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The Stability of Natural Runge-Kutta Methods for Nonlinear Delay Differential Equations

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A natural Runge-Kutta method is a special type of Runge-Kutta method for delay differential equations (DDEs); it is known that any collocation method is equivalent to one of such methods. In this paper, stability properties of natural Runge-Kutta methods are studied using nonlinear DDEs which have a quadratic Liapunov functional. A discrete analogue of the functional is defined for each method, and the stability of the method is examined on the basis of this analogue. In particular, it is shown that an algebraically stable method, if it satisfies an additional condition, preserves the asymptotic properties of the original equations for every stepsize.

Key words: Runge-Kutta methods, algebraic stability, A-stability, delay differential equations, Liapunov functionals

1. Introduction

Various concepts of unconditional stability of Runge-Kutta methods have been introduced and studied from the viewpoint of their application to stiff equations. A-stability for linear autonomous equations and B-stability for contractive nonlinear equations are representative. It is also well known that algebraic stability, which is introduced by an algebraic condition on the parameters of a method, is equivalent to B-stability for most Runge-Kutta methods (see, e.g., [6]). Recently, some studies [5, 15, 16] appeared which revealed a similarity between A-stability and algebraic stability; they have shown that A-stable methods are also characterized by algebraic conditions on their parameters of almost the same form as in the case of algebraically stable methods.

Runge-Kutta methods can be also applied to delay differential equations (DDEs) by using some interpolation procedures. It would be significant to consider what meanings the above stability concepts have in the application of the methods to DDEs. Concerning A-stability, its meaning has been clarified to some extent. Zennaro [19] studied stability properties of Runge-Kutta methods with an interpolation procedure proposed by himself [18] using linear scalar test equations [1]. As a result, he has shown that an A-stable method preserves the asymptotic stability of the zero solution for every stepsize, if the method has a kind of consistency with the interpolation procedure. The same results were obtained in the cases of scalar equations of neutral type [2], multi-dimensional equations [12], and multi-dimensional equations of neutral type [8]. On the other hand, in 't Hout [10] has proposed another type of interpolation procedure; he shows that it can make *every* A-stable method have the above property in the cases of scalar equations [10] and multi-dimensional equations [11].

In those studies, difference equations obtained by the application of methods are directly considered, but it does not seem easy to study nonlinear cases by such an approach. In this paper, we discuss the stability of Runge-Kutta methods for DDEs from a different point of view; we study their stability using nonlinear DDEs which have a quadratic Liapunov functional. More specifically, we define a discrete analogue of the functional for each method, and examine the stability of the method by considering whether the analogue preserves the monotonicity of the original functional. Through such consideration, we show a meaning of algebraic stability and a new significance of A-stability in the application of Runge-Kutta methods to DDEs. Our basic idea and a part of our results in a restricted case are presented in [13] (see also [21] on nonlinear model equations of another type).

2. Preliminaries

2.1. Delay differential equations

We consider initial value problems of the form

$$u'(t) - Ku'(t-\tau) = f(t, u(t), u(t-\tau)), \quad t > 0, \tag{2.1.a}$$

$$u(t) = \varphi(t), \quad -\tau \le t \le 0, \tag{2.1.b}$$

where $u(t) \in R^d$, $\tau > 0$ is a constant delay, K denotes a constant matrix whose spectral radius is less than 1, and $\varphi(t)$ is a given C^1-function. Furthermore, we assume that $f : (0,\infty) \times R^d \times R^d \to R^d$ is locally Lipschitz continuous and satisfies $f(t, 0, 0) = 0$ for all $t > 0$.

For the equation (2.1.a) we consider a functional of the form

$$V(u(t)) = (u(t) - Ku(t-\tau))^T G(u(t) - Ku(t-\tau)) + \int_{t-\tau}^{t} u(\sigma)^T E u(\sigma) d\sigma, \tag{2.2}$$

where G, E are constant symmetric $d \times d$-matrices which satisfy $G > 0$, $E \ge 0$; the symbols '> 0' and '≥ 0' indicate that a matrix is a positive definite and nonnegative definite, respectively. When $u(t)$ is a solution of (2.1.a), we get

$$\frac{d}{dt} V(u(t)) = \psi(t, u(t), u(t-\tau)), \tag{2.3}$$

$$\psi(t, x, y) = 2(x - Ky)^T G f(t, x, y) + x^T E x - y^T E y, \quad x, y \in R^d.$$

Thus, by the standard argument (see, e.g., [14], p.31, Theorem 5.4, see also [7]), it is shown that the solutions of (2.1.a) are *uniformly bounded* if ψ satisfies the following condition (LC).

(LC) There is a continuous nondecreasing function $\omega : [0,\infty) \to [0,\infty)$ such that

$$\psi(t, x, y) \le -\omega(|x|) \quad \text{for} \quad t > 0, \quad x, y \in R^d.$$

Here, $|\cdot|$ denotes a norm on R^d. If, in addition, $\omega(\sigma) > 0$ for $\sigma > 0$, the zero solution of (2.1.a) is *uniformly asymptotically stable* (ibid.).

For example, consider the case f is linear and autonomous, i.e.,

$$f(t, u(t), u(t-\tau)) = Lu(t) + Mu(t-\tau), \tag{2.4}$$

where L, M are constant matrices. Simple computation gives

$$\psi(t, x, y) = -(x^T, y^T)\Omega(x^T, y^T)^T,$$

$$\Omega = \begin{pmatrix} -GL - L^TG - E & L^TGK - GM \\ (L^TGK - GM)^T & K^TGM + M^TGK + E \end{pmatrix}.$$

Hence, (LC) is satisfied if and only if $\Omega \geq 0$, and an $\omega(\sigma)$ is taken so that $\omega(\sigma) > 0$ for $\sigma > 0$ if $\Omega > 0$.

REMARK 1. The functional (2.2) has been used by Slemrod and Infante [17] to obtain sufficient conditions on coefficients in linear DDEs which ensure the asymptotic stability of the zero solution. A discrete analogue of (2.2) based on the θ-method has been also examined in [17].

2.2. Natural Runge-Kutta methods

For the application of a Runge-Kutta method to (2.1), some interpolation procedure is needed which gives approximate values to $u(t-\tau)$, $u'(t-\tau)$ in (2.1.a); we use a *natural continuous extension* [18] of the Runge-Kutta method to approximate those arguments. For simplicity of analysis, we treat only an *aligned mesh*, i.e., a mesh of the form

$$t_n = hn, \ h = \tau/k, \quad n \in Z, \quad k: \text{ positive integer.}$$

An s-stage Runge-Kutta method applied to (2.1) is represented as follows:

$$U'_{n,i} = f(t_n + c_i h, U_{n,i}, \widehat{U}_{n-k,i}) + K \sum_{j=1}^{s} w'_j(c_i) U'_{n-k,j},$$

$$U_{n,i} = u_n + h \sum_{j=1}^{s} a_{ij} U'_{n,j}, \quad \widehat{U}_{n,i} = u_n + h \sum_{j=1}^{s} w_j(c_i) U'_{n,j},$$

$$i = 1, 2, \ldots, s,$$

$$u_{n+1} = u_n + h \sum_{i=1}^{s} b_i U'_{n,i}.$$

Here, u_n denotes an approximate value to $u(t_n)$, and $U_{n,i}$, $\widehat{U}_{n,i}$, $U'_{n,i}$ are intermediate variables; when $n < 0$ they are given, for example, by

$$\widehat{U}_{n,i} = \varphi(t_n + c_i h), \quad U'_{n,i} = \varphi'(t_n + c_i h), \quad 1 \leq i \leq s, \quad -k \leq n < 0.$$

Moreover, a_{ij}, b_i, c_i are the parameters of the method, and $w_i(\sigma)$ are polynomials which satisfy a kind of order condition [18]. In particular, they satisfy

$$w_i(0) = 0, \quad w_i(1) = b_i, \quad 1 \le i \le s, \tag{2.5.a}$$

$$\sum_{i=1}^{s} w_i(\sigma) c_i^{q-1} = \frac{\sigma^q}{q}, \quad 1 \le i \le s, \quad 1 \le q \le \max_{1 \le i \le s} \{\deg(w_i)\}. \tag{2.5.b}$$

In addition, we assume that

$$a_{ij} = w_j(c_i), \quad 1 \le i, j \le s. \tag{2.6}$$

It is easy to see that (2.5) and (2.6) imply

$$\sum_{i=1}^{s} b_i w_j'(c_i) = b_j, \ 1 \le j \le s, \quad \sum_{m=1}^{s} a_{im} w_j'(c_m) = a_{ij}, \quad 1 \le i, \ j \le s. \tag{2.7}$$

A Runge-Kutta method is said to be *natural*, if it satisfies (2.6). It is known that any collocation method can be regarded as a natural Runge-Kutta method [19, 20]. In particular, the Gauss-Legendre, Radau IIA, Lobatto IIIA methods (see, e.g., [3]) are considered as natural Runge-Kutta methods.

For notational convenience, we write the parameters of the method in the form

$$A = (a_{ij}) \quad (1 \le i, j \le s), \quad b = (b_1, b_2, \dots, b_s)^T,$$

$$\Gamma = (w_j'(c_i)) \quad (1 \le i, \ j \le s).$$

Furthermore, define

$$\begin{aligned} T_n &= (t_n + c_1 h, t_n + c_2 h, \dots, t_n + c_s h)^T \in R^s, \\ U_n &= \left((U_{n,1})^T, (U_{n,2})^T, \dots, (U_{n,s})^T\right)^T \in R^{ds}, \\ U_n' &= \left((U_{n,1}')^T, (U_{n,2}')^T, \dots, (U_{n,s}')^T\right)^T \in R^{ds}, \end{aligned}$$

and let

$$F(\theta, X, Y) = \left(f(\theta_1, X_1, Y_1)^T, f(\theta_2, X_2, Y_2)^T, \dots, f(\theta_s, X_s, Y_s)^T\right)^T$$

for

$$\theta = (\theta_1, \theta_2, \dots, \theta_s)^T \in R^s,$$

$$X = \left(X_1^T, X_2^T, \dots, X_s^T\right)^T \in R^{ds}, \quad Y = \left(Y_1^T, Y_2^T, \dots, Y_s^T\right)^T \in R^{ds}.$$

Since $\widehat{U}_{n-k,i} = U_{n-k,i}$ by (2.6), the natural Runge-Kutta method is written in a compact form:

$$U'_n = F(T_n, U_n, U_{n-k}) + (\Gamma \otimes K)U'_{n-k}, \tag{2.8.a}$$

$$U_n = e \otimes u_n + h(A \otimes I_d)U'_n, \tag{2.8.b}$$

$$u_{n+1} = u_n + h(b^T \otimes I_d)U'_n, \tag{2.8.c}$$

where $e = (1, 1, \ldots, 1)^T \in R^s$, I_d is the $d \times d$ identity matrix, and $\otimes$ denotes Kronecker product. The condition (2.7) is also written in the form

$$b^T \Gamma = b^T, \quad A\Gamma = A. \tag{2.7'}$$

We thus get

(2.9)
$$U_n - \widetilde{K}U_{n-k} = e \otimes (u_n - Ku_{n-k}) + h(A \otimes I_d)F(T_n, U_n, U_{n-k}), \quad \widetilde{K} = I_s \otimes K.$$

from (2.8.a) and (2.8.b), where I_s is the $s \times s$ identity matrix. Since U'_n is obtained from U_n by (2.8.a), the approximate values u_n, $n = 1, 2, \ldots$, are succesively obtained if the equation

$$X - \widetilde{K}U_{n-k} = e \otimes (u_n - Ku_{n-k}) + h(A \otimes I_d)F(T_n, X, U_{n-k}) \tag{2.10}$$

has a solution for each $n \geq 0$.

3. Discrete Analogues of Liapunov Functionals

3.1. Conditions on equations and methods

To discuss the properties of the natural Runge-Kutta method (2.8), we prepare further notation. Let S be a nonnegative definite symmetric $s \times s$-matrix, and define

$$\Psi_S(\theta, X, Y) = 2\big(X - \widetilde{K}Y\big)^T (S \otimes G)F(\theta, X, Y) + X^T(S \otimes E)X - Y^T(S \otimes E)Y.$$

The following condition is an analogue of the condition (LC).

(LD) There is a continuous nondecreasing function $\omega_S : [0, \infty) \to [0, \infty)$ such that

$$\Psi_S(\theta, X, Y) \leq -\omega_S(\|X\|) \quad \text{for } \theta \in R^s, \quad X, Y \in R^{ds}.$$

Here, $\|\cdot\|$ denotes a norm on R^{ds}. By the definition of Ψ_S, if S is diagonal, (LC) implies (LD). On the other hand, when f is linear and autonomous, since

$$F(\theta, X, Y) = (I_s \otimes L)X + (I_s \otimes M)Y,$$

we obtain

$$\Psi_S(\theta, X, Y) = -(X^T, Y^T)(S \otimes \Omega)(X^T, Y^T)^T.$$

Therefore, if (LC) is satisfied, (LD) is satisfied for any $S \geq 0$.

Furthermore, let us consider a nonlinear perturbation of (2.4) of the form

$$f(t, u(t), u(t-\tau)) = Lu(t) + Mu(t-\tau) + \varepsilon f_0(t, u(t), u(t-\tau)), \tag{3.1}$$

where $f_0 : (0, \infty) \times R^d \times R^d \to R^d$ is assumed to satisfy

$$|f_0(t, x, y)| \leq C_0(|x| + |y|) \quad \text{for} \quad t > 0, \quad x, y \in R^d \tag{3.2}$$

for a constant C_0 independent of t, x, y. Defining F_0 in the same way as F, we get

$$\begin{aligned}\Psi_S(\theta, X, Y) &= -(X^T, Y^T)(S \otimes \Omega)(X^T, Y^T)^T \\ &\quad + 2\varepsilon\big(X - \widetilde{K}Y\big)^T (S \otimes G) F_0(\theta, X, Y).\end{aligned}$$

Since there is a constant C such that

$$\begin{aligned}&\big|\big(X - \widetilde{K}Y\big)^T (S \otimes G) F_0(\theta, X, Y)\big| \\ &\quad \leq C\left(\|X\|^2 + \|Y\|^2\right) \quad \text{for} \quad \theta \in R^s, \quad X, Y \in R^{ds}\end{aligned}$$

by (3.2), if $S > 0$ and $\Omega > 0$, then (LD) is satisfied for sufficiently small ε.

Concerning Runge-Kutta methods, we consider the following two conditions:

(E) There is a symmetric matrix $P > 0$ such that

$$PA + A^T P > 0. \tag{3.3}$$

(S) There is a symmetric matrix $Q \geq 0$ such that

$$QA + A^T Q - bb^T \geq 0, \quad Qe = b. \tag{3.4}$$

Propositions described below have particular interests when one matrix fulfills both requirements for (E) and (S) simultaneously. Before proceeding the propositions, we show two typical cases where this condition is satisfied.

A Runge-Kutta method is said to be *algebraically stable* if $Q = \operatorname{diag}(b_1, b_2, \ldots, b_s)$ satisfies (3.4) and $Q \geq 0$. Since, if some $Q > 0$ satisfies (3.4) and $b_i > 0$ for all i, then $P = Q$ satisfies (3.3), an algebraically stable method satisfies (E) and (S) simultaneously if $b_i > 0$ for all i. For example, the Gauss-Legendre and Radau IIA methods have this property (see, e.g., [3]); a general condition is presented in [6] which ensures every b_i of an algebraically stable method is positive (p.200, Theorem 12.16).

On the other hand, an s-stage Runge-Kutta method is said to be *minimal* if its stability function is included in $R_{s,s} \setminus R_{s-1,s-1}$, where $R_{m,n}$ denotes the set of all rational functions having degree of the numerator $\leq m$ and degree of denominator $\leq n$. It is shown that a minimal method is A-stable if and only if there is a symmetric matrix $Q > 0$ which satisfies (3.4) (see, e.g., [15]). Therefore, an A-stable minimal method also satisfies both (E) and (S) if $b_i > 0$ for all i.

3.2. Discrete analogues of functionals

The following proposition can be proved by the same argument as in the proof of Proposition 3.4 in [9], which deals with similar problems in the case of usual equations without delayed arguments.

PROPOSITION 1. *Assume that A is invertible. If* (E) *is satisfied and* (LD) *is satisfied for $S = P$, then* (2.10) *has at least one solution for any h, u_n, u_{n-k}, U_{n-k}.*

Proof. Letting $Z = X - \widetilde{K}U_{n-k}$ and $z_n = e \otimes (u_n - Ku_{n-k})$, we write (2.10) as

$$Z - z_n - h(A \otimes I_d)F(T_n, Z + \widetilde{K}U_{n-k}, U_{n-k}) = 0. \tag{3.5}$$

It suffices to show that the function

$$\begin{aligned}\Phi(Z) &= (A^{-1} \otimes I_d)\{Z - z_n - h(A \otimes I_d)F(T_n, Z + \widetilde{K}U_{n-k}, U_{n-k})\} \\ &= (A^{-1} \otimes I_d)(Z - z_n) - hF(T_n, Z + \widetilde{K}U_{n-k}, U_{n-k})\end{aligned}$$

has at least one zero.

Since

$$PA^{-1} + (A^{-1})^T P = (A^{-1})^T (PA + A^T P) A^{-1} > 0$$

by (E), there is a real number $\alpha > 0$ such that

$$\frac{1}{2}\left(PA^{-1} + (A^{-1})^T P\right) - \alpha P \geq 0. \tag{3.6}$$

Hence,

$$Z^T \left(PA^{-1} \otimes G\right) Z \geq \alpha Z^T (P \otimes G) Z \quad \text{for} \quad Z \in R^{ds}. \tag{3.7}$$

Furthermore, let $B(0, R)$ denote the ball centered at 0 with radius R in R^{ds} with a norm defined by

$$\left(Z^T (P \otimes G) Z\right)^{1/2}, \quad Z \in R^{ds}.$$

Using (3.7) and (LD) for $S = P$, we can show that

$$Z^T (P \otimes G) \Phi(Z) > 0 \quad \text{for} \quad Z \in \partial B(0, R)$$

if R is sufficiently large. Hence, by a lemma (see, e.g., [4], p.58, Lemma 7.2) derived from Brouwer's fixed-point theorem, we conclude that Φ has at least one zero.

Q.E.D.

REMARK 2. Proposition 1 treats only the case where A is invertible, but there are some methods with singular A for which the solvability of (2.10) is proved by the same argument as above. For example, we can show that (2.10) is solvable for the Lobatto IIIA methods, by using the argument in the proofs of Theorems 14.7 and 14.8 in [6] together.

In the remainder of the paper, we assume that there are u_n, U_n, U'_n, $n \geq -k$, which satisfy (2.8). Moreover, we assume the condition (S), and define an analogue of the functional (2.2) by

$$V_n = (u_n - Ku_{n-k})^T G(u_n - Ku_{n-k}) + h \sum_{m=1}^{k} U_{n-m}^T (Q \otimes E) U_{n-m}. \tag{3.8}$$

Then, an analogous relation to (2.3) is obtained.

Proposition 2. *For any $n \geq 0$,*

$$V_{n+1} - V_n \leq h\Psi_Q(T_n, U_n, U_{n-k}) \tag{3.9}$$

is satisfied.

Proof. From (2.8), (2.7′) it follows

$$\begin{aligned} u_{n+1} - Ku_{n+1-k} &= u_n - Ku_{n-k} + h\left(b^T \otimes I_d\right)\left(U'_n - \widetilde{K}U'_{n-k}\right) \\ &= u_n - Ku_{n-k} + h\left(b^T \otimes I_d\right)\left(U'_n - (\Gamma \otimes K)U'_{n-k}\right) \\ &= u_n - Ku_{n-k} + h\left(b^T \otimes I_d\right) F_n, \quad F_n = F(T_n, U_n, U_{n-k}). \end{aligned}$$

Thus,

$$\begin{aligned} V_{n+1} - V_n &= (u_{n+1} - Ku_{n+1-k})^T G(u_{n+1} - Ku_{n+1-k}) \\ &\quad - (u_n - Ku_{n-k})^T G(u_n - Ku_{n-k}) \\ &\quad + hU_n^T (Q \otimes E) U_n - hU_{n-k}^T (Q \otimes E) U_{n-k} \\ &= 2h\left(b \otimes (u_n - Ku_{n-k})\right)^T (I_s \otimes G) F_n + h^2 F_n^T (bb^T \otimes G) F_n \\ &\quad + hU_n^T (Q \otimes E) U_n - hU_{n-k}^T (Q \otimes E) U_{n-k}. \end{aligned}$$

Furthermore, multiplying (2.9) by $Q \otimes I_d$ and simplifying it by $Qe = b$, we get

$$b \otimes (u_n - Ku_{n-k}) = (Q \otimes I_d)(U_n - \widetilde{K}U_{n-k}) - h(QA \otimes I_d)F_n.$$

Substituting this in the above equation, we finally obtain

$$V_{n+1} - V_n = -h^2 F_n^T \left((QA + A^T Q - bb^T) \otimes G\right) F_n + h\Psi_Q(T_n, U_n, U_{n-k}).$$

Hence, (3.9) follows from the condition (S). Q.E.D.

4. Stability of Natural Runge-Kutta Methods

We characterize the stability of the natural Runge-Kutta method (2.8) using Proposition 2. To describe our results concisely, we introduce two quantities:

$$W_n = \max_{0\le m\le k} |u_{n-m}| + \sqrt{\tau} \max_{1\le m\le k} \|U_{n-m}\|.$$

$$r_\infty = 1 - b^T A^{-1} e.$$

The former is a measure of initial values for (2.8); the latter is the value of the stability function at the infinity when A is invertible.

THEOREM. *Assume that* (LD) *is satisfied for* $S = Q$.

(i) *There is a constant* C *such that*

$$|u_n| \le C W_{n_0} \quad \text{for} \quad n \ge n_0, \tag{4.1}$$

where n_0 *is any nonnegative integer.*

(ii) *Assume, in addition, that* $\omega_Q(\sigma) > 0$ *for* $\sigma > 0$. *If* A *is invertible and* $|r_\infty| < 1$, *then the following statement* (A) *is true*:

(A) *For any* ρ, $\varepsilon > 0$, *there is a nonnegative integer* N *such that, if* $W_{n_0} \le \rho$, *then*

$$|u_n| \le \varepsilon \quad \text{for} \quad n \ge n_0 + N, \tag{4.2}$$

where n_0 *is any nonnegative integer.*

To prove Theorem, we prepare a simple lemma.

LEMMA. *There are constants* C_0, $0 < \gamma_0 < 1$ *such that, for any positive integer* l *and any real number* β, *if* $x_n \in R^d$ *for* $0 \le n \le kl$ *and* $y_n \in R^d$ *for* $k \le n \le kl$ *satisfy*

$$x_n - K x_{n-k} = y_n, \quad |y_n| \le \beta \quad \text{for} \quad k \le n \le kl,$$

then

$$|x_n| \le C_0 \left(\beta + \gamma_0^{[n/k]} \max_{0\le m\le k-1} |x_m| \right) \quad \text{for} \quad 0 \le n \le kl, \tag{4.3}$$

where $[\,\cdot\,]$ *denotes the Gauss notation.*

Proof. Define p, q by $p = [n/k]$, $q = n - kp$, and let $|\cdot|_*$ be a norm on R^d such that $|K|_* < 1$. Simple computation gives

$$x_n = \sum_{i=0}^{p-1} K^i y_{(p-i)k+q} + K^p x_q.$$

Thus, we have

$$|x_n|_* \leq \beta \sum_{i=0}^{p-1} |K|_*^{\,i} + |K|_*^{\,p} |x_q|_*. \tag{4.4}$$

Since

$$\sum_{i=0}^{p-1} |K|_*^{\,i} < \frac{1}{1-|K|_*},$$

and all norms on R^d are equivalent, (4.4) implies (4.3). Q.E.D.

Proof of Theorem. (i) By Proposition 2 and the condition (LD), we get

$$V_n \leq V_{n_0} \quad \text{for} \quad n \geq n_0,$$

which implies

$$|u_n - Ku_{n-k}| \leq C_1 \Big(|u_{n_0} - Ku_{n_0-k}| + \sqrt{\tau} \max_{1 \leq m \leq k} \|U_{n_0-m}\| \Big)$$

for some C_1, since $G > 0$, $Q \otimes E \geq 0$ and $\tau = kh$. Using Lemma, we obtain (4.1).

(ii) We first show the statement below:

(B) For any $\varepsilon_1 > 0$, there are an integer $N_1 \geq 1$ and a real number $\delta_1 > 0$ such that if

$$\|U_n\| \leq \delta_1 \quad \text{for} \quad n_1 \leq n \leq n_1 + N_1 - 1, \tag{4.5}$$

then $W_{n_1+N_1} \leq \varepsilon_1$, where n_1 is any nonnegative integer.

Let N_2 be an integer which satisfies

$$|r_\infty|^{N_2} C\rho \leq \frac{\varepsilon_1}{4} \tag{4.6}$$

for C in (i), and let $\delta_2 > 0$ be a real number such that

$$\| \left(b^T A^{-1} \otimes I_d \right) U_n \| \leq (1 - |r_\infty|) \frac{\varepsilon_1}{4} \tag{4.7}$$

for any $\|U_n\| \leq \delta_2$. Using N_2 and δ_2 we define N_1 and δ_1 by

$$N_1 = k + N_2, \quad \delta_1 = \min \left\{ \delta_2, \ \frac{\varepsilon_1}{2\sqrt{\tau}} \right\}.$$

Then, these N_1 and δ_1 fulfill the requirement for (B).

In fact, if (4.5) is satisfied, (4.7) is satisfied for $n_1 \leq n \leq n_1 + N_1 - 1$. Since it follows from (2.8.b) and (2.8.c) that

$$u_{n+1} - r_\infty u_n = \left(b^T A^{-1} \otimes I_d \right) U_n, \tag{4.8}$$

the same computation as in the proof of Lemma gives

$$|u_n| \le \frac{\varepsilon_1}{4} + |r_\infty|^{n-n_1}|u_{n_1}| \quad \text{for} \quad n_1 \le n \le n_1 + N_1. \tag{4.9}$$

If $n \ge n_1 + N_1 - k$, then $n - n_1 \ge N_2$ by the definition of N_1. Using (4.9), (i) and (4.6), we get $|u_n| \le \varepsilon_1/4 + \varepsilon_1/4 = \varepsilon_1/2$ for $n_1 + N_1 - k \le n \le n_1 + N_1$. On the other hand, $\|U_n\| \le \varepsilon_1/(2\sqrt{\tau})$ for $n_1 \le n \le n_1 + N_1 - 1$ by the definition of δ_1. Hence, $W_{n_1+N_1} \le \varepsilon_1/2 + \varepsilon_1/2 = \varepsilon_1$.

We now show that (B) implies (A). Take ε_1 in (B) as $\varepsilon_1 = \varepsilon/C$, and take an integer l_1 so that

$$C_2\rho^2 - l_1 h\omega_Q(\delta_1) < 0, \tag{4.10}$$

where C_2 is a constant such that $V_n \le C_2 W_n^2$, $n \ge 0$. Then, $N = l_1 N_1 + 1$ fulfills the requirement for (A).

To show this, consider the disjoint union

$$\{n \in Z : n_0 + 1 \le n \le n_0 + N - 1\} = \bigcup_{p=0}^{l_1-1} J_p,$$

$$J_p = \{n \in Z : n_0 + pN_1 + 1 \le n \le n_0 + (p+1)N_1\}.$$

There is a p such that $\|U_n\| \le \delta_1$ for any $n \in J_p$. If not, an $n(p) \in J_p$ with $\|U_{n(p)}\| > \delta_1$ is taken for each p; hence, $V_{n(l_1)} \le V_{n_0} - l_1 h\omega_Q(\delta_1)$ by Proposition 2. But, this contradicts $V_n \ge 0$ by (4.10). Therefore, it follows from (B) that there is an $n_2 \le n_0 + N$ such that $W_{n_2} \le \varepsilon/C$. This, together with (4.1), implies (4.2).

Q.E.D.

It is suspected that the condition $|r_\infty| < 1$ in (ii) cannot be omitted to ensure that the statement (A) is true for a general f. For example, consider the scalar equation (without delayed arguments)

$$u'(t) = \lambda(t)u(t), \quad t > 0, \quad \lambda(t) = -2\left(t + \frac{3}{2}\right)^3. \tag{4.11}$$

Since $\lambda(t) < -27/4$, $t > 0$, (LC) is satisfied for $V(u(t)) = u(t)^2$ and $\omega(\sigma) = (27/2)\sigma^2$. Hence, if S is diagonal, (LD) is satisfied; if, in addition, $S > 0$, an ω_S can be taken so that $\omega_S(\sigma) > 0$ for $\sigma > 0$.

Applying the implicit midpoint rule (the 1-stage Gauss-Legendre method, $r_\infty = -1$) with $h = 1$ to (4.11), we obtain

$$u_{n+1} = \frac{1 - (n+2)^3}{1 + (n+2)^3}\, u_n.$$

Since

$$\prod_{n=0}^{\infty} \frac{(n+2)^3 - 1}{(n+2)^3 + 1} = \frac{2}{3},$$

we have $|u_n| \to (2/3)|u_0|$ as $n \to \infty$; (A) is thus false.

This example suggests that f must satisfy some uniformity condition on t, for (A) to be ensured in the case of methods lacking $|r_\infty| < 1$. For example, consider the condition below, which is clearly satisfied when f is independent of t, or given by (3.1), (3.2).

(C_0) For any $\varepsilon_0 > 0$, there is a δ_0 such that, if $|x| \le \delta_0$ and $|y| \le \delta_0$, then

$$|f(t, x, y)| \le \varepsilon_0 \quad \text{for} \quad t \in R.$$

Under the condition (C_0), we can use (2.9), instead of (4.8), to estimate $|u_n|$ by $\|U_n\|$ independently of T_n. Thus, when f satisfies (C_0), we can prove the same statement as in (ii) by the same argument without the assumption that A is invertible and $|r_\infty| < 1$.

5. Conclusion

We studied the stability of natural Runge-Kutta methods with aligned meshes using nonlinear DDEs which have a quadratic Liapunov functional. Our main results are, in cases they have particular interests, summarized as follows. An algebraically stable method with all b_i positive preserves the uniform boundedness of the solutions for every stepsize, and preserves also the uniform asymptotic stability of the zero solution if $|r_\infty| < 1$, or if the original equation is autonomous. An A-stable minimal method with all b_i positive preserves the uniform asymptotic stability for every stepsize, if the equation is a sufficiently small perturbation of a linear autonomous equation (see (3.1)).

We thus revealed a meaning of algebraic stability and a new significance of A-stability in the application of Runge-Kutta methods to DDEs. However, Liapunov functionals treated here are very simple and not in general use. To apply our technique to more complicated cases is an important problem in future.

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