretive in $L^2(0,1)$. Next we have to prove that $\mathcal{R}(\lambda I + S) = L^2(0,1)$ for some $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > 0$. Since S is bounded from $L^2(0,1)$ into $L^2(0,1)$, we can take a sufficiently large $\lambda > 0$ such that $\frac{1}{\lambda}\|S\| < 1$. Here and henceforth $\|S\|$ denotes the operator norm of

$S : L^2(0,1) \longrightarrow L^2(0,1)$. By the Neumann series, $I + \frac{1}{\lambda}S$ is surjective from $L^2(0,1)$ onto $L^2(0,1)$, namely, $\mathcal{R}(\lambda I + S)L^2(0,1) = L^2(0,1)$ for such $\lambda > 0$. By Proposition 2.1.4 in Tanabe [26], S is m-accretive in $L^2(0,1)$.

By Lemma 6, we can define the fractional power S^α, $0 \le \alpha \le 1$, which is explicitly given by

LEMMA 7.

$$S^\alpha u = \sum_{n=1}^{\infty} \mu_n^{-\alpha}(u, \phi_n)_{L^2}\phi_n, \qquad u \in L^2(0,1).$$

Proof. By [26], we see that

$$S^\alpha u = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda I + S)^{-1} S u d\lambda$$

for all $u \in \mathcal{D}(S) = L^2(0,1)$. By the definition of S, we have

$$(\lambda I + S)^{-1} S u = \sum_{n=1}^{\infty} \frac{1}{\lambda\mu_n + 1}(u, \phi_n)_{L^2}\phi_n, \quad \lambda > 0,$$

so that

$$\begin{aligned}
&\frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda I + S)^{-1} S u d\lambda \\
&= \sum_{n=1}^{\infty} \frac{\sin \pi\alpha}{\pi} \left\{ \int_0^\infty \lambda^{\alpha-1}(1 + \lambda\mu_n)^{-1} d\lambda \right\} (u, \phi_n)_{L^2}\phi_n \\
&= \sum_{n=1}^{\infty} \frac{\sin \pi\alpha}{\pi} \mu_n^{-\alpha} \left\{ \int_0^1 (1 - \eta)^{\alpha-1}\eta^{-\alpha} d\eta \right\} (u, \phi_n)_{L^2}\phi_n,
\end{aligned}$$

by the change of variables $\eta = (1 + \lambda\mu_n)^{-1}$. The integral occurring in this sum is equal to $B(\alpha, 1-\alpha) = \frac{\pi}{\sin \pi\alpha}$. See, for example, Abramowitz and Stegun [1] for properties of the beta function. We have completed the proof of the lemma.

Combining Lemma 7 with (2.2), we get

$$\|S^\alpha u\|_{L^2} = \|u\|_{X_{-\alpha}}, \qquad u \in L^2(0,1), \quad 0 \le \alpha \le 1. \tag{2.4}$$

Our succeeding arguments are carried out within the Hilbert scale $\{X_\beta\}_{\beta\in\mathbb{R}}$, but another characterization of the spaces X_β, $\beta > 0$, in terms of Sobolev spaces, may be useful, for example, in order to specify the boundary conditions which $u \in X_\beta$ should satisfy at $t = 0$ and $t = 1$.

Henceforth $H^\beta(0,1)$ and $H_0^\beta(0,1)$ denote the usual Sobolev spaces for $\beta > 0$. For details, the reader may consult Adams [2] or Lions and Magenes [21], for example. As norm in $H^q(0,1)$ with $q \in \mathbb{N}$ we simply take

$$\|u\|_{H^q} = \left(\|u\|_{L^2}^2 + \|D^q u\|_{L^2}^2\right)^{1/2},$$

in place of the more complicated expression

$$\left(\|u\|_{L^2}^2 + \sum_{k=1}^{q} \|D^k u\|_{L^2}^2\right)^{1/2}.$$

This is allowed by the interpolation inequality (e.g. Theorem 4.14 in [2]). For $\beta = m + \gamma$ with $m \in \mathbb{N}$ and $\gamma \in (0,1)$, the norm in $H^\beta(0,1)$ is defined as follows (e.g. [2]):

$$\|u\|_{H^\beta} = \left(\|u\|_{H^m}^2 + |D^m u|_{H^\gamma}^2\right)^{1/2}$$

where

$$|v|_{H^\gamma}^2 = \int_0^1 \int_0^1 \frac{|v(t) - v(s)|^2}{|t-s|^{1+2\gamma}} dt\, ds.$$

Moreover, for all $q \in \mathbb{N}$, there exists a constant $C_1 = C_1(q) > 0$, independent of u, such that

$$C_1^{-1} \|D^q u\|_{L^2} \le \|u\|_{H^q} \le C_1 \|D^q u\|_{L^2} \tag{2.5}$$

for all $u \in H_0^q(0,1)$ (e.g. Corollary 4.16 in [2]). Therefore we adopt $\|D^q u\|_{L^2}$ as norm in $H_0^q(0,1)$. In particular, $H_0^1(0,1) = \{u \in H^1 : u(0) = u(1) = 0\}$ and $\|u\|_{H_0^1} = \|D^1 u\|_{L^2}$.

For $\beta > \frac{1}{2}$, the norm $\|u\|_{X_\beta}$ determines the asymptotic behaviour of the Fourier coefficients $(u, \phi_n)_{L^2}$ as $n \to \infty$, which automatically implies that u must satisfy boundary conditions and belong to a Sobolev space. In fact, we establish

LEMMA 8. (i)

$$\left\{ \begin{array}{l} X_\beta = H^\beta(0,1), \quad 0 \le \beta < \dfrac{1}{2} \\ X_{1/2} = \{u \in H^{1/2}(0,1) : \displaystyle\int_0^1 (1-t)^{-1}|u(t)|^2 dt < \infty\} \\ X_\beta = \{u \in H^\beta(0,1) : u(1) = 0\}, \quad \dfrac{1}{2} < \beta \le 1. \end{array} \right\}$$

Moreover there exists a constant $C = C(\beta) > 0$ such that

$$C^{-1}\|u\|_{X_\beta} \le \|u\|_{H^\beta} \le C\|u\|_{X_\beta}, \qquad u \in X_\beta,$$

if $\beta \in [0,1]$ and $\beta \ne \frac{1}{2}$.

(ii) *Let $\beta \ne m + \frac{1}{2}$, $m \in \mathbb{N}$. Then*

$$H_0^\beta(0,1) \subset X_\beta$$

and there exists a constant $C = C(\beta) > 0$ such that

$$\|u\|_{X_\beta} \le C\|u\|_{H^\beta}, \qquad u \in H_0^\beta(0,1).$$

We prove this lemma in Appendix I.

3. Isomorphism between $X_{-\alpha}$ and $L^2(0,1)$ by A_α

In this section, we establish

THEOREM 1. *Let K satisfy the condition* (1.2) *and* $0 < \alpha \le 1$. *Then there exists a constant $C = C(\alpha) > 0$ such that*

$$C^{-1}\|u\|_{X_{-\alpha}} \le \|A_\alpha u\|_{L^2} \le C\|u\|_{X_{-\alpha}}, \qquad u \in L^2(0,1).$$

REMARK. Identifying the space $L^2(0,1)$ with its dual, we consider the triple

$$H_0^\beta(0,1) \subset L^2(0,1) \subset (H_0^\beta(0,1))'$$

(e.g. [21]). Here $(H_0^\beta(0,1))'$ denotes the dual of $H_0^\beta(0,1)$. Then since

$$H_0^\alpha(0,1) = H^\alpha(0,1) \qquad \text{if } 0 \le \alpha < \frac{1}{2}$$

(Theorem 1.11.1 in [21]), according to the usual notation, we have

$$X_{-\alpha} = H^{-\alpha}(0,1) \qquad \text{if } 0 \le \alpha < \frac{1}{2},$$

we can restate the conclusion of Theorem 1 as

$$C^{-1}\|u\|_{H^{-\alpha}} \le \|A_\alpha u\|_{L^2} \le C\|u\|_{H^{-\alpha}}, \quad u \in L^2(0,1)$$

provided that $0 \le \alpha < \frac{1}{2}$. However, if $\frac{1}{2} < \alpha \le 1$, then Lemma 8 implies that $X_\alpha \supsetneqq H_0^\alpha(0,1)$, so that $X_{-\alpha}$ and $H^{-\alpha}(0,1)$ do not coincide for $\frac{1}{2} < \alpha \le 1$.

For the proof of the theorem, we first show

PROPOSITION 1. *There exists a constant $C = C(\alpha) > 0$ such that, for* $0 < \alpha \le 1$,

$$C^{-1}\|u\|_{X_{-\alpha}} \le \|J^\alpha u\|_{L^2} \le C\|u\|_{X_{-\alpha}}, \qquad u \in L^2(0,1).$$

REMARK. In the case of $0 < \alpha < 1/2$, we can directly prove this proposition by Fourier transform since $X_{-\alpha} = H^{-\alpha}(0,1)$ by Lemma 8 (i).

Proof of Proposition 1. By Lemma 1 in §2 and (2.4), we have $\|Ju\|_{L^2} = \|Su\|_{L^2}$, $u \in L^2(0,1)$. By Lemmata 4 and 6, we can apply the Heinz-Kato inequality (e.g. Theorem 2.3.4 in [26]), so that $\mathcal{D}(J^\alpha) = \mathcal{D}(S^\alpha)$ for $0 \le \alpha \le 1$ and there exists a constant $C = C(\alpha) > 0$ such that

$$C^{-1}\|S^\alpha u\|_{L^2} \le \|J^\alpha u\|_{L^2} \le C\|S^\alpha u\|_{L^2}, \qquad u \in \mathcal{D}(J^\alpha).$$

Since $\mathcal{D}(S^\alpha) = L^2(0,1)$, noting (2.4), we have proved Proposition 1.

Proof of Theorem 1. Let us define an operator $L : L^2(0,1) \longrightarrow L^2(0,1)$ by

$$(Ly)(t) = -\frac{\sin \pi\alpha}{\pi} \int_0^t \left\{ \int_\tau^t (s-\tau)^{-\alpha} \times \frac{\partial}{\partial s}\Big((t-s)^{\alpha-1}(K(t,s) - K(t,t))\Big) ds \right\} y(\tau) d\tau, \quad 0 \le t \le 1.$$

Then similarly to Lemma 1 in D.D. Ang, R. Gorenflo and D.D. Hai [3], L and $(I-L)^{-1}$ are bounded from $L^2(0,1)$ to $L^2(0,1)$. Moreover by Lemma 2 in [3] we get

$$A_\alpha u = (I - L) J^\alpha u, \qquad u \in L^2(0,1) \tag{3.1}$$

and

$$J^\alpha u = (I-L)^{-1} A_\alpha u, \qquad u \in L^2(0,1).$$

See also the proof of Theorem 8.3.2 in [11]. Consequently by Proposition 1, for $u \in L^2(0,1)$, we have

$$\|A_\alpha u\|_{L^2} \le \|I - L\| \, \|J^\alpha u\|_{L^2} \le C\|I-L\| \, \|u\|_{X_{-\alpha}}$$

and

$$\|u\|_{X_{-\alpha}} \le C\|J^\alpha u\|_{L^2} \le C\|(I-L)^{-1} A_\alpha u\|_{L^2} \le C\|(I-L)^{-1}\| \, \|A_\alpha u\|_{L^2}.$$

Thus the proof of Theorem 1 is complete (trivilally, by $\|(I-L)^{-1}\| \ne 0$).

4. Upper and Lower Estimate for Singular Values

Since A_α is injective and compact from $L^2(0,1)$ to $L^2(0,1)$ (e.g. Theorem 4.3.3 in Gorenflo and Vessella [11]), the inverse A_α^{-1} is not continuous with respect to the topology of $L^2(0,1)$. That is, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $L^2(0,1)$ such that $\|x_n\|_{L^2} = 1$, $n \in \mathbb{N}$, but $\|A_\alpha x_n\|_{L^2} \longrightarrow 0$ as $n \to \infty$. This means that the problem of solving $A_\alpha u = f$ with respect to $u \in L^2(0,1)$ for a given $f \in L^2(0,1)$, is ill-posed. Let us recall that $(A_\alpha^* A_\alpha)^{1/2} : L^2(0,1) \longrightarrow L^2(0,1)$ is a selfadjoint compact operator. Its eigenvalues $\{s_n(A_\alpha)\}_{n\in\mathbb{N}}$, enumerated in decreasing order with taking account of their multiplicities, are called the singular values of the operator A_α:

$$s_1(A_\alpha) \ge s_2(A_\alpha) \ge \ldots\ldots, \qquad \lim_{n\to\infty} s_n(A_\alpha) = 0.$$

For the discussion of ill-posedness, the knowledge of the properties of the sequence of the singular values is useful. For general presentations, the reader may consult Baumeister [4], Chapter 4 in Colton and Kress [5], Hofmann [18], Kress [19], Louis [22].

For several types of Abel integral operators, Gorenflo and Vu Kim Tuan [10], Vu Kim Tuan and Gorenflo [27], [28], give explicit forms of singular values with corresponding eigenfunctions in weighted L^2-spaces. For the Radon transform, see Natterer [24]. However, for concrete ill-posed problems, it is in general not only extremely difficult to determine the singular values, but also is not easy to give even upper and lower estimation or asymptotic behaviour. For one-sided upper estimates, we refer to Dostanić [6], Faber and Wing [7], Hille and Tamarkin [17], Gorenflo and Yamamoto [13], Yamamoto [30]. For determination of the principal term in the asymptotic behaviour of singular values of Abel integral operators, the reader may consult Vu Kim Tuan and Gorenflo [28], [29] which are most closely related to our result:

THEOREM 2. *Let K satisfy the condition* (1.2) *and* $0 < \alpha \le 1$. *Then there exists a constant* $C = C(\alpha) > 0$ *such that*

$$C^{-1}n^{-\alpha} \le s_n(A_\alpha) \le Cn^{-\alpha}, \qquad n \in \mathbb{N}.$$

REMARK. We should compare this theorem with Theorems 2–4 in Vu Kim Tuan and Gorenflo [29]. There, for a more general Volterra integral operator $(Lu)(t) = (t^{-\beta}A_\alpha u)(t)$ where the kernel function K satisfies extra conditions, it is proved that

$$C^{-1}n^{-\alpha} \le s_n(t^{-\beta}A_\alpha) \le Cn^{-\alpha}$$

if $\alpha > 0$ and $0 \le \beta \le \frac{\alpha}{2}$ (Theorem 2). Moreover, under a stronger condition on K and suitable restrictions on α and β, they prove

$$s_n(t^{-\beta}A_\alpha) = (n\pi)^{-\alpha}(1 + o(1))$$

(Theorems 3 and 4). By our method, we cannot determine the principal term as precisely as these authors .

Proof. The keys are Theorem 1 in §3 and a comparison property of singular values (e.g. Lemma 2.46 in Chapter 2 in Hofmann [18]): Let H_1, H_2, H_3 be separable infinite-dimensional Hilbert spaces, $A : H_1 \longrightarrow H_2$ be a linear compact operator such that $\dim \mathcal{R}(A) = \infty$, and let $H_1 \subset H_3$ such that the embedding $E : H_1 \longrightarrow H_3$ is a compact operator with singular values $s_n(E)$, $n \in \mathbb{N}$. If there exist constants $C_2, C_3 > 0$ such that

$$C_2\|x\|_{H_3} \le \|Ax\|_{H_2} \le C_3\|x\|_{H_3}, \qquad x \in H_1,$$

then

$$C_2 s_n(E) \le s_n(A_\alpha) \le C_3 s_n(E), \qquad n \in \mathbb{N}.$$

To proceed further we state

LEMMA 9. *Let $\beta > 0$. The embedding $E_\beta : X_0 \longrightarrow X_{-\beta}$ is compact. Moreover*

$$s_n(E_\beta) = \left(\left(n - \frac{1}{2}\right)\pi\right)^{-\beta}, \qquad n \in \mathbb{N}.$$

Proof of Lemma 9. For the compactness, see, e.g. Lemma 5.1 in Baumeister [4]. Trivially,

$$E_\beta \phi_n = \mu_n^{-\beta}(\mu_n^\beta \phi_n), \qquad n \in \mathbb{N},$$

and from the facts that $\{\phi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of X_0 and $\{\mu_n^\beta \phi_n\}_{n\in\mathbb{N}}$ is one of $X_{-\beta}$ it follows that $\{\mu_n^{-\beta}, \phi_n, \mu_n^\beta \phi_n\}_{n\in\mathbb{N}}$ is a singular system of E_β. Thus the proof of Lemma 9 is complete.

Now the proof of Theorem 2 is straightforward. In the comparison property of singular values, we set $H_1 = H_2 = L^2(0,1)$ and $H_3 = X_{-\alpha}$ and we apply Lemma 9 to complete the proof of Theorem 2.

5. Conditional Stability

Because of the compactness of $A_\alpha : L^2(0,1) \longrightarrow L^2(0,1)$, we cannot give an upper estimate of $\|u\|_{L^2}$ by $\|A_\alpha u\|_{L^2}$ for all $u \in L^2(0,1)$. However if we assume that u lies in a compact subset of $L^2(0,1)$, then we can restore the continuity of A_α^{-1}. This is meant by the notion of the conditional stability, here of the problem $A_\alpha u = f$, and it is important to discuss the order of the restored continuity. In this section, assuming u to lie in a bounded set in X_q, $q > 0$, we derive such orders:

THEOREM 3. *Let $0 < \alpha \le 1$. For given $q > 0$ and $M > 0$, set*

$$\mathcal{U}_{M,q} = \{u \in X_q : \|u\|_{X_q} \le M\}.$$

Then there exists a constant $C = C(q) > 0$ such that

$$\|u\|_{L^2} \le C(q) M^{\alpha/(q+\alpha)} \|A_\alpha u\|_{L^2}^{q/(q+\alpha)}, \qquad u \in \mathcal{U}_{M,q}.$$

REMARK. Let $0 < q \le 1$ and $q \ne \frac{1}{2}$. By Lemma 8 (i), we can replace the definition of $\mathcal{U}_{M,q}$ in this theorem by

$$\widetilde{\mathcal{U}_{M,q}} = \begin{cases} \{u \in H^q(0,1) : \|u\|_{H^q} \le M\}, & 0 < q < \dfrac{1}{2} \\ \{u \in H^q(0,1) : \|u\|_{H^q} \le M,\ u(1) = 0\}, & \dfrac{1}{2} < q \le 1. \end{cases}$$

Moreover, by Lemma 8 (ii), if $q \ne m + \frac{1}{2}$, $m \in \mathbb{N}$, then we can similarly replace $\mathcal{U}_{m,q}$ by

$$\widetilde{\mathcal{U}_{M,q}} = \{u \in H_0^q(0,1) : \|u\|_{H^q} \le M\}.$$

Proof. The proof is based on the the argument in p. 100 of Baumeister [4]. That is, by the interpolation inequality (e.g. Theorem 5.3 in [4]), we see that

$$\|u\|_{L^2} \le \|u\|_{X_q}^{\alpha/(q+\alpha)} \|u\|_{X_{-\alpha}}^{q/(q+\alpha)}, \qquad u \in X_q.$$

Noting that $u \in \mathcal{U}_{M,q}$, we apply Theorem 1 in §3, so that the proof is complete.

Notice for example that $u \in \mathcal{U}_{M,1}$ implies $u(1) = 0$. In cases of $q = 1$ and $q = 2$, we should compare this theorem with Theorems 8.3.2 and 8.3.4 in Gorenflo and Vessella [11], where no boundary conditions at $t = 0, 1$ such as $u(0) = u(1) = 0$ are assumed for u. The advantage of this theorem is, however, that we can derive conditional stability also for "nonsmooth" functions which are not covered by the above-mentioned theorems in [11].

Example. For given $M > 0$ and $N \in \mathbb{N}$, let us set

$$\mathcal{V}_{N,M} = \left\{ \sum_{j=1}^{N} \nu_j \chi_{[a_j,b_j]} : |\nu_j| \le M,\ 0 \le a_j < b_j \le 1,\ 1 \le j \le N \right\}$$

where $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$. Let us estimate $\|\chi_{[a,b]}\|_{X_q}$ for $0 < q < \frac{1}{2}$. By (2.2) we have

$$\begin{aligned}
\|\chi_{[a,b]}\|_{X_q} &= \left(\sum_{n=1}^{\infty} \mu_n^{2q} \left(\int_a^b \sqrt{2} \cos \mu_n t dt \right)^2 \right)^{1/2} \\
&= \sqrt{2} \left(\sum_{n=1}^{\infty} \mu_n^{2q-2} (\sin \mu_n b - \sin \mu_n a)^2 \right)^{1/2} \\
&\le 2\sqrt{2}\pi^{q-1} \left(\sum_{n=1}^{\infty} \left(n - \frac{1}{2} \right)^{2q-2} \right)^{1/2} \equiv \widetilde{C(q)} < \infty.
\end{aligned}$$

Here we note that $\widetilde{C(q)} < \infty$ by $2q - 2 < -1$, and $\widetilde{C(q)}$ is independent of $a, b \in [0, 1]$ and $\widetilde{C(q)} \longrightarrow \infty$ as $q \uparrow \frac{1}{2}$.

Thus for fixed $q \in (0, \frac{1}{2})$, we see that $\chi_{[a,b]} \in X_q$ and $\|\chi_{[a,b]}\|_{X_q}$ is uniformly bounded for all $a, b \in [0, 1]$. Therefore

$$\|u_1 - u_2\|_{X_q} \le 2MN\widetilde{C(q)}$$

if $u_1, u_2 \in \mathcal{V}_{N,M}$, so that Theorem 3 is applicable: for $q \in (0, \frac{1}{2})$, there exists a constant $\widehat{C(q)} > 0$ such that

$$\|u_1 - u_2\|_{L^2} \le \widehat{C(q)} M^{\alpha/(q+\alpha)} \|A_\alpha u_1 - A_\alpha u_2\|_{L^2}^{q/(q+\alpha)}, \qquad u_1, u_2 \in \mathcal{V}_{N,M}.$$

In particular,

$$|[a_1, b_1] \ominus [a_2, b_2]|^{1/2} \le \widehat{C(q)} \|A_\alpha \chi_{[a_1,b_1]} - A_\alpha \chi_{[a_2,b_2]}\|_{L^2}^{q/(q+\alpha)}$$

if $0 \le a_1 < b_1 \le 1$ and $0 \le a_2 < b_2 \le 1$. Here $[a_1, b_1] \ominus [a_2, b_2]$ denotes the symmetric difference between $[a_1, b_1]$ and $[a_2, b_2]$, namely, $[a_1, b_1] \ominus [a_2, b_2] = ([a_1, b_1] \setminus [a_2, b_2]) \cup ([a_2, b_2] \setminus [a_1, b_1])$, and $|\cdot|$ is the sum of lengths of intervals.

Now what is the meaning of conditional stability within our framework if for $u \in H^q(0,1)$ we cannot assume any boundary conditions at $t = 1$? Unfortunately for general $u \in H^q(0,1)$, Theorem 3 cannot give results as good as those presented in Theorems 8.3.2 and 8.3.4 of Gorenflo and Vessella [11] for special cases $q = 1$ and $q = 2$. It yields, however, an interior estimate.

THEOREM 4. *Let $0 < \alpha \le 1$ and $\epsilon \in (0, \frac{1}{2})$. Then for $q > 0$ there exists a constant $C(q) > 0$ such that*

$$\|u\|_{L^2(\epsilon, 1-\epsilon)} \le C(q) \epsilon^{-[q]-1} M^{\alpha/(q+\alpha)} \|A_\alpha u\|_{L^2}^{q/(q+\alpha)}$$

provided that $u \in H^q(0,1)$ and $\|u\|_{H^q} \le M$. Here $[\cdot]$ denotes the greatest integer not exceeding q.

Proof. First we state a lemma that we shall prove in Appendix II.

LEMMA 10. *Take a function $\chi \in C^\infty(0,1)$ and set $\tilde{u}(t) = \chi(t) u(t)$, $0 \le t \le 1$ for $u \in H^q(0,1)$. Then there exists a constant $C_4 = C_4(q, \chi) > 0$, independent of u, such that*

$$\|\tilde{u}\|_{H^q} \le C_4(q, \chi) \|u\|_{H^q}.$$

Let us now carry out the proof of of Theorem 4. We set

$$l = [q] + 1, \qquad \chi(t) = t^l (1-t)^l.$$

Then $(D^k \chi)(0) = (D^k \chi)(1) = 0$, $0 \le k \le [q]$. Let $\tilde{u} = \chi u$ for $u \in H^q(0,1)$ satisfy $\|u\|_{H^q} \le M$. Then by Lemma 10, we see

$$\tilde{u} \in H_0^q(0,1), \qquad \|\tilde{u}\|_{H_0^q} \le C_4(q) M. \tag{5.1}$$

On the other hand, by integration by parts, we have

$$(J\tilde{u})(t) = J(\chi u)(t) = \chi(t) Ju(t) - \int_0^t Ju(s)(D^1 \chi)(s) ds.$$

By induction, noting that $(J^k u)(0) = 0$, $u \in L^2(0,1)$, $k \in \mathbb{N}$ and $(D^{2l+1}\chi)(t) = 0$, $0 \le t \le 1$, we get

$$(J\tilde{u})(t) = \sum_{k=0}^{2l} (-1)^k (D^k \chi)(t)(J^{k+1} u)(t), \quad 0 \le t \le 1. \tag{5.2}$$

Since $J : L^2(0,1) \longrightarrow L^2(0,1)$ is bounded, the relation (5.2) implies

$$\|J\tilde{u}\|_{L^2} \le C_5 \|Ju\|_{L^2}.$$

Let us define $B : L^2(0,1) \longrightarrow L^2(0,1)$ by $(Bu)(t) = \chi(t)u(t)$, $0 < t < 1$. Then B is bounded from $L^2(0,1)$ into itself. Moreover, since J is an m-accretive operator in $L^2(0,1)$, we can apply the Heinz-Kato inequality (Tanabe [26]). Therefore we get

$$\|J^\alpha \widetilde{u}\|_{L^2} = \|J^\alpha Bu\|_{L^2} \le C_6(\alpha)\|J^\alpha u\|_{L^2}, \qquad u \in L^2(0,1).$$

On the other hand, by (3.1),

$$\|A_\alpha \widetilde{u}\|_{L^2} = \|(I-L)J^\alpha \widetilde{u}\|_{L^2} \le \|I-L\|\,\|J^\alpha \widetilde{u}\|_{L^2}$$

and

$$\|J^\alpha u\|_{L^2} = \|(I-L)^{-1}A_\alpha u\|_{L^2} \le \|(I-L)^{-1}\|\,\|A_\alpha u\|_{L^2}$$

follow, so that

$$\|A_\alpha \widetilde{u}\|_{L^2} \le C_6(\alpha)\|I-L\|\,\|J^\alpha u\|_{L^2} \le C_6(\alpha)\|I-L\|\,\|(I-L)^{-1}\|\,\|A_\alpha u\|_{L^2}.$$

Moreover by (5.1) and Lemma 8 (ii), we can apply Theorem 3 to obtain

$$\begin{aligned}\|\widetilde{u}\|_{L^2} &\le C(q)M^{\alpha/(q+\alpha)}\|A_\alpha \widetilde{u}\|_{L^2}^{q/(q+\alpha)}\\ &\le C(q)(C_6\|I-L\|\,\|(I-L)^{-1}\|)^{q/(q+\alpha)}M^{\alpha/(q+\alpha)}\|A_\alpha u\|_{L^2}^{q/(q+\alpha)}.\end{aligned}$$

Recalling that $\widetilde{u}(t) = t^l(1-t)^l u(t)$, $0 \le t \le 1$, we see that

$$|\widetilde{u}(t)| \ge \epsilon^l(1-\epsilon)^l|u(t)| \ge 2^{-l}\epsilon^l|u(t)|, \qquad \epsilon \le t \le 1-\epsilon,$$

and so

$$\|\widetilde{u}\|_{L^2(0,1)} \ge 2^{-l}\epsilon^l\|u\|_{L^2(\epsilon,1-\epsilon)}.$$

Thus the proof of Theorem 4 is complete.

6. Tikhonov Type Regularization

As we have already observed, the problem of determining u from the equation

$$A_\alpha u = f,$$

$f \in L^2(0,1)$ given, is ill-posed. We now discuss recovery of $u \in L^2(0,1)$ from an approximate equation

$$A_\alpha u \approx \phi, \qquad \phi \in L^2(0,1),$$

which means that instead of the true right hand side we have at our disposal only a perturbed right hand side ϕ, the perturbation happening within $L^2(0,1)$ and being bounded by the "noise level" $\delta > 0$ as follows:

$$\|f-\phi\|_{L^2} \le \delta.$$

As a-priori information about the true solution u, we assume

$$(6.1) \qquad u \in X_q, \quad \|u\|_{X_q} \le M, \quad q > \alpha,$$

with given positive numbers M and q. The number q contains information on the smoothness (or regularity) of u.

Our Tikhonov type regularization method for recovering u consists of minimizing the functional

$$\|A_\alpha v - \phi\|_{L^2}^2 + \gamma\|v\|_{X_p}^2$$

by appropriate choice of v in X_p.

Here $p > 0$ is fixed and the positive number γ is the regularization parameter. It is known (see e.g. Baumeister [4], Groetsch [14]) that there exists a unique minimizer $u^{\delta,\gamma,p}$ for given $\phi \in L^2(0,1)$. The next question then is the convergence of $u^{\delta,\gamma,p}$ towards the true solution u as the noise level δ tends to zero and γ is properly chosen in dependency on δ and M. Of course, we want this convergence to be as fast as possible. There are several possible strategies to achieve this aim (see e.g. [4] and [14]). We follow the idea of Natterer [23] (see also §3 of Chapter 6 in [4]) to get the convergence rate.

THEOREM 5. *Assume* (6.1) *and take*

$$p \ge \frac{q-\alpha}{2}, \qquad \gamma = c_7\left(\frac{\delta}{M}\right)^{2(\alpha+p)/(\alpha+q)}$$

with a constant $c_7 > 0$. *Then*

$$\|u - u^{\delta,\gamma,p}\|_{L^2} = O\left(\delta^{q/(\alpha+q)}M^{\alpha/(\alpha+q)}\right)$$

as $\delta \longrightarrow 0$.

COMMENTS. This theorem gives the optimal convergence rate

$$O\left(\delta^{q/(\alpha+q)}M^{\alpha/(\alpha+q)}\right)$$

by the proposed choice of γ. The larger we can take q (that is, the more we know the degree of smoothness of u), the better this rate is. However, the exponent $\frac{q}{\alpha+q}$ cannot reach the ideal ("best thinkable") value 1.

Proof of Theorem 5. This theorem is, by Theorem 1, an immediate consequence of Natterer's Theorem 1 in [23] (or of the arguments presented in Baumeister [4], Section 6.3). This ends the proof.

If $q > \frac{1}{2}$, then $u \in X_q$, by Lemma 8, implies that $u(t)$ must vanish at the boundary $t = 1$. What can be said about the convergence of the regularized solutions $u^{\delta,\gamma,p}$ if $u \in H^q(0,1)$ but $u \notin X_q$? In this case, as in §5, our method does not give the best result. As for overcoming the constraint of the zero boundary condition at $t = 1$, we can, in the special case $q = 1$, refer to Dang Ding Hai and Dang Dinh Ang [15] and to Gorenflo and Yamamoto [12]. Now, by a method different from theirs we shall derive a local estimate.

Let us set

$$l = [q] + 1, \qquad \chi(t) = t^l(1-t)^l \quad \text{for } \ 0 \le t \le 1$$

and

$$R_l\phi = (I-L)\sum_{k=0}^{2l}(-1)^k D^{1-\alpha}(D^k\chi J^{k+1-\alpha}(I-L)^{-1}\phi).$$

Here and henceforth we define the operator D^β of fractional differentiation by

$$D^\beta v = DJ^{1-\beta}v, \quad \text{for } \ 0<\beta<1, \tag{6.2}$$

as long as the right hand side is well-defined. Concerning the operator $I - L$ that is a bounded operator from $L^2(0,1)$ onto itself, the reader should recall formula (3.1), namely, $A_\alpha = (I-L)J^\alpha$. In order to obtain convergence rates of regularized solutions in a compact subinterval of $(0,1)$, we modify the regularizing functional, so that we have to minimize

$$\|A_\alpha v - R_l\phi\|_{L^2}^2 + \gamma\|v\|_{X_p}^2$$

by appropriate choice of v in X_p. We denote the uniquely existing minimizer by $u_l^{\delta,\gamma,p}$.

THEOREM 6. *Assume*

$$u \in H^q(0,1), \quad \|u\|_{H^q} \le M, \quad q > \alpha,$$

with given numbers $M > 0$ *and* $q > \alpha$, *and take*

$$p \ge \frac{q-\alpha}{2}, \qquad \gamma = c_7\left(\frac{\delta}{M}\right)^{2(\alpha+p)/(\alpha+q)}$$

(*as in Theorem* 5). *Then for every* $\epsilon \in (0, \frac{1}{2})$, *we have*

$$\left\| u - \frac{u_l^{\delta,\gamma,p}}{\chi}\right\|_{L^2(\epsilon,1-\epsilon)} = O\left(\epsilon^{-[q]-1}\delta^{q/(\alpha+q)}M^{\alpha/(\alpha+q)}\right)$$

as $\delta \longrightarrow 0$.

Proof. Set $A_\alpha u = f$ and $\widetilde{u}(t) = \chi(t)u(t) = t^l(1-t)^l u(t)$ for $0 \le t \le 1$. By (6.2) and the semi-group property of the powers of J, we have

$$D^{1-\alpha}J = (DJ^\alpha)(J^{1-\alpha}J^\alpha) = D(J^\alpha J^{1-\alpha})J^\alpha = DJJ^\alpha = J^\alpha.$$

Hence (5.2) implies

$$\begin{aligned} J^\alpha \widetilde{u} &= \sum_{k=0}^{2l} (-1)^k D^{1-\alpha}(D^k \chi J^{k+1} u) \\ &= \sum_{k=0}^{2l} (-1)^k D^{1-\alpha}(D^k \chi J^{k+1-\alpha}(I-L)^{-1} f). \end{aligned}$$

For the last "=" we have used

$$\begin{aligned} J^{k+1} u &= J^{k+1-\alpha}(I-L)^{-1}(I-L) J^\alpha u \\ &= J^{k+1-\alpha}(I-L)^{-1} A_\alpha u = J^{k+1-\alpha}(I-L)^{-1} f. \end{aligned}$$

By (3.1) we now see that

$$A_\alpha \widetilde{u} = (I-L) \sum_{k=0}^{2l} (-1)^k D^{1-\alpha}(D^k \chi J^{k+1-\alpha}(I-L)^{-1} f),$$

so that

$$A_\alpha \widetilde{u} - R_l \phi = (I-L) \sum_{k=0}^{2l} (-1)^k D^{1-\alpha}(D^k \chi J^{k+1-\alpha}(I-L)^{-1}(f-\phi)).$$

Since $I-L$ is bounded, we get

$$\|A_\alpha \widetilde{u} - R_l \phi\|_{L^2} \leq C_8 \sum_{k=0}^{2l} \|D^{1-\alpha}(D^k \chi J^{k+1-\alpha} \psi)\|_{L^2}$$

with $\psi = (I-L)^{-1}(f-\phi)$. For the estimation of the terms

$$\|D^{1-\alpha}(D^k \chi J^{k+1-\alpha} \psi)\|_{L^2}, \qquad 0 \leq k \leq 2l,$$

we use the Leibniz formula for the fractional derivative (see e.g. (17.11) in [25]), and we obtain

$$\begin{aligned} & D^{1-\alpha}((D^k\chi)(t)(J^{k+1-\alpha}\psi)(t)) \\ &= (D^{1-\alpha} J^{k+1-\alpha} \psi)(t)(D^k \chi)(t) \\ &\quad - \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} (J^{k+1-\alpha}\psi)(s) \left((D^k\chi)(t) - (D^k\chi)(s)\right) ds \\ &= (J^k \psi)(t)(D^k\chi)(t) \\ &\quad - \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} (J^{k+1-\alpha}\psi)(s) \left((D^k\chi)(t) - (D^k\chi)(s)\right) ds. \end{aligned}$$

For the last "=" we have used (see (6.2))

$$D^{1-\alpha} J^{k+1-\alpha} = D J^\alpha J^{k+1-\alpha} = D J^{k+1} = J^k.$$

By $\chi \in C^\infty[0,1]$, we obtain

$$
\begin{aligned}
&\left|D^{1-\alpha}\left((D^k\chi)(t)(J^{k+1-\alpha}\psi)(t)\right)\right| \\
&\le C_8\left((J^k|\psi|)(t) + \int_0^t (t-s)^{\alpha-1}(J^{k+1-\alpha}|\psi|)(s)ds\right) \\
&\le C_8\left\{(J^k|\psi|)(t) + \frac{1}{\Gamma(k+1-\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-\eta)^{k-\alpha}|\psi(\eta)|d\eta\right)ds\right\} \\
&= C_8\left\{(J^k|\psi|)(t) + \frac{1}{\Gamma(k+1-\alpha)}\int_0^t |\psi(\eta)|\left(\int_\eta^t (t-s)^{\alpha-1}(s-\eta)^{k-\alpha}ds\right)d\eta\right\} \\
&= C_8\left\{(J^k|\psi|)(t) + \frac{B(\alpha,k+1-\alpha)}{\Gamma(k+1-\alpha)}\int_0^t (t-\eta)^k|\psi(\eta)|d\eta\right\} \\
&\le C_9'(J^k+J)|\psi|(t).
\end{aligned}
$$

To obtain the last "=" we have substituted $\xi = \frac{s-\eta}{t-\eta}$ in the integral. Now we find

$$\|D^{1-\alpha}(D^k\chi J^{k+1-\alpha}\psi)\|_{L^2} \le C_9\|\psi\|_{L^2},$$

because J^k and J are bounded operators from $L^2(0,1)$ into itself. Consequently, since $(I-L)^{-1}$ is also bounded and $\|f-\phi\|_{L^2} \le \delta$, we obtain

$$
\begin{aligned}
\|A_\alpha\widetilde{u} - R_l\phi\|_{L^2} &\le (2l+1)C_8C_9\|\psi\|_{L^2} \\
&= (2l+1)C_8C_9\|(I-L)^{-1}(f-\phi)\|_{L^2} \le C_{10}\|f-\phi\|_{L^2} \le C_{10}\delta.
\end{aligned}
$$

Moreover, by Lemma 10, we get $\widetilde{u} \in H_0^q(0,1)$ and $\|\widetilde{u}\|_{H^q} \le C_4(q)M$, and applying Theorem 5 with u and ϕ replaced by $\widetilde{u}$ and $R_l\phi$ we find

$$\|\widetilde{u} - u_l^{\delta,\gamma,p}\|_{L^2} = O\left(\delta^{q/(\alpha+q)}M^{\alpha/(\alpha+q)}\right).$$

We complete the proof of Theorem 6 now by recalling that $\widetilde{u}(t) = \chi(t)u(t) = t^{[q]+1}(1-t)^{[q]+1}u(t)$ for $0 \le t \le 1$.

Appendix I. Proof of Lemma 8

Proof of the part (i). Let

$$\mu_n = \left(n - \frac{1}{2}\right)\pi, \quad \lambda_n = n\pi, \qquad n \in \mathbb{N},$$

$$x_n(t) = \cos\mu_n t = \frac{1}{\sqrt{2}}\phi_n(t), \quad y_n(t) = \sin\lambda_n t, \qquad 0 \le t \le 1,\ n \in \mathbb{N}.$$

As is easily seen, $\{\mu_n^2 : n \in \mathbb{N}\} \cup \{\lambda_n^2 : n \in \mathbb{N}\}$ is the set of all the eigenvalues of

$$\left\{\begin{array}{ll} (D^2u)(t) = -\lambda u(t), & -1 < t < 1 \\ u(-1) = u(1) = 0 & \end{array}\right\}$$

and x_n and y_n are eigenfunctions respectively for μ_n^2 and λ_n^2, $n \in \mathbb{N}$. Similarly to (2.2), we see that

$$B \equiv \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\} \quad \text{is an orthonormal basis in } L^2(-1,1). \tag{1}$$

Now for $\beta \in \mathbb{R}$, let us form a Hilbert scale $\{Y_\beta\}_{\beta\in\mathbb{R}}$ over B similarly to $\{X_\beta\}_{\beta\in\mathbb{R}}$. Let $\beta \in \mathbb{R}$. In $\operatorname{span} B$, we define scalar products and norms by

$$(u,v)_{Y_\beta} = \sum_{n=1}^{\infty} \mu_n^{2\beta}(u,x_n)_{L^2}\overline{(v,x_n)_{L^2}} + \sum_{n=1}^{\infty} \lambda_n^{2\beta}(u,y_n)_{L^2}\overline{(v,y_n)_{L^2}}, \tag{2}$$
$$\|u\|_{Y_\beta} = (u,u)_{Y_\beta}^{1/2},$$

for all $u, v \in \operatorname{span} B$. The completion of $\operatorname{span} B$ in the norm $\|\cdot\|_{Y_\beta}$ is denoted by Y_β. By (1), we see

$$Y_0 = L^2(-1,1).$$

Let us define an operator $-A$ in $L^2(-1,1)$ by

$$(-Au)(t) = (D^2u)(t), \qquad -1 < t < 1$$

and

$$\mathcal{D}(-A) = \{u \in H^2(-1,1) : u(-1) = u(1) = 0\}.$$

Then, as is known (e.g. §3 in Chapter 1 of Henry [16]), the fractional power A^γ, $\gamma \geq 0$, is well-defined and

$$A^{\beta/2}u = \sum_{n=1}^{\infty} \mu_n^{\beta}(u,x_n)_{L^2}x_n + \sum_{n=1}^{\infty} \lambda_n^{\beta}(u,y_n)_{L^2}y_n, \quad u \in \mathcal{D}(A^{\beta/2}).$$

Therefore by the definition of (2), we see that

$$\mathcal{D}(A^{\beta/2}) = Y_\beta, \qquad \beta \geq 0 \tag{3}$$

and

$$\|A^{\beta/2}u\|_{L^2(-1,1)} = \|u\|_{Y_\beta}, \quad u \in Y_\beta. \tag{4}$$

On the other hand, by Fujiwara [9], we have

$$\mathcal{D}(A^{\beta/2}) = H^\beta(-1,1), \quad 0 \leq \beta < \frac{1}{2}$$

$$\begin{aligned}\mathcal{D}(A^{1/4}) &= \left\{u \in H^{1/2}(-1,1) : \int_{-1}^{1} (\min\{1-t, 1+t\})^{-1} |u(t)|^2 dt < \infty\right\}\\ &=: E^{0,1/2}(-1,1),\end{aligned}$$

$$\mathcal{D}(A^{\beta/2}) = H_0^\beta(-1,1), \quad \frac{1}{2} < \beta \le 1,$$

and for $\beta \ne \frac{1}{2}$, there exists a constant $C_{11} = C_{11}(\beta) > 0$ such that

$$C_{11}^{-1}\|u\|_{H^\beta(-1,1)} \le \|A^{\beta/2}u\|_{L^2(-1,1)} \le C_{11}\|u\|_{H^\beta(-1,1)}, \quad u \in \mathcal{D}(A^{\beta/2}).$$

Therefore we have

$$\left\{\begin{array}{ll} Y_\beta = H^\beta(-1,1), & 0 \le \beta < \frac{1}{2} \\ Y_{1/2} = E^{0,1/2}(-1,1), & \\ Y_\beta = H_0^\beta(-1,1), & \frac{1}{2} < \beta \le 1 \end{array}\right\} \tag{5}$$

and for some $C_{11} = C_{11}(\beta) > 0$, we get

$$C_{11}\|u\|_{Y_\beta} \le \|u\|_{H^\beta(-1,1)} \le C_{11}^{-1}\|u\|_{Y_\beta}, \quad u \in Y_\beta,\ \beta \ne \frac{1}{2}. \tag{6}$$

Finally we define an isomorphism L from X_β onto a closed subspace of Y_β:

$$(Lu)(t) = \begin{cases} u(t), & 0 \le t \le 1 \\ u(-t), & -1 \le t < 0. \end{cases}$$

That is, Lu is an even extension of the function in $(0,1)$ to one in $(-1,1)$. Then we see: for $\beta > 0$, $u \in X_\beta$ if and only if $Lu \in Y_\beta$, and

$$\|u\|_{X_\beta} = \frac{1}{\sqrt{2}}\|Lu\|_{Y_\beta}, \qquad u \in X_\beta. \tag{7}$$

In fact, since Lu is an even function, we have $(u, y_n)_{L^2(-1,1)} = 0$, $n \in \mathbb{N}$, so that

$$\begin{aligned}\|Lu\|_{Y_\beta}^2 &= \sum_{n=1}^{\infty} \mu_n^{2\beta}|(Lu, x_n)_{L^2(-1,1)}|^2\\ &= 2\sum_{n=1}^{\infty} \mu_n^{2\beta}|(u, \phi_n)_{L^2(0,1)}|^2 = 2\|u\|_{X_\beta}^2.\end{aligned}$$

Further we need

LEMMA 11. (a) *For* $0 \le \beta \le 1$,

$$\|u\|_{H^\beta(0,1)} \le \|Lu\|_{H^\beta(-1,1)} \le 2\|u\|_{H^\beta(0,1)}, \quad u \in H^\beta(0,1). \tag{8}$$

(b) *We have*

$$\int_{-1}^{1} (\min\{1-t, 1+t\})^{-1} |Lu(t)|^2 dt = 2\int_{0}^{1} (1-t)^{-1} |u(t)|^2 dt.$$

Proof of Lemma 11. The part (b) is easily seen. Let us prove the part (a). The cases $\beta = 0$ and $\beta = 1$ are readily verified. Let $0 < \beta < 1$. Then we have

$$\|Lu\|^2_{H^\beta(-1,1)} = \|Lu\|^2_{L^2(-1,1)} + |Lu|^2_{H^\beta(-1,1)}$$

and

$$\|u\|^2_{H^\beta(0,1)} = \|u\|^2_{L^2(0,1)} + |u|^2_{H^\beta(0,1)},$$

with

$$|v|^2_{H^\beta(-1,1)} = \int_{-1}^{1}\int_{-1}^{1} \frac{|v(t)-v(s)|^2}{|t-s|^{1+2\beta}} dt\, ds$$

and

$$|v|^2_{H^\beta(0,1)} = \int_{0}^{1}\int_{0}^{1} \frac{|v(t)-v(s)|^2}{|t-s|^{1+2\beta}} dt\, ds$$

(e.g. Adams [2]). Therefore $\|u\|_{H^\beta(0,1)} \le \|Lu\|_{H^\beta(-1,1)}$ is straightforward. For the second inequality, since $\|Lu\|_{L^2(-1,1)} = \sqrt{2}\|u\|_{L^2(0,1)}$, it is sufficient to prove

$$|Lu|_{H^\beta(-1,1)} \le 2|u|_{H^\beta(0,1)}.$$

Noting the definition of Lu, we have

$$\begin{aligned} &|Lu|^2_{H^\beta(-1,1)} \\ &= \left(\int_0^1\int_0^1 + \int_{-1}^0\int_0^1 + \int_0^1\int_{-1}^0 + \int_{-1}^0\int_{-1}^0\right)\left(\frac{|Lu(t)-Lu(s)|^2}{|t-s|^{1+2\beta}}\right) dt\, ds \\ &= 2|u|^2_{H^\beta(0,1)} + 2\int_0^1\int_0^1 \frac{|u(t)-u(s)|^2}{|t+s|^{1+2\beta}} dt\, ds. \end{aligned}$$

Here we get

$$\begin{aligned} \int_0^1\int_0^1 \frac{|u(t)-u(s)|^2}{|t+s|^{1+2\beta}} dt\, ds &= \int_0^1 \left(\int_0^1 \frac{|u(t)-u(s)|^2}{|t-s|^{1+2\beta}} \frac{|t-s|^{1+2\beta}}{|t+s|^{1+2\beta}} dt\right) ds \\ &\le \int_0^1 \left(\int_0^1 \frac{|u(t)-u(s)|^2}{|t-s|^{1+2\beta}} dt\right) ds, \end{aligned}$$

so that

$$|Lu|^2_{H^\beta(-1,1)} \le 4|u|^2_{H^\beta(0,1)}.$$

Thus the proof of Lemma 11 is complete.

Now let us complete the proof of Lemma 8 (i). For $\beta \neq \frac{1}{2}$, the estimates (6)–(8) imply the estimates in the part (i). Finally we have to prove the relations between X_β and the Sobolev spaces.

First let $0 \le \beta < \frac{1}{2}$. By (5) and (7), $u \in X_\beta$ if and only if $Lu \in H^\beta(-1,1)$, and by (a) in Lemma 11, we see that $u \in X_\beta$ if and only if $u \in H^\beta(0,1)$. Second, let $\beta = \frac{1}{2}$. By (5) and (7), $u \in X_{1/2}$ if and only if $Lu \in E^{0,1/2}(-1,1)$, which, by Lemma 11, is equivalent to

$$u \in H^{1/2}(0,1), \qquad \int_0^1 (1-t)^{-1}|u(t)|^2 dt < \infty.$$

Finally let $\frac{1}{2} < \beta \le 1$. By (5) and (7), $u \in X_\beta$ if and only if $Lu \in H_0^\beta(-1,1)$, namely, $Lu \in H^\beta(-1,1)$ and $(Lu)(-1) = (Lu)(1) = 0$, which is equivalent to $u \in \{v \in H^\beta(0,1) : v(1) = 0\}$ by Lemma 11. Thus we have completed the proof of the part (i) of Lemma 8.

Proof of the part (ii) *of Lemma* 8. The cases $0 \le \beta \le 1$ are already proved in the part (i). Let $1 < \beta \le 2$. First let $u \in H_0^2(0,1)$. Then, since $u(0) = u(1) = (D^1u)(0) = (D^1u)(1) = 0$, by integration by parts, we have

$$(u, \phi_n)_{L^2} = \frac{1}{\mu_n^2}(D^2u, \phi_n)_{L^2}, \qquad n \in \mathbb{N},$$

so that

$$\|u\|_{X_2}^2 = \sum_{n=1}^\infty \mu_n^4 \frac{1}{\mu_n^4}|(D^2u, \phi_n)_{L^2}|^2 = \|D^2u\|_{L^2}^2, \qquad u \in H_0^2(0,1),$$

by (2.1). That is, noting (2.5), we obtain

$$\|u\|_{X_2} \le C_1\|u\|_{H^2}, \qquad u \in H_0^2(0,1).$$

Therefore $E : H_0^2(0,1) \longrightarrow X_2$ is well-defined and bounded. Moreover, by the part (i), $E : H_0^1(0,1) \longrightarrow X_1$ is also bounded. We denote the intermediate space between the Hilbert spaces H_1 and H_2 by $[H_1, H_2]_\theta$, $\theta \in [0,1]$ (e.g. Definition 1.2.1 in Lions and Magenes [21]). If $\theta \neq \frac{1}{2}$ and $0 < \theta < 1$, then

$$[H_0^2(0,1), H_0^1(0,1)]_\theta = H_0^{2-\theta}(0,1)$$

(Theorem 1.11.6 in [21]). Furthermore by the definition of X_β, we can directly see that

$$[X_2, X_1]_\theta = X_{2-\theta}, \qquad 0 \le \theta \le 1.$$

Consequently by the boundedness of $E : H_0^2(0,1) \longrightarrow X_2$ and $E : H_0^1(0,1) \longrightarrow X_1$, we can apply Theorem 1.5.1 in [21], so that also $E : H_0^{2-\theta}(0,1) \longrightarrow X_{2-\theta}$ is bounded if $\theta \in [0,1]$ and $\theta \neq \frac{1}{2}$. For $\beta > 2$, we can proceed similarly. Thus the proof of Lemma 8 is complete.

Appendix II. Proof of Lemma 10

First let $q \in \mathbb{N}$. Then for some constant $C_{12} > 0$ which is independent of u but dependent on χ, we have

$$\begin{aligned}\|\widetilde{u}\|_{H^q}^2 &= (\|\chi u\|_{L^2}^2 + \|D^q(\chi u)\|_{L^2}^2) \le C_{12}\left(\|u\|_{L^2}^2 + \sum_{k=1}^{q} \|D^k u\|_{L^2}^2\right) \\ &\le C_{12}\|u\|_{H^q}^2.\end{aligned}$$

Next let $q \notin \mathbb{N}$, that is, $q = m + \gamma$ where $m \in \mathbb{N}$ and $0 < \gamma < 1$. Then

$$\|\widetilde{u}\|_{H^q}^2 = (\|\widetilde{u}\|_{L^2}^2 + \|D^m\widetilde{u}\|_{L^2}^2 + |D^m\widetilde{u}|_{H^\gamma}^2)$$

(e.g. Adams [2]). Setting $\chi_k(t) = (D^k\chi)(t)$ and $u_{m-k}(t) = (D^{m-k}u)(t)$, $0 \le k \le m$, we get

$$|D^m\widetilde{u}|_{H^\gamma}^2 \le C_{13}\sum_{k=0}^{m} |\chi_k u_{m-k}|_{H^\gamma}^2.$$

On the other hand, for $0 \le k \le m$, we have

$$\begin{aligned}|\chi_k u_{m-k}|_{H^\gamma}^2 &= \int_0^1\int_0^1 \frac{|\chi_k(t)u_{m-k}(t) - \chi_k(s)u_{m-k}(s)|^2}{|t-s|^{1+2\gamma}}dtds \\ &\le C_{13}\int_0^1\int_0^1 \frac{|\chi_k(t)|^2|u_{m-k}(t) - u_{m-k}(s)|^2}{|t-s|^{1+2\gamma}}dtds \\ &\quad + C_{13}\int_0^1\left(\int_0^1 \frac{|\chi_k(t) - \chi_k(s)|^2}{|t-s|^{1+2\gamma}}dt\right)|u_{m-k}(s)|^2ds \\ &\le C_{14}\int_0^1\int_0^1 \frac{|u_{m-k}(t) - u_{m-k}(s)|^2}{|t-s|^{1+2\gamma}}dtds \\ &\quad + C_{14}\int_0^1\left(\int_0^1 |t-s|^{1-2\gamma}dt\right)|u_{m-k}(s)|^2ds \quad \text{by } \chi_k \in C^\infty(0,1) \\ &\le C_{14}|D^{m-k}u|_{H^\gamma}^2 + \frac{C_{14}}{1-\gamma}\int_0^1 |u_{m-k}(s)|^2ds \\ &\le C_{15}\|u\|_{H^q}^2.\end{aligned}$$

At the first inequality, we have used

$$\begin{aligned}&|\chi_k(t)u_{m-k}(t) - \chi_k(s)u_{m-k}(s)|^2 \\ &= |\chi_k(t)(u_{m-k}(t) - u_{m-k}(s)) + (\chi_k(t) - \chi_k(s))u_{m-k}(s)|^2 \\ &\le 2|\chi_k(t)(u_{m-k}(t) - u_{m-k}(s))|^2 + 2|(\chi_k(t) - \chi_k(s))u_{m-k}(s)|^2,\end{aligned}$$

and at the last inequality, $\|u\|_{H^{m-k+\gamma}} \le C_{15}\|u\|_{H^{m+\gamma}}$. Therefore we obtain

$$|D^m\widetilde{u}|_{H^\gamma}^2 \le C_{16}\|u\|_{H^q}^2,$$

so that we have

$$\|\widetilde{u}\|_{H^q}^2 \le C_{16}(\|u\|_{L^2}^2 + \|D^m u\|_{L^2}^2 + \|u\|_{H^q}^2) \le 2C_{16}\|u\|_{H^q}^2,$$

which is the conclusion of the lemma.

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