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Variable Coefficient A-stable Explicit Runge-Kutta Methods

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The explicit Runge-Kutta methods possessing extended stability regions have already been discussed by many authors including van der Houwen [6]. It is the purpose of the present paper to derive a class of variable coefficients A-stable explicit Runge-Kutta methods. Some numerical tests justifying the results are given.

Key words: initial-value problem, stability, Runge-Kutta method, variable coefficients formulae

1. Introduction

The present paper is concerned with the numerical integration for the stiff system of ordinary differential equations:

$$y' = f(x,y), \quad y(x_0) = y_0, \quad y \in R^m, \tag{1.1}$$

or if we rewrite (1.1) in it's component form

$${}^l y' = {}^l f(x, {}^1y, {}^2y, {}^3y, \ldots, {}^m y), \quad y(x_0) = y_0 \quad (l = 1, 2, \ldots, m). \tag{1.2}$$

A basic difficulty in the numerical solution of stiff systems is the satisfying of the requirement of stability. By the restriction of stability, implicit methods have been proposed, and some explicit methods imposing stability conditions have been derived; however, there is still a stability problem for an explicit method. So it is the purpose of the present paper to derive the explicit A-stable Runge-Kutta methods (abb. R-K methods) with respect to the model equation defined in (2.4). Using the idea of deriving stabilized R-K methods explored by P.J. van der Houwen [6], we propose variable coefficients A-stable R-K methods. The outline of this paper is as follows: In §2, we consider two-stage first order and three-stage second order R-K methods. In §3, we propose some numerical tests.

2. Derivation of the Formulae

Consider r-stage explicit R-K methods:

$${}^l y_{n+1} = {}^l y_n + h \sum_{i=1}^{r} {}^l b_i {}^l k_i, \tag{2.1}$$

$$
\begin{aligned}
{}^l k_1 &= {}^l f(x_n, {}^1y_n, {}^2y_n, \ldots, {}^m y_n), \\
{}^l k_i &= {}^l f(x_n + c_i h, {}^1y_n + h\sum a_{ij}{}^1k_j, \ldots, {}^m y_n + h\sum a_{ij}{}^m k_j), \\
& \qquad c_i = \sum a_{ij} \qquad (i = 2, \ldots, r)\ (l = 1, 2, \ldots, m).
\end{aligned}
$$

Here the weights $\{{}^l b_i\}$ vary with the component. Introducing the vector notations

$$
\begin{aligned}
Y_p &= [{}^1y_p, {}^2y_p, \ldots, {}^m y_p], \quad K_i = [{}^1k_i, {}^2k_i, \ldots, {}^m k_i], \\
B_i &= \mathrm{diag}[{}^1b_i, {}^2b_i, \ldots, {}^m b_i],
\end{aligned}
$$

we may write (2.1) in the form

$$
Y_{n+1} = Y_n + h\sum_{i=1}^{r} B_i K_i.
$$

The order conditions of Runge-Kutta methods which are discussed in [1], are listed up to second order:

$$
\text{order 1}: \qquad \sum {}^l b_i = 1, \tag{2.2}
$$

$$
\text{order 2}: \qquad \sum {}^l b_i c_i = 1/2. \tag{2.3}
$$

Let us now apply the r-stage, p-th order R-K methods (2.1) to the test equation

$$
{}^l y' = \lambda_l {}^l y, \quad \mathrm{Re}(\lambda_l) < 0 \qquad (l = 1, 2, \ldots, m), \tag{2.4}
$$

then we have the stability polynomial

$$
{}^l y_{n+1} = S(z_l){}^l y_m, \tag{2.5}
$$

where $S(z_l)$ takes the form

$$
S(z_l) = \sum_{i=0}^{p} \frac{(z_l)^i}{i!} + \sum_{\kappa=p+1}^{r} \gamma_\kappa z_l^\kappa \qquad (z_l = \lambda_l h). \tag{2.6}
$$

Thus, the method (2.1) is A-stable if for all $h > 0$

$$
|S(z_l)| < 1.
$$

In this paper, the case of $(p, r) = (1, 2)$ and (2,3) in (2.6) are discussed.

2.1. Two-stage first order Runge-Kutta formulae

Setting $p = 1$ and $r = 2$ in (2.6), we have

$$
{}^l y_{n+1} = (1 + z_l + {}^l b_2 a_{21} z_l^2){}^l y_n. \tag{2.7}
$$

If ${}^{l}b_2a_{21}$ takes the form

$$
{}^{l}b_2a_{21} = \frac{\sum_{i=0}^{s} \widetilde{b}_i z_l^i}{\sum_{i=0}^{m} \widetilde{a}_i z_l^i} \qquad (\widetilde{a}_i, \widetilde{b}_i;\ \text{constants}), \tag{2.8}
$$

then, putting (2.8) to (2.7), it is required that

$$
m \leq s - 1,
$$

where s is any positive integer. Many formulas may be considered in determining (2.8), in this paper, we study the case $s = 2, 3$ in the first order and $s = 2, 3, 4$ in the second order. Hereafter, for simplicity of expression, we abbreviate z_l by z. Assuming ${}^{l}b_2a_{21}$ in the form

$$
{}^{l}b_2a_{21} = \frac{\delta + \rho z}{\alpha + \beta z + \gamma z^2}, \tag{2.9}
$$

then we have

$$
{}^{l}y_{n+1} = \frac{\alpha + (\alpha + \beta)z + (\beta + \gamma + \delta)z^2 + (\gamma + \rho)z^3}{(\alpha + \beta z + \gamma z^2)}\, {}^{l}y_n, \tag{2.10}
$$

where the undetermined parameters $\alpha, \beta, \gamma, \delta$ and ρ must be chosen so that (2.10) satisfies the A-stability conditions.

Case (I) $(s = 1)$: Firstly, we consider the case $\gamma = \rho = 0$ in (2.9), putting those values in (2.10), we have the stability conditions

$$
\beta + \delta = 0 \quad \text{and} \quad |R(z)| < 1
$$

with

$$
R(z) = \frac{\alpha + (\alpha + \beta)z}{\alpha + \beta z},
$$

which lead to the conditions

$$
\alpha\beta < 0 \quad \text{and} \quad \alpha(\alpha + 2\beta) < 0.
$$

If we take, for example, $\alpha = 1$, $\beta = -1$, and $\gamma = \rho = 0$, then (2.9) reduces to

$$
{}^{l}b_2a_{21} = \frac{1}{1 - z}, \tag{2.11}
$$

and (2.10)

$$
{}^{l}y_{n+1} = \frac{1}{1 - z}\, {}^{l}y_n,
$$

which is an A-stable algorithm. In determining the coefficient ${}^l b_2$ from order condition (2.2), we leave ${}^l b_2$ as a free parameter which is specified in the form

$$ {}^l b_2 a_{21} = \frac{{}^l y_n}{{}^l y_n - h\,{}^l u_1} \tag{2.12} $$

where ${}^l u_1 = {}^l f(x_n, {}^1 y_n, {}^2 y_n, \ldots, {}^m y_n)$.

It is easily seen that (2.12) applied to the test function (2.4) gives the same value as that of (2.11) and is bounded by

$$ \left| \frac{1}{1-z} \right| < 1 \quad \text{for} \quad \mathrm{Re}(z) < 0. $$

From those results, we may set ${}^l b_2$ as follows:

(i) If $-1 < D_1 = \dfrac{{}^l y_n}{{}^l y_n - h\,{}^l z_1} < 1$, then

$$ {}^l b_2 = D_1 / a_{21}, \tag{2.13} $$

(ii) If $|D_1| \geq 1$, then

$$ {}^l b_2 = \mathrm{sgn}(D_1)/a_{21}, \tag{2.14} $$

where ${}^l z_1 = {}^l f(x_n, {}^1 y_n, {}^2 y_n, \ldots, {}^m y_n)$.
Therefore, solving the first order condition (2.2) with (2.13) and (2.14), we have the l-th component coefficients of R-K methods (2.1)

(A) if $|D_1| < 1$, then

$$ {}^l b_2 = \frac{D_1}{a_{21}}, \quad {}^l b_2 = 1 - {}^l b_1, \quad c_2 = a_{21} \quad (a_{21} : \text{ free parameter}), \tag{2.15} $$

(B) if $|D_1| \geq 1$, then

$$ {}^l b_2 = \frac{\mathrm{sgn}(D_1)}{a_{21}}, \quad {}^l b_1 = 1 - {}^l b_2, \quad c_2 = a_{21} \quad (a_{21} : \text{ free parameter}). \tag{2.16} $$

Case (II) $(s = 2)$: From (2.10), we have the stability conditions

$$ \gamma + \rho = 0 \quad \text{and} \quad |S(z)| < 1 \tag{2.17} $$

with

$$ S(z) = \frac{\alpha + (\alpha + \beta)z + (\beta + \gamma + \delta)z^2}{\alpha + \beta z + \gamma z^2}, \tag{2.18} $$

which lead to

(i) $\gamma + \rho = 0$,

(ii) $\beta^2 - 4\alpha\gamma < 0$, or $\beta^2 - 4\alpha\gamma > 0$, $\beta\gamma < 0$, $\alpha\gamma > 0$,

(iii) $(\beta + \delta)(2\gamma + \beta + \delta) < 0, \quad \alpha(2\delta - \alpha) > 0$.

If, for example, we take the parameters α, β, γ, δ and ρ satisfying the conditions (i), (ii) and (iii)

$$\alpha = 1, \quad \beta = -2, \quad \gamma = 1, \quad \delta = 1 \quad \text{and} \quad \rho = -1,$$

then (2.9) reduces to

$$^lb_2 a_{21} = \frac{1 - z}{1 - z + z^2}, \tag{2.19}$$

and (2.10)

$$^ly_{n+1} = \frac{1 - z}{1 - z + z^2}\, ^ly_n,$$

which is an A-stable algorithm. By the same reason stated in the case (I), we may replace (2.19) by the function

$$^lb_2 a_{21} = \frac{^ly_n - h^lu_1}{^ly_n - 3h^lu_1 + 2h^lu_2} \tag{2.20}$$

with $^lu_1 = {}^lf(x_n, {}^1y_n, \ldots, {}^my_n)$, $^lu_2 = {}^lf(x_n + h/2,\ {}^1y_n + h^1u_1/2, \ldots,\ {}^my_n + h^mu_1/2)$. It is easily seen that the function (2.19) is bounded by

$$\left| \frac{1 - z}{1 - z + z^2} \right| < 1 \quad \text{for } \operatorname{Re}(z) < 0.$$

From those results, we may set $^lb_2$ as follows:

(i) if $-1 < D_2 = \dfrac{^ly_n - h^lu_1}{^ly_n - 3h^lu_1 + 2h^lu_2} < 1$, then

$$^lb_2 = D_2/a_{21}, \tag{2.21}$$

(ii) if $|D_2| > 1$, then

$$^lb_2 = \operatorname{sgn}(D_2)/a_{21}, \tag{2.22}$$

where $^lu_1 = {}^lf(x_n, {}^1y_n, \ldots, {}^my_n)$, $^lu_2 = {}^lf(x_n + h/2,\ {}^1y_n + h^1u_1/2, \ldots,\ {}^my_n + h^mu_1/2)$.

Solving the order conditions in the component form, from (2.2), (2,21) and (2.22), we have the l-th component coefficients

(A) if $-1 < D_2 = \dfrac{^ly_n - h^lu_1}{(^ly_n - 3h^lu_1 + 2h^lu_2)} < 1$, then we take

$$^lb_2 = \frac{D_2}{a_{21}}, \ {}^lb_1 = 1 - {}^lb_2, \ c_2 = a_{21} \tag{2.23}$$

$$(a_{21} : \text{ free parameter}),$$

(B) if $|D_2| \geq 1$, then we take

$$ {}^l b_2 = \frac{\text{sgn}(D_2)}{a_{21}},\ {}^l b_1 = 1 - {}^l b_2,\ c_2 = a_{21} \qquad (2.24) $$

$(a_{21}$: free parameter).

2.2. Three stage second order Runge-Kutta formulae

In this section, we discuss the second order methods. Applying the test function (2.4) to the second order R-K methods, we have the numerical processes

$$ {}^l y_{n+1} = (1 + z + \frac{z^2}{2} + {}^l b_3 a_{32} a_{21} z^3) {}^l y_n. \qquad (2.25) $$

Assuming ${}^l b_3 a_{32} a_{21}$ in the form

$$ {}^l b_3 a_{32} a_{21} = \frac{\delta + \rho z + \tau z^2}{2!(\alpha + \beta z + \gamma z^2 + \eta z^3)}, \qquad (2.26) $$

we have

$$ {}^l y_{n+1} = \frac{2\alpha + 2(\alpha+\beta)z + (\alpha+2\beta+2\gamma)z^2 + (\beta+2\gamma+2\eta+\delta)z^3 + (\gamma+2\eta+\rho)z^4 + (\eta+\tau)z^5}{2!(\alpha+\beta z+\gamma z^2+\eta z^3)} {}^l y_n. \qquad (2.27) $$

Proceeding in the same way as in the case of the first order, we give some A-stable algorithms.

Case (I) $(s = 1)$: Firstly, we consider the case $\gamma = \eta = \rho = \tau = 0$ in (2.26), putting those values in (2.27), we have the stability conditions

$$ \alpha = -2\beta \quad \text{and} \quad \delta = -\beta, $$

which lead to

$$ {}^l b_3 a_{32} a_{21} = \frac{1}{2(2 - z)}, \qquad (2.28) $$

and (2.27) takes the form

$$ {}^l y_{n+1} = \frac{2 + z}{2 - z} {}^l y_n, $$

which is an A-stable algorithm. It is easily seen that if we set

$$ {}^l b_3 a_{32} a_{21} = \frac{{}^l y_n}{2(2 {}^l y_n - h {}^l u_1)} \qquad (2.29) $$

with ${}^l u_1 = {}^l f(x_n, {}^1 y_n, {}^2 y_n, \ldots, {}^m y_n)$, then (2.29) applied to the test function (2.4) gives the same value as that of (2.28). It can be shown that the function (2.28) is bounded by

$$ \left| \frac{1}{2(2 - z)} \right| < \frac{1}{4} \quad \text{for} \quad \text{Re}(z) < 0. $$

From those results, we may set ${}^l b_3$ as follows:

(i) if $-\frac{1}{4} < D_3 = \frac{{}^l y_n}{2(2{}^l y_n - h{}^l u_1)} < \frac{1}{4}$, then we take

$$ {}^l b_3 = D_3 / a_{32} a_{21}, \tag{2.30} $$

(ii) if $|D_3| \geq \frac{1}{4}$, then we take

$$ {}^l b_3 = \mathrm{sgn}(D_3)/(4a_{32}a_{21}), \tag{2.31} $$

where ${}^l u_1 = {}^l f(x_n, {}^1 y_n, {}^2 y_n, \ldots, {}^m y_n)$.
Therefore, solving the second order conditions (2.2), (2.3) with (2.30) and (2.31), we have the l-th components coefficients of R-K methods (2.1).

(A) if $|D_3| < \frac{1}{4}$, then we set

$$ {}^l b_3 = \frac{D_3}{a_{32}a_{21}}, \quad {}^l b_2 = \frac{1}{c_1}\left(\frac{1}{2} - {}^l b_3 c_3\right), \tag{2.32} $$
$$ {}^l b_1 = 1 - ({}^l b_2 + {}^l b_3), $$
$$ c_2 = a_{21}, \quad c_3 = a_{31} + a_{32} $$
$$ (a_{21}, a_{31}, a_{32} : \text{ free parameters}), $$

(B) if $|D_3| \geq \frac{1}{4}$, then we set

$$ {}^l b_3 = \frac{\mathrm{sgn}(D_3)}{4a_{32}a_{21}}, \quad {}^l b_2 = \frac{1}{c_1}\left(\frac{1}{2} - {}^l b_3 c_3\right), \tag{2.33} $$
$$ {}^l b_1 = 1 - ({}^l b_2 + {}^l b_3), $$
$$ c_2 = a_{21}, \quad c_3 = a_{31} + a_{32} $$
$$ (a_{21}, a_{31}, a_{32} : \text{ free parameters}). $$

Case (II) $(s = 2)$: Secondly, we consider the case $\eta = \tau = 0$ in (2.26), we put those values to (2.27) which leads to the following stability conditions

$$ \beta + 2\gamma + \delta = 0, \ \gamma + \rho = 0 \quad \text{and } |S(z)| < 1 \tag{2.34} $$

with

$$ S(z) = \frac{2\alpha + 2(\alpha+\beta)z + (\alpha + 2\beta + 2\gamma)z^2}{2!(\alpha + \beta z + \gamma z^2)}, \tag{2.35} $$

which lead to

(i) $\beta + 2\gamma + \delta = 0, \gamma + \rho = 0,$

(ii) $\beta^2 - 4\alpha\gamma < 0$, or $\beta^2 - 4\alpha\gamma > 0$, $\beta\gamma < 0$, $\alpha\gamma > 0$,

(iii) $(\alpha + 2\beta)(\alpha + 2\beta + 4\gamma) < 0$.

If, for example, we take the parameters α, β, γ, δ and ρ satisfying the conditions (i), (ii) and (iii)

$$\alpha = 1,\ \beta = -1,\ \gamma = 1/2,\ \delta = 0 \text{ and } \rho = -1/2,$$

then (2.26) reduces to

$$^{l}b_3 a_{32} a_{21} = \frac{-z}{2(2 - 2z + z^2)}. \tag{2.36}$$

Substituting (2.36) into (2.25) yields

$$^{l}y_{n+1} = \frac{2}{2 - 2z + z^2}\, {}^{l}y_n,$$

which is an A-stable algorithm. As the same reason stated in case (I), we may replace (2.36) by the function

$$^{l}b_3 a_{32} a_{21} = \frac{-h^{l}u_1}{4(^{l}y_n - 2h^{l}u_1 + h^{l}u_2)} \tag{2.37}$$

with $^{l}u_1 = {}^{l}f(x_n, {}^{1}y_n, \ldots, {}^{m}y_n)$, $^{l}u_2 = {}^{l}f(x_n + h/2, y_n + h^{1}u_1/2, \ldots, {}^{m}y_n + h^{m}u_1/2)$.
It is easily seen that the function (2.36) is bounded by

$$\left| \frac{-z}{2(2 - 2z + z^2)} \right| \le \frac{1}{4\sqrt{2}} \quad \text{for} \quad \mathrm{Re}(z) < 0,$$

so we may set $^{l}b_3$ as follows

(i) if $-\dfrac{1}{4\sqrt{2}} \le D_4 = \dfrac{-h^{l}u_1}{4(^{l}y_n - 2h^{l}u_1 + h^{l}u_2)} \le \dfrac{1}{4\sqrt{2}}$, then

$$^{l}b_3 = D_4/(a_{32}a_{21}), \tag{2.38}$$

(ii) if $|D_4| > \dfrac{1}{4\sqrt{2}}$, then

$$^{l}b_3 = \mathrm{sgn}(D_4)/(4\sqrt{2}\, a_{32}a_{21}), \tag{2.39}$$

where $^{l}u_1 = {}^{l}f(x_n, {}^{1}y_n, \ldots, {}^{m}y_n)$, $^{l}u_2 = {}^{l}f(x_n + h/2, {}^{1}y_n + h^{1}u_1/2, \ldots, {}^{m}y_n + h^{m}u_1/2)$.

Thus, solving the second order conditions with (2.38) and (2.39) in the same way as in case (I), we have the l-th component coefficients of R-K methods (2.1)

(A) if $|D_4| \le \dfrac{1}{4\sqrt{2}}$, then we have

$$^{l}b_3 = \frac{D_4}{a_{32}a_{21}},\ {}^{l}b_2 = \frac{1}{c_2}\left(\frac{1}{2} - {}^{l}b_3 c_3\right), \tag{2.40}$$

$$^{l}b_1 = 1 - (^{l}b_2 + {}^{l}b_3),\ c_2 = a_{21},\ c_3 = a_{31} + a_{32}$$

$$(a_{21}, a_{31}, a_{32} : \text{ free parameters}),$$

(B) if $|D_4| > \frac{1}{4\sqrt{2}}$, then we have

$$^{l}b_3 = \frac{\mathrm{sgn}(D_4)}{4\sqrt{2}a_{32}a_{21}},\ ^{l}b_2 = \frac{1}{c_2}\left(\frac{1}{2} - {}^{l}b_3c_3\right), \tag{2.41}$$

$$^{l}b_1 = 1 - ({}^{l}b_2 + {}^{l}b_3),\ c_3 = a_{31} + a_{32}$$

$(a_{21}, a_{31}, a_{32}:$ free parameters).

Case (III) ($s = 3$): From (2.27), we have the stability conditions

$$\gamma + 2\eta + \rho = 0,\ \eta + \tau = 0 \quad \text{and } |S(z)| < 1 \tag{2.42}$$

with

$$S(z) = \frac{2\alpha + 2(\alpha+\beta)z + (\alpha + 2\beta + 2\gamma)z^2 + (\beta + 2\gamma + 2\eta + \delta)z^3}{2!(\alpha + \beta z + \gamma z^2 + \eta z^3)}, \tag{2.43}$$

which lead to

(i) $\gamma + 2\eta + \rho = 0,\ \eta + \tau = 0,$

(ii) $\alpha > 0,\ \beta < 0,\ \gamma > 0,\ \eta < 0,$

or

(ii)′ $\alpha < 0,\ \beta > 0,\ \gamma < 0,\ \eta > 0,$

or

(ii)″ $\alpha\eta < 0,\ D < 0,$

or

(ii)‴ $\alpha < 0,\ \eta > 0,\ D \geq 0,\ -2DS_1 + (9\alpha\eta - \beta\gamma) < 0,$

or

(ii)$^{\text{iv}}$ $\alpha > 0,\ \eta < 0,\ D \geq 0,\ -2DS_2 + (9\alpha\eta - \beta\gamma) > 0,$
with $D = \gamma^2 - 3\beta\eta,\ S_1 = -(\gamma + D)/3\eta,\ S_2 = -(\gamma + D)/3\eta,$

(iii) $\alpha(-\alpha + 4\gamma + 2\eta + 4\delta) + 4\beta\delta > 0,\ (\beta + 2\gamma + \delta)(\beta + 2\gamma + 4\eta + \delta) < 0.$

If, for example, we take the parameters α, β, γ, δ and η satisfying the conditions (i), (ii) and (iii)

$$\alpha = 1,\ \beta = 1,\ \gamma = 0,\ \eta = -50,\ \delta = 99,$$

then (2.26) reduces to

$$^{l}b_3a_{32}a_{21} = \frac{99 + 100z + 50z^2}{2(1 + z - 50z^3)}. \tag{2.44}$$

Putting (2.44) into (2.25) yields

$$^{l}y_{n+1} = \frac{2 + 4z + 3z^2}{2(1 + z - 50z^3)}\,{}^{l}y_n,$$

which is A-stable algorithm. Analyzing in the same way as in case (I), we may replace (2.44) by the function

$$^{l}b_3a_{32}a_{21} = \frac{99\,{}^{l}y_n + 100h\,{}^{l}u_2}{2({}^{l}y_n - 99h\,{}^{l}u_1 + 200h\,{}^{l}u_2 - 100h\,{}^{l}u_3)} \tag{2.45}$$

with ${}^l u_1 = {}^l f(x_n, {}^1 y_n, \ldots, {}^m y_n)$, ${}^l u_2 = {}^l f(x_n + h/2,\ {}^1 y_n + h^1 u_1/2, \ldots,\ {}^m y_n + h^m u_1/2)$, ${}^l u_3 = {}^l f(x_n + h, {}^1 y_n + h^1 u_2, \ldots, {}^m y_n + h^m u_2)$.
The function (2.44) is bounded by

$$\left| \frac{99 + 100z + 50z^2}{2(1 + z - 50z^3)} \right| < 49.5 \quad \text{for } \mathrm{Re}(z) < 0.$$

So we may set ${}^l b_3$ as follows

(i) if $-49.5 < D_5 = \dfrac{99^l y_n + 100h^l u_2}{2({}^l y_n - 99h^l u_1 + 200h^l u_2 - 100h^l u_3)} < 49.5$, then

$$ {}^l b_3 = D_5/(a_{32}a_{21}), \tag{2.46}$$

(ii) if $|D_5| \geq 49.5$, then

$$ {}^l b_3 = \mathrm{sgn}(D_5)/(a_{32}a_{21}). \tag{2.47}$$

where ${}^l u_1 = {}^l f(x_n, {}^1 y_n, \ldots, {}^m y_n)$, ${}^l u_2 = {}^l f(x_n + h/2,\ {}^1 y_n + h^1 u_1/2, \ldots,\ {}^m y_n + h^m u_1/2)$, ${}^l u_3 = {}^l f(x_n + h, {}^1 y_n + h^1 u_2, \ldots, {}^m y_n + h^m u_2)$.
In the same way as previously, from (2.2), (2.3), (2.46) and (2.47) we have the l-th component coefficients of R-K methods (2.1)

(A) if $|D_5| < 1$, then we have

$$ {}^l b_3 = \frac{D_5}{a_{32}a_{21}}, \quad {}^l b_2 = \frac{1}{c_2}\left(\frac{1}{2} - {}^l b_3 c_3\right), \tag{2.48}$$

$$ {}^l b_1 = 1 - ({}^l b_2 + {}^l b_3), \ c_2 = a_{21}, \ c_3 = a_{31} + a_{32},$$
$$ (a_{21}, a_{31}, a_{32} : \text{ free parameters}),$$

(B) if $|D_5| \geq 1$, then we have

$$ {}^l b_3 = \frac{\mathrm{sgn}(D_5)}{a_{32}a_{21}}, \quad {}^l b_2 = \frac{1}{c_2}\left(\frac{1}{2} - {}^l b_3 c_3\right), \tag{2.49}$$

$$ {}^l b_1 = 1 - ({}^l b_2 + {}^l b_3), \ c_3 = a_{31} + a_{32},$$
$$ (a_{21}, a_{31}, a_{32} : \text{ free parameters}).$$

3. Computational Results

We propose numerical methods whose numerical processes applied to the test function are bounded. We restrict our study on the number of function evaluations and the largest step-size whose numerical processes are stable, i.e., using the error bound, the largest step-size whose absolute error is bounded. By some numerical comparison the methods described in the preceding section with Runge-Kutta type methods indicate that our methods are much more efficient than other R-K methods for the numerical solution of stiff equations. The test problems are taken from the examples of A.I. Johnson and J.R. Barney [1] and Hull, Enright and Sedgwick [3]. The methods considered are

Explicit Runge-Kutta Type Formulas.

Euler 1: Euler's First-Order Formulas.

Heun 2: Heun Second-Order Formulas.

Nyström 3: Nyström Third-Order Formulas.

Implicit Runge-Kutta Type Formulas.

Euler 1: Euler's First-Order Formulas.

Gauss 2: Gauss 1-point Second-Order Formulas.

Radau 3: Radau 2-point Third-Order Formulas.

Variable Coefficients Explicit Runge-Kutta Type Formulas.

VCRK1:

$$\text{case I}: \quad \begin{array}{c|cc} \frac{1}{2} & \frac{1}{2} & \\ \hline & b_1 & b_2 \end{array} \qquad \text{case II}: \quad \begin{array}{c|cc} \frac{1}{2} & \frac{1}{2} & \\ \hline & \widetilde{b}_1 & \widetilde{b}_2 \end{array}$$

where b_1 and b_2 are given by (2.15), (2.16) and $\widetilde{b}_1$, $\widetilde{b}_2$ by (2.23), (2.24).

VCRK2:

$$\text{case I}: \begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & & \\ 1 & \frac{6}{7} & \frac{1}{7} & \\ \hline & b_1 & b_2 & b_3 \end{array} \quad \text{case II}: \begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & & \\ 1 & \frac{6}{7} & \frac{1}{7} & \\ \hline & \widetilde{b}_1 & \widetilde{b}_2 & \widetilde{b}_3 \end{array} \quad \text{case III}: \begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & & \\ 1 & \frac{6}{7} & \frac{1}{7} & \\ \hline & \overline{b}_1 & \overline{b}_2 & \overline{b}_3 \end{array}$$

where b_1, b_2 and b_3 are given by (2.32), (2.33), $\widetilde{b}_1$, $\widetilde{b}_2$, $\widetilde{b}_3$ by (2.40), (2.41) and $\overline{b}_1$, $\overline{b}_2$, $\overline{b}_3$ by (2.48), (2.49).

We set the heading

Fev: number of function evaluations necessary to integrate the given integration interval.

SDj: the absolute error of the j-th component with respect to a given reference solution.

Problems

(1) A linear, modestly stiff system with real eigenvalues:

$$Y' = AY, \quad Y(0) = (1,1,1,1),$$

with

$$A = \begin{pmatrix} -0.5 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 0 \\ 0 & 0 & -100.0 & 0 \\ 0 & 0 & 0 & -90.0 \end{pmatrix},$$

its theoretical solution is

$$Y(x) = \left(e^{-0.5x},\ e^{-x},\ e^{-100x},\ e^{-90x}\right).$$

(2) A linear, modestly stiff system with real eigenvalues:

$$Y' = AY, \quad Y(0) = (0, 2),$$

with

$$A = \begin{pmatrix} -500.5 & 499.5 \\ 499.5 & -500.5 \end{pmatrix},$$

its theoretical solution is

$$y_1(x) = \exp(-x) - \exp(-1000x),$$
$$y_2(x) = \exp(-x) + \exp(-1000x).$$

(3) A linear, highly stiff system with real eigenvalues:

$$Y' = AY, \quad Y(0) = (0, 2),$$

with

$$A = \begin{pmatrix} -500000.5 & 499999.5 \\ 499999.5 & -500000.5 \end{pmatrix},$$

its theoretical solution is

$$y_1(x) = \exp(-x) - \exp(-1000000x),$$
$$y_2(x) = \exp(-x) + \exp(-1000000x).$$

(4) A linear, highly stiff system with real eigenvalues:

$$Y' = AY, \quad Y(0) = (1, 1, 1, 1)$$

with

$$A = \begin{pmatrix} -10^5 & 10^2 & -10 & 1 \\ 0 & -10^4 & 10 & -10 \\ 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

its theoretical solution is

$$Y(x) = (u(x),\ v(x),\ w(x),\ z(x)),$$

where

$$\begin{aligned}
u(x) &= c_1 \exp(-10^5 x) + c_2 \exp(-10^4 x) + c_3 \exp(-10x) + c_4 \exp(-x),\\
v(x) &= 900 c_2 \exp(-10^4 x) - \exp(-10x)/8991 + 10\exp(-x)/89991,\\
w(x) &= (10\exp(-x) - \exp(-10x))/9,\\
z(x) &= \exp(-x),\\
c_2 &= \frac{1}{900}\left(1 + \frac{1}{12208779}\right),\ c_3 = \frac{1}{89991}\left(1 - \frac{10}{999}\right),\\
c_4 &= \frac{1}{99999}\left(1 - \frac{100}{9} + \frac{1000}{89991}\right),\ c_1 = 1 - c_2 - c_3 - c_4.
\end{aligned}$$

(5) A linear, highly stiff system with complex eigenvalues:

$$Y' = AY, \quad Y(0) = (1,1,1,1)$$

with

$$A = \begin{pmatrix} -10^4 & 10^3 & 0 & 0 \\ -10^3 & -10^4 & 0 & 0 \\ 0 & 0 & -10 & 100 \\ 0 & 0 & -100 & -10 \end{pmatrix},$$

its theoretical solution is

$$Y(x) = (u(x),\ v(x),\ w(x),\ z(x)),$$

where

$$\begin{aligned}
u(x) &= \exp(-10^4 x)(\cos(10^3 x) + \sin(10^3 x)),\\
v(x) &= \exp(-10^4 x)(\cos(10^3 x) - \sin(10^3 x)),\\
w(x) &= \exp(-10x)(\cos(10^2 x) + \sin(10^2 x)),\\
z(x) &= \exp(-10x)(\cos(10^2 x) - \sin(10^2 x)).
\end{aligned}$$

CONCLUSION. The purpose of this paper is to develop efficient numerical algorithms of explicit R-K methods for solving stiff differential equations. From the numerical result of example I, we know that our methods are stable for the test problem. By comparing the results of Tables I, II, we see that among the first order methods, (2.23) and (2.24) are more efficient than (2.15) and (2.16), and among the second order algorithms, (2.48) and (2.49) are more efficient.

4. Additional Comments

There remain some problems which we plan to investigate. First, We have to analyze how to determine the parameters which appeared in (2.9) and (2.26). Second, we have to study the step control policy for the methods.

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TABLE I

Absolute Error

Problem I.

Explicit Methods:

	$x = 0.156E-1$		$h = 1/2^6$
	Euler 1	Heun 2	Nystrom 3
SD1	0.301E−4	0.392E−7	0.374E−10
SD2	0.118E−3	0.309E−6	0.602E−9
SD3	0.416E−1	0.314E−1	0.491E−2
SD4	0.522E−1	0.275E−1	0.420E−2
Fev	8	16	24

	$x = 0.484$		$h = 1/2^6$
	Euler 1	Heun 2	Nystrom 3
SD1	0.746E−3	0.194E−5	0.377E−8
SD2	0.235E−2	0.122E−4	0.479E−7
SD3	0.179E−7	0.233E−5	0.920E−21
SD4	0.745E−12	0.530E−7	0.116E−18
Fev	124	248	372

Implicit Methods:

	$x = 0.781E-1$	$h = 1/2^7$
	Euler 1	Gauss 2
SD1	0.540E+0	0.494E−8
SD2	0.785E+0	0.394E−7
SD3	0.457E+0	0.196E−1
SD4	0.495E+0	0.152E−1
Fev	808	64

$x = 0.523$		$h = 1/2^7$
	Euler 1	Gauss 2
SD1	0.572E+0	0.257E−6
SD2	0.553E+0	0.159E−5
SD3	0.183E−22	0.184E−22
SD4	0.347E−20	0.343E−20
Fev	1928	1240

Radau 3 $h = 1/2^7$

X	SD1	SD2	SD3	SD4	Fev
0.585E−2	0.229E−10	0.345E−9	0.128E−3	0.148E−3	69
0.242E+0	0.143E−9	0.198E−8	0.846E−11	0.642E−10	2860

Variable Coefficients Explicit Runge-Kutta Methods:

VCRK1: case I $h = 1$

X	SD1	SD2	SD3	SD4	Fev
1.00	0.601E−1	0.132E+0	0.990E−2	0.109E−1	8
7.00	0.283E−1	0.690E−2	0.932E−14	0.193E−13	56

VCRK1: case II $h = 1$

X	SD1	SD2	SD3	SD4	Fev
1.00	0.107E+0	0.298E+0	0.990E+0	0.989E+0	8
7.00	0.649E−1	0.576E−1	0.932E+0	0.925E+0	56

VCRK2: case I $h = 1$

X	SD1	SD2	SD3	SD4	Fev
1.00	0.653E−2	0.345E−1	0.960E+0	0.956E+0	12
7.00	0.220E−2	0.454E−3	0.755E+0	0.732E+0	84

VCRK2: case II $h = 1$

X	SD1	SD2	SD3	SD4	Fev
1.00	0.885E−2	0.321E−1	0.196E−3	0.241E−3	12
7.00	0.322E−2	0.726E−3	0.111E−25	0.0	84

VCRK2: case III $\underline{h = 1}$

X	SD1	SD2	SD3	SD4	Fev
1.00	0.550E+0	0.357E+0	0.296E−3	0.328E−3	16
7.00	0.301E−1	0.911E−3	0.199E−24	0.412E−24	112

Problem II.

Explicit Method:

	$x = 0.390\text{E} - 2$		$h = 1/2^9$
	Euler 1	Heun 2	Nystrom 3
SD1	0.888E+0	0.890E+0	0.625E−1
SD2	0.888E+0	0.890E+0	0.625E−1
Fev	4	8	14

	$x = 0.048..$		$h = 1/2^9$
	Euler 1	Heun 2	Nystrom 3
SD1	0.301E+0	0.309E+0	0.130E−10
SD2	0.316E+0	0.309E+0	0.137E−10
Fev	62	124	186

Implicit Methods:

Euler 1 $\underline{h = 1/2^9}$

X	SD1	SD2	Fev
0.195E−2	0.135E+0	0.321E+0	104
0.117E−1	0.438E+0	0.438E+0	624

	$x = 0.976\text{E}{-3}$		$x = 0.537\text{E}{-1}$	$h = 1/2^{10}$
	Gauss 2	Radau 3	Gauss 2	Radau 3
SD1	0.327E−1	0.396E−2	0.404E−8	0.172E−11
SD2	0.327E−1	0.396E−2	0.404E−8	0.172E−11
Fev	96	92	650	624

Variable Coefficients Explicit Runge-Kutta Methods:

VCRK1: case I $h = 1/2^{10}$

X	SD1	SD2	Fev
0.195E−2	0.125E+0	0.458E+0	8
0.537E−1	0.772E−1	0.116E+0	220

VCRK1: case II $h = 1/2^{10}$

X	SD1	SD2	Fev
0.195E−2	0.149E+0	0.468E+0	8
0.537E−1	0.105E+0	0.147E+0	220

VCRK2: case I $h = 1/2^{9}$

X	SD1	SD2	Fev
0.976E−2	0.942E+0	0.917E+1	20
0.195E−1	0.912E+0	0.912E+0	40

VCRK2: case II $h = 1/2^{4}$

X	SD1	SD2	Fev
0.125E+0	0.881E+0	0.879E+0	12
0.156E+1	0.209E+0	0.209E+0	150

VCRK2: case III $h = 1$ $h = 1/2^{4}$

X	SD1	SD2	Fev	X	SD1	SD2	Fev
1.0	0.357E+0	0.357E+0	8	0.125E+0	0.882E+0	0.882E+0	16
5.0	0.673E−2	0.673E−2	40	0.812E+0	0.443E+0	0.443E+0	104

Problem III.

Explicit Method:

$x = 0.381\text{E} - 5$ $h = 1/2^{19}$

	Euler 1	Heun 2	Nystrom 3
SD1	0.801E+0	0.809E+0	0.379E−1
SD2	0.801E+0	0.809E+0	0.379E−1
Fev	4	8	12

	$x = 0.104\text{E}-3$		$h = 1/2^{19}$
	Euler 1	Heun 2	Nystrom 3
SD1	0.475E−2	0.617E−2	0.693E−16
SD2	0.475E−2	0.617E−2	0.693E−16
Fev	62	124	186

Implicit Methods:

Euler 1 $h = 1/2^{19}$

X	SD1	SD2	Fev
0.195E−5	0.155E+0	0.155E+0	104
0.514E−4	0.187E−3	0.106E−3	262

$x = 0.953\text{E}{-6}$		$h = 1/2^{20}$
	Gauss 2	Radau 3
SD1	0.310E−1	0.369E−2
SD2	0.310E−1	0.696E−2
Fev	66	64

$x = 0.524\text{E}{-4}$	$h = 1/2^{20}$
Gauss 2	Radau 3
0.194E−15	0.693E−16
0.194E−15	0.693E−16
920	908

Variable Coefficients Explicit Runge-Kutta Methods:

VCRK1: case I $h = 1/2^{20}$

X	SD1	SD2	Fev
0.195E−5	0.117E+0	0.443E+0	8
0.524E−4	0.128E+0	0.146E+0	220

VCRK1: case II $h = 1/2^{20}$

X	SD1	SD2	Fev
0.195E−5	0.142E+0	0.453E+0	8
0.524E−4	0.161E+0	0.180E+0	220

VCRK2: case I $h = 1/2^{19}$

X	SD1	SD2	Fev
0.953E−5	0.948E+0	0.933E+0	30
0.476E−4	0.939E+0	0.939E+0	150

$h = 1/2^{20}$

X	SD1	SD2	Fev
0.381E−5	0.235E+0	0.235E+0	24
0.104E−3	0.164E+0	0.164E+0	654

VCRK2: case II $h = 1/2^9$

X	SD1	SD2	Fev
0.195E−2	0.905E+0	0.905E+0	6
0.761E−1	0.926E+0	0.926E+0	234

VCRK2: case III $h = 1$

X	SD1	SD2	Fev
1.0	0.357E+0	0.357E+0	8
5.0	0.673E−2	0.673E−2	40

$h = 1/2^4$

X	SD1	SD2	Fev
0.625E−1	0.929E+0	0.929E+0	8
0.562E+0	0.569E+0	0.569E+0	72

Problem IV.

Explicit Methods:

$x = 0.305E-4$ $h = 1/2^{16}$

	Euler 1	Heun 2	Nystrom 3
SD1	0.229E+0	0.359E+0	0.450E−1
SD2	0.188E−1	0.979E−3	0.376E−4
SD3	0.232E−8	0.130E−12	0.555E−16
SD4	0.232E−9	0.119E−14	0.277E−16
Fev	8	16	24

	$x = 0.10E-2$		$h = 1/2^{16}$
	Euler 1	Heun 2	Nystrom 3
SD1	0.234E−7	0.183E−8	0.730E−10
SD2	0.210E−4	0.165E−5	0.620E−7
SD3	0.771E−7	0.432E−11	0.212E−14
SD4	0.779E−8	0.399E−13	0.319E−15
Fev	268	536	804

Implicit Methods:

Euler 1 $h = 1/2^{16}$

X	SD1	SD2	SD3	SD4	Fev
0.152E−4	0.218E+0	0.852E+0	0.456E−2	0.999E+0	132
0.503E−3	0.152E−4	0.548E−2	0.941E−2	0.999E+0	388

	$x = 0.726E{-4}$	$h = 1/2^{17}$
	Gauss 2	Radau 3
SD1	0.185E−1	0.187E−2
SD2	0.343E−4	0.427E−6
SD3	0.410E−14	0.416E−16
SD4	0.277E−16	0.138E−16
Fev	92	80

	$x = 0.511E{-3}$	$h = 1/2^{17}$
	Gauss 2	Radau 3
SD1	0.167E−7	0.211E−9
SD2	0.149E−4	0.186E−6
SD3	0.271E−12	0.416E−15
SD4	0.285E−14	0.693E−16
Fev	2364	2088

Variable Coefficients Explicit Runge-Kutta Methods:

VCRK1: case I $h = 1$

X	SD1	SD2	SD3	SD4	Fev
1.0	0.809E+2	0.105E−2	0.940E+1	0.132E+0	8
7.0	0.529E+2	0.296E+1	0.155E+0	0.690E−2	56

$h = 1/2^{10}$

X	SD1	SD2	SD3	SD4	Fev
0.976E−3	0.353E+0	0.287E+0	0.0	0.0	8
0.244E−1	0.557E−1	0.170E+0	0.0	0.0	200

VCRK1: case II $h = 1$

X	SD1	SD2	SD3	SD4	Fev
1.00	0.100E+1	0.999E+0	0.170E+1	0.298E+0	8
7.00	0.100E+1	0.101E+1	0.224E+1	0.576E−1	56

$h = 1/2^3$

X	SD1	SD2	SD3	SD4	Fev
0.125	0.100E+1	0.999E+0	0.140E+0	0.791E−2	8
3.125	0.101E+1	0.983E+0	0.441E−1	0.109E−1	200

VCRK2: case I $h = 1/2^9$

X	SD1	SD2	SD3	SD4	Fev
0.195E−2	0.162E+2	0.814+0	0.689E−7	0.619E−9	12
0.154E+0	0.195E+2	0.617E−2	0.115E−5	0.420E−7	948

VCRK2: case II $h = 1/2^9$

X	SD1	SD2	SD3	SD4	Fev
0.976E−2	0.602E+2	0.358E+0	0.318E−6	0.307E−8	60
0.781E−1	0.502E+2	0.417E−2	0.127E−5	0.229E−7	480

VCRK2: case III $h = 1$

X	SD1	SD2	SD3	SD4	Fev
1.00	0.464E−4	0.378E−4	0.398E+0	0.357E+0	16
3.00	0.502E−5	0.553E−5	0.553E−1	0.497E−1	48

$h = 1/2^8$

X	SD1	SD2	SD3	SD4	Fev
0.390E−2	0.175E−3	0.738E−3	0.323E−3	0.294E−5	16
0.976E−1	0.865E−4	0.583E−4	0.300E−2	0.669E−4	400

Problem V.

Explicit Methods:

	$x = 0.244E-3$ Euler 1	$h = 1/2^{13}$ Heun 2	$x = 0.524E-2$ Euler 1	$h = 1/2^{13}$ Heun 2
SD1	0.125E+0	0.133E+0	0.558E−23	0.707E−12
SD2	0.242E−1	0.230E+0	0.219E−22	0.831E−13
SD3	0.178E−3	0.394E−6	0.434E−2	0.521E−6
SD4	0.114E−3	0.774E−6	0.300E−3	0.177E−4
Fev	8	16	172	244

Nystrom 3 $x = 0.109$ $h = 1/2^{12}$

X	SD1	SD2	SD3	SD4	Fev
0.488E−3	0.677E−1	0.119E+1	0.399E−7	0.149E−7	24
0.46 E−2	0.228E+0	0.202E+0	0.387E−6	0.223E−7	204

Implicit Methods:

$x = 0.122E-3$ $h = 1/2^{13}$

	Euler 1	Gauss 2	Radau 3
SD1	0.328E+0	0.415E−1	0.528E−2
SD2	0.256E+0	0.637E−1	0.928E−2
SD3	0.554E−1	0.992E−7	0.412E−9
SD4	0.998E+0	0.193E−6	0.160E−9
Fev	848	160	188

$x = 0.524E-2$ $h = 1/2^{13}$

	Euler 1	Gauss 2	Radau 3
SD1	0.556E−23	0.174E−22	0.339E−23
SD2	0.219E−22	0.108E−21	0.358E−23
SD3	0.121E+1	0.341E−6	0.178E−7
SD4	0.352E+0	0.887E−5	0.257E−8
Fev	7556	1648	1888

Variable Coefficients Explicit Runge-Kutta Methods:

VCRK1: case I $h = 1/2^{10}$

X	SD1	SD2	SD3	SD4	Fev
0.976E−3	0.927E−1	0.772E−1	0.557E−2	0.323E−2	8
0.185E−1	0.127E−3	0.590E−6	0.436E−1	0.881E−1	152

VCRK1: case II $h = 1/2^8$

X	SD1	SD2	SD3	SD4	Fev
0.781E−2	0.896E+0	0.922E+0	0.198E+0	0.624E−1	16
0.195E+0	0.328E−1	0.301E+0	0.230E+0	0.314E+0	400

VCRK2: case I $h = 1/2^{11}$

X	SD1	SD2	SD3	SD4	Fev
0.244E−2	0.342E−1	0.130E−1	0.217E−4	0.591E−4	60
0.488E−2	0.557E−2	0.386E−3	0.122E−4	0.117E−3	1320

VCRK2: case II $h = 1/2^{10}$

X	SD1	SD2	SD3	SD4	Fev
0.976E−3	0.948E+0	0.675E+0	0.114E−3	0.170E−3	12
0.585E−1	0.344E−6	0.290E−6	0.212E−3	0.197E−2	720

VCRK2: case III $h = 1$

X	SD1	SD2	SD3	SD4	Fev
1.0	0.592E−3	0.312E−3	0.590E−2	0.296E−2	16
4.0	0.342E−13	0.348E−13	0.385E−9	0.315E−9	64

$h = 1/2^8$

X	SD1	SD2	SD3	SD4	Fev
0.390E−2	0.288E−3	0.390E−3	0.740E+0	0.601E+0	16
0.976E−1	0.0	0.0	0.480E+0	0.229E+0	400

TABLE II

The largest step-size whose numerical processes are stable and functional evaluation per step of s-system equation (Fevs)

	Problem	I	II	III	IV	V	Fevs
Explicit R-K Methods	Euler 1	$1/2^6$	$1/2^9$	$1/2^{19}$	$1/2^{16}$	$1/2^{13}$	s
	Heun 2	$1/2^6$	$1/2^9$	$1/2^{19}$	$1/2^{16}$	$1/2^{13}$	2s
	Nyström 3	$1/2^6$	$1/2^9$	$1/2^{19}$	$1/2^{16}$	$1/2^{12}$	3s
Implicit R-K Methods	Euler 1	$1/2^7$	$1/2^9$	$1/2^{19}$	$1/2^{16}$	$1/2^{13}$	
	Gauss 2	$1/2^7$	$1/2^{10}$	$1/2^{20}$	$1/2^{17}$	$1/2^{13}$	
	Radau 3	$1/2^7$	$1/2^{10}$	$1/2^{20}$	$1/2^{17}$	$1/2^{13}$	
Variable Coefficients Explicit R-K Methods	VCRK1: case I	$1/2^0$	$1/2^{10}$	$1/2^{20}$	$1/2^{10}$	$1/2^{10}$	2s
	VCRK1: case II	$1/2^0$	$1/2^{10}$	$1/2^{20}$	$1/2^3$	$1/2^8$	2s
	VCRK2: case I	$1/2^0$	$1/2^9$	$1/2^{19}$	$1/2^9$	$1/2^{11}$	3s
	VCRK2: case II	$1/2^0$	$1/2^4$	$1/2^9$	$1/2^9$	$1/2^{10}$	3s
	VCRK2: case III	$1/2^0$	$1/2^0$	$1/2^0$	$1/2^0$	$1/2^0$	4s

Remark: (a) We define that the numerical process whose absolute error is bounded to be stable.

(b) We take step size in the form $h = 1/2^k \quad (k = 0, 1, \cdots)$.

(c) The computations of implicit function $k_j^{(i)} = f(x_n + c_j h, y_n + h\sum_{l=1}^{j} a_{jl} k_j^{(i-1)})$ $(i = 0, 1, 2, 3, \cdots, m)$ are done by the simplified Newton iteration starting the first approximation to $k_j^{(0)} = f(x_n, y_n)$.

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