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Compressible Flow with Damping and Vacuum*

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We consider the compressible Euler equations with damping. The singular behavior of the flow near vacuum and the large-time states are of particular interest. A class of solutions is constructed and shown to converge to the self-similar solutions of the porous media equation. The porous media equation is derived from the Euler equations through Darcy's law. Thus we have justified Darcy's law for the compressible flow time-asymptotically.

Key words: compressible Euler, vacuum, friction

1. Introduction

Consider the compressible Euler equations with damping

$$\rho_t + \nabla \cdot (\rho\vec{u}) = 0, \tag{1.1$_1$}$$

$$(p\vec{u})_t + \nabla \cdot \rho(\vec{u} \otimes \vec{u}) + \nabla p(\rho) + -\alpha\rho\vec{u}, \tag{1.1$_2$}$$

where ρ, $\vec{u}$, p are the density, velocity, and pressure, respectively, of the gas, and the friction constant α is positive. We assume that the gas is polytropic

$$p(\rho) = k\rho^\gamma, \quad k > 0, \quad \gamma > 1. \tag{1.2}$$

It is well-known that the Euler equations possess shock waves. Our interest, though, is in the singular behavior of the flow near the vacuum $\rho = 0$. We also want to study the large-time behavior of solutions to (1.1) and relate it, time-asymptotically, to the often-called "*porous media equation*":

$$\rho_t = \alpha^{-1}\triangle p(\rho), \tag{1.3}$$

when $(1.1)_2$ is simplified to Darcy's law

$$\nabla p(\rho) = -\alpha\rho\vec{u}. \tag{1.4}$$

For (1.3) basic understanding is provided by the self-similar solutions of Barenblatt, [2]. We will construct a class of particular solutions for (1.1) which tend to the Barenblatt solutions time-asymptotically. This establishes Darcy's law for the compressible flows in the time-asymptotic sense.

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For gas flows away from the vacuum, $\rho \geq \rho_0 > 0$, the damping prevents shocks from forming, [7]. No other singularity emerges from the flow and it can be shown relatively easily that (1.1) and (1.3) are time-asymptotically equivalent, [3]. The main interest, however, is in the vacuum, and this is the concern of the present paper. The study of vacuum is also important in other physical situations such as astro-physics, [5], [6]. As in the present situation, the external forces dictate the singular behavior of the solutions near the vacuum. It would be interesting to study the general solutions for these physical models.

In the next section we give a heuristic analysis of the nature of the singularity near vacuum for general solutions of (1.1). This motivates the construction of explicit solutions in the following section. The time-asymptotic behavior of these solutions is then studied in Section 4.

REMARK. Some of the results in the present paper have been announced in [4]. The author would like to thank Professor Tetu Makino for his urging to write this paper.

2. Boundary Singularity

The sound speed c is given by

$$c^2 = p'(\rho) = k\gamma\rho^{\gamma-1}. \tag{2.1}$$

The characteristic speeds for (1.1) are formed by $\vec{u}$ and c. Since we do not consider shocks, there is no need to keep the equations in conservation form and it is natural to rewrite (1.1) in terms of $\vec{u}$ and c:

$$(c^2)_t + \nabla(c^2)\cdot\vec{u} + (\gamma-1)c^2\nabla\cdot\vec{u} = 0, \qquad (2.2)_1$$

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + (\gamma-1)^{-1}\nabla(c^2) = -\alpha\vec{u}. \qquad (2.2)_2$$

We now investigate the behavior of a solution near vacuum $\rho = c = 0$. The trajectory of the free boundary

$$\Gamma \equiv \{(\vec{x},t) : \rho(\vec{x},t) \geq 0\} \cap \{(\vec{x},t) : \rho(\vec{x},t) = 0\}$$

coincides with the gas particle paths:

$$\frac{d\vec{x}(t)}{dt} = \vec{u}(\vec{x}(t),t). \qquad (2.3)_1$$

Thus $(2.2)_2$, specified on Γ, becomes

$$\frac{d\vec{u}}{dt} + \alpha\vec{u} = -(\gamma-1)^{-1}\nabla(c^2). \qquad (2.3)_2$$

Generically, the acceleration $d\vec{u}/dt$ of Γ would be finite and $(2.3)_2$ would yield

$$c^2(\vec{x},t) = \eta(\vec{x},t)\cdot|\vec{x}(t)-\vec{x}|,$$

for some function $\eta(\vec{x}, t)$ differentiable up to Γ. Thus we have

$$c(\vec{x}, t) \cong |\vec{x}(t) - \vec{x}|^{1/2} \tag{2.4$_1$}$$

near Γ. In particular, the sound speed is not Lipschitz continuous. The characteristic speed of (1.1) are therefore not Lipschitz continuous near vacuum. This implies that the characteristic surfaces hit the vacuum surface Γ tangentially and then bounce back. It is interesting to note that for $\rho > 0$, (1.1) is hyperbolic and for $\rho = 0$, it is nondiagonizable and parabolic; while the reverse is true for (1.3).

From (2.4) we have

$$\rho(\vec{x}, t) \cong |\vec{x}(t) - \vec{x}|^{1/2(\gamma-1)},$$
$$p(\vec{x}, t) \cong |\vec{x}(t) - \vec{x}|^{\gamma/2(\gamma-1)}$$

near Γ. These singularities are the same as for the solutions of the porous media equation (1.3), [2]. Such a singularity would hold generically when the free boundary Γ moves. Same as for (1.3), the phenomenon of waiting time, before which the free boundary Γ does move, also occures for (1.1) when the initial value is smooth, [1]. This follows from the local existence theory of smooth solutions, cf. [5].

3. Explicit Solutions

We look for two types of particular solutions, spherical and plane-wave solutions, of (1.1). For spherical solutions, (2.2) becomes

$$(c^2)_t + u(c^2)_x + \frac{n-1}{x}(\gamma-1)c^2 u + (\gamma-1)c^2 u_x = 0, \tag{3.1$_1$}$$

$$u_t + uu_x + \alpha u + \frac{1}{\gamma-1}(c^2)_x = 0, \tag{3.1$_2$}$$

$$x = \left(\sum_{i=1}^{n} {x_i}^2\right)^{1/2}, \quad \vec{u} = (\vec{x}/x)u.$$

We consider solutions with finite mass:

$$\rho(x,t) \equiv 0, \quad \text{for } |x| > (e(t)/b(t))^{1/2},$$

and with the ansatz, cf. (2.4),

$$c^2(x,t) = e(t) - b(t)x^2, \tag{3.2$_1$}$$

$$u(x,t) = a(t)x. \tag{3.2$_2$}$$

Plug (3.2) into (3.1) and compare the coefficients for x^i, $i = 0, 1, 2$, to yield the following ordinary differential equations for the functions $a(t)$, $b(t)$, and $e(t)$:

$$e' + n(\gamma-1)ea = 0, \tag{3.3$_1$}$$

$$b' + (n\gamma - n + 2)ab = 0, \tag{3.3$_2$}$$

$$a' + a^2 + \alpha a - \frac{2}{\gamma - 1} b = 0. \tag{3.3$_3$}$$

It can be shown through simple phase-plane analysis of $(3.3)_2$ and $(3.3)_3$ for $a(t)$ and $b(t)$ that $a(t)$, $b(t)$ and $c(t)$ exist for all time. This yields solutions for (1.1) for each given initial value $a(0)$, $b(0)$, and $c(0)$.

There also exist explicit plane wave solutions for (1.1), $x \in R^1$. Consider

$$c^2(x,t) = D(e(t) - x), \quad x < e(t),$$

$$u(x,t) = a(t),$$

for a constant D. From (2.2) with $n = 1$ we have

$$a' + \alpha a = \frac{D}{\gamma - 1}, \quad e' = a,$$

which can easily be solved to obtain

$$a(t) = a(0)e^{-\alpha t} + \frac{D}{\alpha(\gamma - 1)}(1 - e^{-\alpha t}),$$

$$e(t) = e(0) + \left(\frac{a(0)}{\alpha} - \frac{D}{\alpha^2(\gamma - 1)}\right)(1 - e^{-\alpha t}) + \frac{D}{\alpha(\gamma - 1)} t.$$

This class of solutions tend, as $t \to \infty$,

$$c^2(x,t) = (\gamma - 1)\alpha(u_0 t - x),$$

$$u(x,t) = u_0, \quad x < u_0 t.$$

These are travelling wave solutions of (1.1) with any given speed $u_0 > 0$. They are also the travelling wave solutions of the porous media equation (1.3). In the next section we will show that solutions with finite mass (3.2), (3.3) of (1.1) are also time-asymptotic solutions of (1.3).

4. Time-asymptotic Behavior

To analyze the phase-plane diagram of (3.2) and (3.3) we set the curve Γ_1, Γ_2 in the (a, b)-plane as follows:

$$\Gamma_1 : b = \frac{\gamma - 1}{2}(a^2 + \alpha a),$$

$$\Gamma_2 : b = \frac{\gamma - 1}{2}\alpha a.$$

Along the trajectories of (3.3)

$$b' < 0, \quad a' > 0, \quad \text{between } b\text{-axis and } \Gamma_1, \tag{4.1$_1$}$$

$$b' < 0, \quad a' = 0, \quad \text{on } \Gamma_1, \tag{4.1$_2$}$$

$$b' < 0, \quad a' < 0, \quad \text{between } a\text{-axis and } \Gamma_1, \tag{4.1$_3$}$$

$$\frac{db}{da} = \frac{1}{2}(n\gamma - n + 2)(\gamma - 1)\alpha, \quad \text{on } \Gamma_2. \tag{4.1$_4$}$$

From (4.1) we see that $a(t)$, $b(t) \to 0$, as $t \to \infty$. As we will see later, the Barenblatt's self-similar solutions for (1.3) move along the curve Γ_2. We therefore call Γ_2 the Darcy's line. $(4.1)_4$ says that all trajectories of (3.3) are transversal to Γ_2 at a constant angle θ:

$$\tan\theta = \frac{2(n\gamma - n + 1)(\gamma - 1)\alpha}{(n\gamma - n + 2)(\gamma - 1)^2\alpha^2 + 4}.$$

Nevertheless, we will show that the trajectories of (3.3) will come around and approach the Darcy's line Γ_2 as $t \to \infty$.

PROPOSITION. *Solutions of* (3.3) *tend to the Darcy's line* Γ_2 :

$$\frac{b(t)}{a(t)} \to \frac{\gamma - 1}{2}\alpha, \quad \text{as } t \to \infty.$$

Proof. From (3.3)

$$\left(\frac{b}{a}\right)' = \frac{b'a - a'b}{a^2} = -(n\gamma - n + 1)b + \frac{\alpha b}{a} - \frac{2}{\gamma - 1}\left(\frac{b}{a}\right)^2.$$

Given any small $\varepsilon > 0$, consider the curve

$$\Gamma_\varepsilon : -(n\gamma - n + 1)ba^2 + \alpha ba - \frac{2}{\gamma - 1}b^2 = \varepsilon a^2,$$

which is below Γ_2 and has slope

$$\frac{\gamma - 1}{2}\alpha + \frac{\gamma - 1}{4}\left(\left(\alpha^2 - \frac{8}{\gamma - 1}\varepsilon\right)^{1/2} - \alpha\right) > \frac{\gamma - 1}{2}\alpha - \frac{2\varepsilon}{\alpha}$$

at origin (0,0). The slope tends to that of Γ_2 as $\varepsilon \to 0$. Below Γ_ε, $\left(\frac{b}{a}\right)' > \varepsilon$. Thus any trajectory can stay below Γ_ε at most for finite time. Since ε is arbitrary, each trajectory eventually approaches Γ_2. Q.E.D.

The Barenblatt's self-similar solutions of (1.3) can be obtained by the same ansatz as (3.2):

$$c^2(x,t) = \overline{e}(t) - \overline{b}(t)x^2, \tag{4.2$_1$}$$

$$u(x,t) = \overline{a}(t)x. \tag{4.2$_2$}$$

Plugging this into the porous media equation, $(1.1)_1$ and (1.4), or, equivalently, $(3.1)_1$ and

$$\alpha u + \frac{1}{\gamma - 1}(c^2)_x = 0 \tag{4.3}$$

we obtain the system of ordinary differential equations

$$\overline{e}' + n(\gamma - 1)\overline{e}\,\overline{a} = 0, \tag{4.4$_1$}$$

$$\overline{b}' + (n\gamma - n + 2)\overline{a}\overline{b} = 0, \tag{4.4$_2$}$$

$$\alpha\overline{a} = \frac{2\overline{b}}{\gamma - 1}. \tag{4.4$_3$}$$

This (4.4) is the same as (3.3) except for the third equation $(4.4)_3$, which says that the solutions move aong Γ_2. $(4.4)_3$ follows from the Darcy's law (4.3) and so we call Γ_2 the Darcy's line. (4.4) can be integrated directly to yield the Barenblatt's solutions:

$$\overline{a}(t) = \frac{1}{n\gamma - n + 2}t^{-1}, \tag{4.5$_1$}$$

$$\overline{b}(t) = \frac{(\gamma - 1)\alpha}{2(n\gamma - n + 2)}t^{-1}, \tag{4.5$_2$}$$

$$\overline{e}(t) = e_0 t^{-n(\gamma-1)/(n\gamma-n+2)}. \tag{4.5$_3$}$$

Here we have made the normalization that the support of the solution is a point $x = 0$ at the initial time $t = 0$ so that $\overline{a}(0) = \overline{b}(0) = \infty$. The positive constant e_0 is related to the total mass m through

$$\begin{aligned} m &= \Omega_{n-1} \int_0^{\sqrt{\overline{e}(t)/\overline{b}(t)}} \rho(x,t)x^{n-1}dx \\ &= \Omega_{n-1}(k\gamma)^{-1/(\gamma-1)} \int_0^{\sqrt{\overline{e}(t)/\overline{b}(t)}} (c^2(x,t))^{1/(\gamma-1)}x^{n-1}dx \\ &= \Omega_{n-1}(k\gamma)^{-1/(\gamma-1)} \int_0^{\sqrt{\overline{e}(t)/\overline{b}(t)}} (\overline{e}(t) - \overline{b}(t)x^2)^{1/(\gamma-1)}x^{n-1}dx \end{aligned}$$

$$= \Omega_{n-1}(k\gamma)^{-1/(\gamma-1)} \left(\frac{2(n\gamma - n + 2)}{(\gamma-1)\alpha}\right)^{n/2} e_0^{(n\gamma-n+2)/2(\gamma-1)} \cdot \int_0^1 (1-y^2)^{1/(\gamma-1)} y^{n-1} dy. \tag{4.6}$$

THEOREM. *The solutions* (3.2), (3.3) *of the system* (1.1) *with total mass* m *tend to the self-similar Barenblatt solutions* (4.5), (4.6) *of the porous media equation* (1.3), (1.4) :

$$(a, b, c)(t) = (\bar{a}, \bar{b}, \bar{e})(t) + O(1)\frac{\ln t}{t},$$

as $t \to \infty$. *Here the bound* $O(1)$ *is independent of* $t \geq 1$, *but varies with the trajectories of* (3.3). *In particular, the supports* $\{x : x^2 < e(t)/b(t)\}$ *and* $\{x : x^2 < \bar{e}(t)/\bar{b}(t)\}$ *are time-asymptotically the same.*

Proof. From the Proposition and $(3.3)_2$ we have

$$b' + \frac{2(n\gamma - n + 2)}{\alpha(\gamma-1)}(1 + 0(1))b^2 = 0,$$

for some $0(1) \to 0$ as $t \to \infty$. This implies

$$b(t) = D(t)t^{-1}, \quad b'(t) = O(1)t^{-2},$$

for some function $D(t)$ positive and bounded away from zero. Thus we have from $(3.3)_3$ and the Proposition that $a(t) = O(1)t^{-1}$, and so

$$f' + \alpha f + \frac{2}{\alpha(\gamma-1)}b' = -a^2 - \frac{2}{\alpha(\gamma-1)}b' = O(1)t^{-2},$$

$$f \equiv a - \frac{2}{\alpha(\gamma-1)}b.$$

This can be solved to yield

$$f(t) = a(t) - \frac{2}{\alpha(\gamma-1)}b(t) = O(1)t^{-2} = O(1)b(t)t^{-1}.$$

Thus we conclude

$$|a(t)| + |b(t)| = O(1)t^{-1},$$

$$\left|a(t) - \frac{2}{\alpha(\gamma-1)}b(t)\right| = O(1)b(t)t^{-1}, \quad \text{as } t \to \infty. \tag{4.7}$$

From the above identities and $(3.3)_2$ we may compare the function $b(t)$ with $\bar{b}(t)$ of $(4.5)_2$:

$$b' + (n\gamma - n + 2)(1 + O(1)t^{-1})b^2 = 0,$$

$$b(t) = \left[b(t_0)^{-1} + \int_{t_0}^{t} (n\gamma - n + 2)(1 + O(1)s^{-1})ds\right]^{-1}$$
$$= \bar{b}(t)\left(1 + O(1)\frac{\ln t}{t}\right). \tag{4.8}$$

Plug (4.7), (4.8) into $(3.3)_1$ to obtain

$$e' + n(\gamma - 1)e\bar{a}\left(1 + O(1)\frac{\ln t}{t}\right) = 0,$$

$$e(t) = e(t_0)e^{-\int_{t_0}^{t} n(\gamma-1)\bar{a}(s)ds} e^{-\int_{t_0}^{t} O(1)(\ln s/s^2)ds}$$
$$= A\bar{e}(t)\left(1 + O(1)\int_t^{\infty} \frac{\ln s}{s^2} ds\right)$$
$$= A\bar{e}(t)\left(1 + O(1)\frac{\ln t}{t}\right), \tag{4.9}$$

for some constant A. Note that both (1.1) and (1.3) satisfy the same conservation law

$$m = \int_{-\infty}^{\infty} \rho(x, t)dx.$$

When we apply this time-asymptotically it follows from (4.6), (4.8) and (4.9) that $A = 1$. This completes the proof of the theorem. Q.E.D.

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