Japan J. Indust. Appl. Math., 14 (1997), 199–213

Free Boundary Problem for the Equation of Spherically Symmetric Motion of Viscous Gas (III)

Šárka MATUŠŮ-NEČASOVÁ*†, Mari OKADA†† and Tetu MAKINO††

† *Mathematical Institute, Academy of Sciences, Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic*
†† *Department of Applied Science, Faculty of Engineering, Yamaguchi University, Ube 755, Japan*

Received June 26, 1995
Revised October 19, 1995

We study the spherically symmetric motion of viscous barotropic gas surrounding a solid ball. We are interested in the density distribution which contacts the vacuum at a finite radius. The equilibrium is asymptotically stable with respect to small perturbation, provided that $\gamma > \frac{4}{3}$ and a is sufficiently small, when the equation of state is $p = a\rho^\gamma$, p being the pressure and ρ the density.

Key words: Navier-Stokes equation, asymptotic stability of equilibria, free boundary problem, spherically symmetric motion

We are investigating the equations

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial r} + \rho\dfrac{\partial u}{\partial r} + \dfrac{2}{r}\rho u = 0, \\ \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial r}\right) + \dfrac{\partial p}{\partial r} = \nu\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r} - \dfrac{2}{r^2}u\right) - \dfrac{\rho M}{r^2}, \\ p = a\rho^\gamma, \end{cases}$$

where ν, a, γ are positive constants and $1 < \gamma \leq 2$. These equations govern the spherically symmetric motion of a viscous barotropic gas. We consider these equations in $r \geq 1$ with the boundary condition

$$u|_{r=1} = 0$$

and the initial conditions

$$\rho|_{t=0} = \rho^0(r), \quad u|_{t=0} = u^0(r).$$

* The first author was supported by Grant of Czech Academy n° 201.93.2177.

These equations admit the equilibria

$$\rho = \begin{cases} \left[\dfrac{(\gamma-1)M}{a\gamma}\left(\dfrac{1}{r}-\dfrac{1}{R}\right)\right]^{1/(\gamma-1)} & (r \le R) \\ 0 & (R < r), \end{cases} \qquad u = 0.$$

Here $R > 1$ is arbitrary. Since we are interested in the class of initial data which includes these equilibria, we introduce the Lagrange coordinates

$$x = 4\pi \int_1^r \rho(s,t)s^2 ds.$$

Then the equations turn out to be

$$\frac{\partial \rho}{\partial t} + 4\pi\rho^2 \frac{\partial}{\partial x}(r^2 u) = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + 4\pi r^2 \frac{\partial p}{\partial x} = 16\pi^2 \nu \frac{\partial}{\partial x}\left(r^4 \rho \frac{\partial u}{\partial x}\right) - 2\nu \frac{u}{r^2 \rho} - \frac{M}{r^2}, \tag{2}$$

$$p = a\rho^\gamma, \tag{3}$$

where

$$r = \left[1 + \frac{3}{4\pi}\int_0^x \frac{d\xi}{\rho(\xi,t)}\right]^{1/3}. \tag{4}$$

Normalizing the total mass, we consider the equations (1) (2) (3) (4) in $0 \le x \le 1$ with the boundary conditions

$$u|_{x=0} = 0, \quad 4\pi r^2 p - 16\pi^2 \nu r^4 \rho \frac{\partial u}{\partial x}\bigg|_{x=1} = 0 \tag{5}$$

and the initial conditions

$$\rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x). \tag{6}$$

In this case the equilibrium is unique. We denote it by $\rho = \overline{\rho}(x)$, $p = \overline{p}(x) = a\overline{\rho}(x)^\gamma$ and $r = \overline{r}(x)$, who satisfy

$$4\pi \overline{r}^2 \frac{\partial \overline{p}}{\partial x} = -\frac{M}{\overline{r}^2}, \tag{7}$$

$$\frac{1}{C}(1-x) \le \overline{p}(x) \le C(1-x). \tag{8}$$

In the paper [2], we constructed global solutions under the following assumptions:

(A.0) $\rho_0 \in C[0,1]$, $\rho_0(x) > 0$ for $0 \le x < 1$, $\rho_0(0) = 0$, total variation $[\rho_0] < +\infty$;

(A.1) There exists a monotone decreasing function $\lambda(x)$ such that $0 \le \lambda(x) \le \rho_0(x)$ and $\int_0^1 \frac{dx}{\lambda(x)} < +\infty$;

(A.2) $u_0 \in C[0,1]$;

(A.3) This is a slightly complicated assumption concerning ρ_0 and u_0. But it is satisfied at least if $p_0 = a\rho^\gamma \in C^1[0,1]$ and $u_0 = 0$. See [2] for the details.

The global solution (ρ, u) constructed in [2] satisfies that, for any T,

$$\rho, u \in L^\infty([0,T] \times [0,1]) \cap C^1([0,T]; L^2(0,1)), \tag{9}$$

$$\rho u_x \in L^\infty([0,T] \times [0,1]) \cap C^{1/2}([0,T]; L^2(0,1)), \tag{10}$$

and there exists a constant $C(T)$ such that

$$\frac{1}{C(T)}\rho_0(x) \le \rho(x,t) \le C(T)\rho_0(x) \quad \text{for} \quad 0 \le t \le T, \quad 0 \le x \le 1. \tag{11}$$

In the last paper [3], we showed that such a solution is unique.

In this paper we will show that the solution tends to the equilibrium as $t \to +\infty$ under some additional assumptions.

First we prepare some preliminary estimates. Here we apply the argument of I. Straskraba [4].

PROPOSITION 1. *There exists a constant C such that*

$$\rho(x,t) \le C \quad \textit{for} \quad 0 \le x \le 1 \quad \textit{and} \quad 0 \le t < +\infty. \tag{12}$$

Here and hereafter C denotes various constants depending on the parameters γ, ν, M, a and the initial conditions ρ_0 and u_0.

Proof. We rewrite the equation (2) as

$$u_t = 4\pi r^2(4\pi\nu\rho(r^2u)_x - p)_x - \frac{M}{r^2}. \tag{2$'$}$$

Integrating (2)$'$ with respect to x from x to 1 and using the boundary condition (5), we get

$$\int_x^1 \frac{u_t}{r^2}dx = -4\pi(4\pi\nu\rho(r^2u)_x - p) - \int_x^1 \frac{M}{r^4}dx.$$

But, since $4\pi\rho(r^2u)_x = -(\log\rho)_t$ from (1), this can be rewritten

$$\frac{\partial}{\partial t}\log\rho = \frac{1}{\nu}\left[\frac{1}{4\pi}\frac{\partial}{\partial t}\int_x^1 \frac{u}{r^2}dx - p + H\right], \tag{13}$$

where

$$H = \frac{1}{2\pi}\int_x^1 \frac{u^2}{r^3}dx + \frac{1}{4\pi}\int_x^1 \frac{M}{r^4}dx. \tag{14}$$

Here we use the fact that $r_t = u$.

Now we use the energy equality

$$\frac{d}{dt}\int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{\gamma-1}\frac{p}{\rho} - \frac{M}{r}\right)dx + Y(t) = 0, \tag{15}$$

where

$$Y(t) = \int_0^1 \left(16\pi^2\nu r^4\rho u_x^2 + 2\nu\frac{u^2}{r^2\rho}\right)dx. \tag{16}$$

Let us denote

$$E_0 = \int_0^1 \left(\frac{1}{2}u_0^2 + \frac{1}{\gamma-1}\frac{p_0}{\rho_0} - \frac{M}{r_0}\right)dx. \tag{17}$$

Since $r \geq 1$, the energy equality (15) implies the boundedness of H, say $H \leq H^*$.

Suppose $p(x_0, T) > H^*$ for some T. Then there exists $t_1 < T$ such that i) $t_1 > 0$, $p(x_0,t) \geq H^*$ for $t \in [t_1, T]$, $p(x_0, t_1) = H^*$ or ii) $t_1 = 0$ and $p(x_0,t) \geq H^*$ for $t \in [0,T]$. Integrating (13) with respect to t from t_1 to T, we see

$$\log\rho(x_0,T) = \log\rho(x_0,t_1) + \frac{1}{\nu}\left[\frac{1}{4\pi}\int_{x_0}^1 \frac{u}{r^2}\Big|_{t_1}^T dx + \int_{t_1}^T (H-p)dt\right].$$

But, since $H - p \leq 0$ along $x = x_0$, $t_1 \leq t \leq T$, and since

$$\left|\int_{x_0}^1 \frac{u}{r^2}dx\right| \leq \left[\int_{x_0}^1 u^2 dx\right]^{1/2} \leq [2(E_0+M)]^{1/2},$$

we get

$$\log\rho(x_0,T) \leq \max\left(\log\rho_0(x_0), \log(H^*/a)^{1/\gamma}\right) + \frac{1}{2\pi\nu}[2(E_0+M)]^{1/2}.$$

This completes the proof, since $\rho_0 \in C[0,1]$ is bounded.

PROPOSITION 2. *We have*

$$\int_0^1 u(x,t)^2 dx \longrightarrow 0 \qquad \text{as} \quad t \to +\infty. \tag{18}$$

Proof. We start with the energy equality (15), which can be rewritten as

$$\frac{d}{dt}\int_0^1\left(\frac{1}{2}u^2dx\right)dx+Y(t)-\int_0^1 4\pi p(r^2u)_xdx+\int\frac{Mu}{r^2}dx=0. \tag{19}$$

Put

$$\epsilon(t)=\left[\int_{t-1}^t Y(s)ds\right]^{1/2}.$$

Since $\int_0^{+\infty}Y(t)dt\le E_0+M<+\infty$, we see $\epsilon(t)\to 0$ as $t\to+\infty$. Now we integrate (19) with respect to t from s to t, and then integrate it with respect to s from $t-1$ to t. The result is

$$\int_0^1\frac{1}{2}u^2(t)dx-\int_{t-1}^t\int_0^1\frac{1}{2}u^2(s)dxds+\int_{t-1}^t\int_s^t Y(\tau)d\tau ds$$
$$=\int_{t-1}^t ds\int_s^t d\tau\left[\int_0^1 4\pi p(r^2u)_xdx-\int_0^1\frac{M}{r^2}dx\right](\tau).$$

We see

$$0\le\int_{t-1}^t\int_s^t Y(\tau)d\tau ds=\int_{t-1}^t Y(\tau)(\tau-(t-1))d\tau\le\int_{t-1}^t Y(\tau)d\tau=\epsilon(t)^2\to 0.$$

On the other hand, since

$$u^2=\int_0^x 2uu_xdx\le CY, \tag{20}$$

we see

$$\int_{t-1}^t\int_0^1\frac{1}{2}u^2(s)dxds\le C\int_{t-1}^t Y(s)ds\le C\epsilon(t)^2\to 0.$$

Noting

$$p(r^2u)_x=\frac{1}{2\pi}\frac{1}{r}\frac{p}{\rho}u+pr^2u_x,$$

we easily see

$$\left|\int_{t-1}^t\int_s^t\int_0^1 p(r^2u)_xdxd\tau ds\right|\le C\left[\int_{t-1}^t\int_0^1\left(\frac{u^2}{r^2\rho}+r^4\rho u_x^2\right)dxds\right]^{1/2}\le C'\epsilon(t).$$

Here we use the result of Proposition 1, say $\rho\le C$. Similarly

$$\left|\int_{t-1}^t\int_s^t\int_0^1\frac{Mu}{r^2}dxd\tau ds\right|\le C\left[\int_{t-1}^t\int_0^1\frac{u^2}{r^2\rho}dxds\right]^{1/2}\le C'\epsilon(t).$$

This completes the proof.

Now we are going to estimate ρ from below to show $\rho \to \overline{\rho}$. Then we must leave the argument of I. Straskraba [4]. The reason is that [4] supposes that the external force $f(r)$ satisfies $f'(r) \leq 0$ and this plays an important role in his argument, but in our case $f(r) = -\frac{M}{r^2}$ satisfies $f'(r) = \frac{2M}{r^3} > 0$.

We assume

$$\gamma > \frac{4}{3} \quad \text{and} \quad M > 0. \tag{A.4}$$

Moreover we assume temporarily

$$p_0(x) \leq M(1-x). \tag{a.1}$$

PROPOSITION 3. *There exists a constant $B(\gamma, \nu, E^*, M^*, R^*)$ such that*

$$B = \sup_{\substack{0 \leq t < +\infty \\ 0 \leq x < 1}} \frac{\rho_0(x)}{\rho(x,t)} \leq B(\gamma, \nu, E^*, M^*, R^*), \tag{21}$$

provided that $E_0 \leq E^*$, $M \leq M^*$ and $R_0 \leq R^*$. Here

$$R_0 = r(1,0) = \left[1 + \frac{3}{4\pi} \int_0^1 \frac{dx}{\rho_0(x)}\right]^{1/3}.$$

Proof. We write (13) (14) as

$$\frac{\partial}{\partial t} \log p + \frac{\gamma}{\nu} p = A(x,t), \tag{22}$$

where

$$A(x,t) = \frac{\gamma}{4\pi\nu} \frac{\partial}{\partial t} \int_x^1 \frac{u}{r^2} dx + \frac{\gamma}{2\pi\nu} \int_x^1 \frac{u^2}{r^3} dx + \frac{\gamma}{4\pi\nu} \int_x^1 \frac{M}{r^4} dx. \tag{23}$$

Solving (22), we get

$$\frac{1}{p(x,t)} = \frac{1}{p_0(x)} \left[\exp\left(-\int_0^t A ds\right) + \frac{\gamma}{\nu} p_0(x) \int_0^t \exp\left(-\int_s^t A d\tau\right) ds\right]. \tag{24}$$

We put

$$\beta(T) = \sup_{\substack{0 \leq t \leq T \\ 0 \leq x < 1}} \frac{p_0(x)}{p(x,t)}.$$

This is finite by (11). Then we have

$$R(t) = r(1,t) = \left[1 + \frac{3}{4\pi} \int_0^1 \frac{dx}{\rho(x,t)}\right]^{1/3} \leq K_1 \left(1 + \beta(T)^{1/\gamma}\right)^{1/3},$$

where $K_1 = K_1(R^*)$ and

$$-\int_s^t A d\tau \leq -\frac{\gamma}{4\pi\nu}\int_x^1 \frac{u}{r^2}\Big|_s^t dx - \frac{\gamma}{4\pi\nu}\int_s^t d\tau \int_x^1 \frac{M}{r^4}dx$$
$$\leq K_2 - \frac{\gamma M}{4\pi\nu {K_1}^4}(1-x)\left(1+\beta(T)^{1/\gamma}\right)^{-4/3}(t-s)$$

for $0 \leq s \leq t \leq T$, where $K_2 = \frac{\gamma}{2\pi\nu}\sqrt{2(E^*+M^*)}$. Applying this estimate to (24), we see

$$\frac{1}{p(x,t)} \leq \frac{e^{K_2}}{p_0(x)}\left(1 + \frac{4\pi\nu {K_1}^4}{\gamma}\left(1+\beta(T)^{1/\gamma}\right)^{4/3}\right).$$

Here we use (a.1). Thus we get

$$\beta(T) \leq K_3\left(1+\left(1+\beta(T)^{1/\gamma}\right)^{4/3}\right), \quad \text{with} \quad K_3 = K_3(\gamma,\nu,E^*,M^*,R^*).$$

Consider the function

$$\varphi(\beta) = \frac{\beta}{K_3\left(1+\left(1+\beta^{1/\gamma}\right)^{4/3}\right)}.$$

Then $\varphi(\beta) \to +\infty$ as $\beta \to +\infty$, since $\gamma > \frac{4}{3}$ by (A.4). Thus $\varphi(\beta) \leq 1$ implies $\beta \leq \beta^* = \beta^*(\gamma,\nu,E^*,M^*,R^*)$. Putting $B(\gamma,\nu,E^*,M^*,R^*) = (\beta^*)^{1/\gamma}$ we get (21). This completes the proof.

Now we write (13) (14) as

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} + \frac{1}{\nu}p = \frac{1}{4\pi\nu}\frac{\partial}{\partial t}\int_x^1 \frac{u}{r^2}dx + \frac{1}{2\pi\nu}\int_x^1 \frac{u^2}{r^3}dx + \frac{M}{4\pi\nu}\int_x^1 \frac{dx}{r^4},$$

or

(25)
$$\frac{\partial}{\partial t}\left(\frac{1}{\rho} - \frac{1}{\overline{\rho}}\right) - \frac{1}{\nu}\frac{p}{\rho} = -\frac{1}{4\pi\nu}\frac{1}{\rho}\frac{\partial}{\partial t}\int_x^1 \frac{u}{r^2}dx - \frac{1}{2\pi\nu}\frac{1}{\rho}\int_x^1 \frac{u^2}{r^3}dx - \frac{M}{4\pi\nu}\frac{1}{\rho}\int_x^1 \frac{dx}{r^4}.$$

The equilibrium satisfies

(26)
$$-\frac{1}{\nu}\frac{\overline{p}}{\overline{\rho}} = -\frac{M}{4\pi\nu}\frac{1}{\overline{\rho}}\int_x^1 \frac{dx}{\overline{r}^4}.$$

Taking $\{(25)-(26)\} \times \left(\frac{1}{\rho} - \frac{1}{\overline{\rho}}\right)\rho_0$, we get

(27)

$$\frac{\partial}{\partial t}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2-\frac{1}{\nu}\left(\frac{p}{\rho}-\frac{\overline{p}}{\overline{\rho}}\right)\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\rho_0$$
$$=-\frac{1}{4\pi\nu}\frac{\partial}{\partial t}\left(\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\int_x^1\frac{u}{r^2}dx\right)+\frac{1}{\nu}(r^2u)_x\left(\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)+\frac{\rho_0}{\rho}\right)\int_x^1\frac{u}{r^2}dx$$
$$-\frac{1}{2\pi\nu}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\int_x^1\frac{u^2}{r^3}dx-\frac{M}{4\pi\nu}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\int_x^1\frac{dx}{\overline{r}^4}$$
$$-\frac{M}{4\pi\nu}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\int_x^1\left(\frac{1}{r^4}-\frac{1}{\overline{r}^4}\right)dx.$$

Now we see

$$-\frac{1}{\nu}\left(\frac{p}{\rho}-\frac{\overline{p}}{\overline{\rho}}\right)\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\rho_0=\frac{a(\gamma-1)}{\nu}\widetilde{\rho}^{\gamma}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\rho_0\geq 0, \tag{28}$$

where $\widetilde{\rho}=\left(\frac{1}{\overline{\rho}}+\theta\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\right)^{-1}$ with $0<\theta<1$.
We assume temporarily

(a.2) $$\rho_0(x)\leq C\overline{\rho}(x)\quad\text{for}\quad 0\leq x\leq 1.$$

Then we get

$$\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\leq\frac{C}{\rho_0},\quad \frac{\rho_0}{\rho}\leq C,\quad \rho_0\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|\leq C,\quad \frac{\rho_0}{\rho}\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|\leq\frac{C}{\rho_0}\quad\text{and so on.}$$

Moreover we assume

(a.3) $$\rho_0(x)\geq\frac{(1-x)^{1/\gamma}}{C}.$$

Now, for $0<\mu$ and $0\leq\delta\leq 1$, we put

$$Q_{\delta,\mu}=\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{\mu}}. \tag{29}$$

$Q_{\delta,\mu}$ is finite as least if $\delta>0$.

Let $\delta>0$. Then it follows from (28) that

$$\frac{d}{dt}Q_{\delta,\mu}\leq\frac{d}{dt}F_{\delta,\mu}+G_{\delta,\mu}-\frac{M}{4\pi\nu}\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{1}{(1-x)^{\mu}}\int_x^1\frac{d\xi}{\overline{r}^4}dx$$
$$-\frac{M}{4\pi\nu}\int_0^{1-\delta}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\frac{1}{(1-x)^{\mu}}\int_x^1\left(\frac{1}{r^4}-\frac{1}{\overline{r}^4}\right)d\xi dx,$$

where

$$F_{\delta,\mu} = -\frac{1}{4\pi\nu}\int_0^{1-\delta}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\frac{1}{(1-x)^\mu}\int_x^1\frac{u}{r^2}d\xi dx,$$

$$G_{\delta,\mu} = \frac{1}{\nu}\int_0^{1-\delta}(r^2u)_x\left(\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)+\frac{\rho_0}{\rho}\right)\frac{1}{(1-x)^\mu}\int_x^1\frac{u}{r^2}d\xi dx$$
$$-\frac{1}{2\pi\nu}\int_0^{1-\delta}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\frac{1}{(1-x)^\mu}\int_x^1\frac{u^2}{r^3}d\xi dx.$$

First we consider the case $\mu = \frac{5}{4}$. Then it is easy to see

$$|F_{\delta,5/4}| \le C\int_0^{1-\delta}\frac{(1-x)^{1/2}dx}{(1-x)^{1/\gamma+5/4}}\sqrt{\int_x^1 u^2 d\xi} \le C'\delta^{1/4-1/\gamma}\sqrt{\int_0^1 u^2 dx}$$

by (a.3), and

$$|G_{\delta,5/4}| \le C\delta^{-1/4}Y \quad \text{(see (20))}$$

Here and hereafter C stands for various constants independent of δ. On the other hand

$$-\frac{M}{4\pi\nu}\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{5/4}}\int_x^1\frac{d\xi}{\overline{r}^4}$$
$$\le -\frac{M}{4\pi\nu}\frac{1}{\overline{R}^4}\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}},$$

where $\overline{R} = \overline{r}(1)$, and

$$-\frac{M}{4\pi\nu}\int_0^{1-\delta}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\frac{dx}{(1-x)^{5/4}}\int_x^1\left(\frac{1}{r^4}-\frac{1}{\overline{r}^4}\right)d\xi$$
$$\le \frac{MB}{4\pi^2\nu}\int_0^{1-\delta}\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|\frac{dx}{(1-x)^{1/4}}\left(\int_0^{1-\delta}\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|dx+\int_{1-\delta}^1\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|dx\right)$$
$$\le \frac{MB}{4\pi^2\nu}\left[\int_0^1\frac{dx}{\rho_0(1-x)^{1/4}}\right]^{1/2}\left[\int_0^1\frac{(1-x)^{1/4}}{\rho_0}dx\right]^{1/2}\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\frac{dx}{(1-x)^{1/4}}$$
$$+C\delta^{(\gamma-1)/\gamma}\left[\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}}\right].$$

Here we use the estimate

$$\int_{1-\delta}^1\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|dx \le C\delta^{(\gamma-1)/\gamma},$$

by (a.3). Let us suppose

$$B\left[\int_0^1 \frac{dx}{\rho_0(1-x)^{1/4}}\right]^{1/2}\left[\int_0^1 \frac{(1-x)^{1/4}}{\rho_0}dx\right]^{1/2} \le \frac{\pi}{2}\frac{1}{\overline{R}^4}. \tag{a.4}$$

Then

$$\frac{d}{dt}Q_{\delta,5/4} \le \frac{d}{dt}F_{\delta,5/4} + G_{\delta,5/4} - \alpha\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}}$$
$$+ C\delta^{(\gamma-1)/\gamma}\left[\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}}\right]^{1/2},$$

where

$$\alpha = \frac{M}{8\pi\nu\overline{R}^4}.$$

Suppose

$$\frac{2C\delta^{(\gamma-1)/\gamma}}{\alpha} \le \left[\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}}\right]^{1/2} \quad \text{for} \quad t \ge T.$$

Then

$$\frac{d}{dt}Q_{\delta,5/4} \le \frac{d}{dt}F_{\delta,5/4} + G_{\delta,5/4} - \frac{\alpha\delta}{2}Q_{\delta,5/4} \quad \text{for} \quad t \ge T.$$

Since $F_{\delta,5/4} \to 0$ as $t \to +\infty$ by Proposition 2 and

$$\int_0^{+\infty}|G_{\delta,5/4}(t)|dt < +\infty,$$

it follows that $Q_{\delta,5/4}(t) \to 0$ as $t \to +\infty$ from the above differential inequality. This is a contradiction, since

$$Q_{\delta,5/4} \ge \int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}}.$$

Therefore there exists a sequence $t_n(\delta) \to +\infty$ $(n \to +\infty)$ such that

$$\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}} \le C\delta^{2(\gamma-1)/\gamma} \quad \text{at} \quad t = t_n(\delta). \tag{30}$$

Next we take $\mu = \frac{3}{4}$. Then

$$|F_{\delta,3/4}| \le C\int_0^1 \frac{(1-x)^{1/2}dx}{(1-x)^{1/\gamma+3/4}}\sqrt{\int_x^1 u^2 d\xi} \le C'\sqrt{\int_0^1 u^2 dx},$$

since $\frac{1}{2}-\frac{1}{\gamma}-\frac{3}{4}+1=\frac{3}{4}-\frac{1}{\gamma}>0$ by (A.4)

$$|G_{\delta,3/4}|\leq CY,$$

since $1-\frac{3}{4}=\frac{1}{4}>0$. And by a similar argument we get

$$\begin{aligned}\frac{d}{dt}Q_{\delta,3/4} \leq\ & \frac{d}{dt}F_{\delta,3/4}+G_{\delta,3/4}-\alpha\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\bar{\rho}}\right)^2(1-x)^{1/4}dx\\ &+C\delta^{(\gamma-1)/\gamma}\left[\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\bar{\rho}}\right)^2(1-x)^{1/4}dx\right]^{1/2}\end{aligned}$$

under the assumption (a.4). Then, in this case, we have

$$\begin{aligned}Q_{\delta,3/4}(t) \leq\ & Q_{\delta,3/4}(t_0)+F_{\delta,3/4}(t)-F_{\delta,3/4}(t_0)+\int_{t_0}^t G_{\delta,3/4}(\tau)d\tau\\ &+C\delta^{(\gamma-1)/\gamma}\int_{t_0}^t\left[\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\bar{\rho}}\right)^2(1-x)^{1/4}dx\right]^{1/2}(\tau)d\tau\\ \leq\ & Q_{\delta,3/4}(t_0)+F_{\delta,3/4}(t)-F_{\delta,3/4}(t_0)\\ &+\int_{t_0}^t G_{\delta,3/4}(\tau)d\tau+C'\delta^{(\gamma-1)/\gamma}(t-t_0), \qquad \text{for} \quad 0\leq t_0\leq t.\end{aligned}$$

Take $t_0=0$. Assume

$$\text{(a.5)} \qquad Q_{0,3/4}(0)=\int_0^1\rho_0\left(\frac{1}{\rho_0}-\frac{1}{\bar{\rho}}\right)^2\frac{dx}{(1-x)^{3/4}}<+\infty.$$

Then, since $F_{\delta,3/4}\to F_{0,3/4}$ and $G_{\delta,3/4}\to G_{0,3/4}$, we see

$$Q_{0,3/4}(t)\leq C \quad \text{for} \quad t\geq 0$$

and

$$\text{(31)} \qquad \begin{aligned}Q_{0,3/4}(t) \leq\ & Q_{0,3/4}(t_0)+F_{0,3/4}(t)-F_{0,3/4}(t_0)\\ &+\int_{t_0}^t G_{0,3/4}(\tau)d\tau \quad \text{for} \quad 0\leq t_0\leq t.\end{aligned}$$

Here we note

$$\text{(32)} \qquad Q_{\delta,3/4}(t_n(\delta))\leq\frac{1}{\delta^{1/2}}\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\bar{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}}\leq C\delta^{(3\gamma-4)/2\gamma},$$

from (30).

Finally take μ such that

$$\text{(33)} \qquad \frac{3}{4}<\mu<\frac{3}{2}-\frac{1}{\gamma}.$$

Since $\frac{3}{2} - \frac{1}{\gamma} - \frac{3}{4} = \frac{3}{4} - \frac{1}{\gamma} > 0$ by (A.4), this is possible, and we see

$$|F_{\delta,\mu}| \leq C \int_0^1 \frac{(1-x)^{1/2}dx}{(1-x)^{1/\gamma+\mu}} \sqrt{\int_x^1 u^2 d\xi} \leq C' \sqrt{\int_0^1 u^2 dx},$$

since $\frac{1}{2} - \mu - \frac{1}{\gamma} + 1 = \frac{3}{2} - \frac{1}{\gamma} - \mu > 0$, and

$$|G_{\delta,\mu}| \leq CY,$$

since $1 - \mu > 1 - \frac{3}{2} + \frac{1}{\gamma} = \frac{2-\gamma}{2\gamma} \geq 0$. Then, assuming

$$B\left[\int_0^1 \frac{(1-x)^{1-\mu}}{\rho_0} dx\right]^{1/2} \left[\int_0^1 \frac{dx}{\rho_0 (1-x)^{1-\mu}}\right]^{1/2} \leq \frac{\pi}{2} \frac{1}{\overline{R}^4}, \tag{a.6}$$

we get

$$Q_{0,\mu}(t) \leq C$$

by a similar argument to the case of $\mu = \frac{3}{4}$. Here we assume

$$Q_{0,\mu}(0) = \int_0^1 \rho_0 \left(\frac{1}{\rho_0} - \frac{1}{\overline{\rho}}\right)^2 \frac{dx}{(1-x)^\mu} < +\infty \tag{a.7}$$

and use $1 - \mu > 1 - \left(\frac{3}{2} - \frac{1}{\gamma}\right) = \frac{2-\gamma}{\gamma} \geq 0$.

Then we get

$$\begin{aligned} Q_{0,3/4}(t_n(\delta)) &= Q_{\delta,3/4}(t_n(\delta)) + \int_{1-\delta}^1 \rho_0 \left(\frac{1}{\rho} - \frac{1}{\overline{\rho}}\right)^2 \frac{dx}{(1-x)^{3/4}} \\ &\leq Q_{\delta,3/4}(t_n(\delta)) + \delta^{\mu-3/4} Q_{0,\mu}(t_n(\delta)) \\ &\leq C\left(\delta^{(3\gamma-4)/2\gamma} + \delta^{\mu-3/4}\right). \end{aligned}$$

It follows from (31) that

$$\limsup_{t\to+\infty} Q_{0,3/4}(t) \leq C\left(\delta^{(3\gamma-4)/2\gamma} + \delta^{\mu-3/4}\right).$$

Since $\delta > 0$ is arbitrary, we get

$$Q_{0,3/4}(t) \longrightarrow 0 \quad \text{as} \quad t \to +\infty.$$

This is our final goal.

We are going to give a sufficient condition for (a.1) (a.2) (a.3) (a.4) (a.5) (a.6).

We put the assumption

$$\int_0^1 \rho_0 \left(\frac{1}{\rho_0} - \frac{1}{\overline{\rho}}\right)^2 \frac{dx}{(1-x)^\mu} < +\infty \quad \text{for some} \quad \mu > \frac{3}{4}. \tag{A.5}$$

This assumption (A.5) guarantees (a.5) (a.7).

On the other hand we consider the condition

$$La(1-x) \leq p_0 \leq M(1-x), \tag{a.8}$$

where L is a suitable constant depending upon γ, ν, E^*, M^* and $\overline{R}$ specified as follows, where $\overline{R} = \overline{R}(\gamma, \frac{a}{M})$ is the solution of

$$\frac{1}{4\pi}\left(\frac{\gamma}{\gamma-1}\frac{a}{M}\right)^{1/(\gamma-1)} = \int_1^{\overline{R}} \left(\frac{1}{r} - \frac{1}{\overline{R}}\right)^{1/(\gamma-1)} r^2 dr.$$

Under the condition (a.8), we have

$$B \leq B(\gamma, \nu, E^*, M^*, R^*)$$

provided that $E_0 \leq E^*$ and $M \leq M^*$. We see $R_0 \leq 2^{1/3}$ provided $\frac{3}{4\pi}\frac{\gamma}{\gamma-1}\left(\frac{1}{L}\right)^{1/\gamma} \leq 1$, since

$$\frac{1}{\rho_0} \leq \left(\frac{1}{L}\right)^{1/\gamma} \frac{1}{(1-x)^{1/\gamma}}. \tag{34}$$

It is easy to see (a.4) and (a.6) hold if we take L sufficiently large so that

$$B\left(\frac{1}{L}\right)^{1/\gamma} \frac{1}{\mu} \leq \frac{\pi}{2}\frac{1}{\overline{R}^4},$$

using (34). The conditions (a.1) (a.2) (a.3) are direct consequences of the condition (a.8).

We note that, for the equilibrium,

$$\frac{M}{4\pi}\overline{R}^{(-2\gamma+1)/(\gamma-1)}(1-x) \leq \overline{p} \leq \frac{M}{4\pi}\overline{R}^{(-3\gamma+4)/(\gamma-1)}(1-x) \quad \text{if} \quad \gamma \leq \frac{3}{2}$$

or

$$\frac{M}{4\pi}\overline{R}^{-4}(1-x) \leq \overline{p} \leq \frac{M}{4\pi}\overline{R}^{-1}(1-x) \quad \text{if} \quad \gamma \geq \frac{3}{2}.$$

Note that $\overline{R} \to 1$ as $\frac{a}{M} \to 0$. Therefore p_0 near $\overline{p}$ satisfies (a.8) for sufficiently small a. Thus we have proved the following:

THEOREM. *Under the assumptions* (A.0) (A.2) (A.3) (A.4) (A.5), *suppose that a is so small that*

$$La < M, \quad \textit{where} \quad L = L(\gamma, \nu, E^*, M^*, \overline{R}),$$

provided that $E_0 \leq E^*$ and $M \leq M^*$, and that the initial pressure p_0 satisfies

$$La(1-x) \leq p_0 \leq M(1-x).$$

Then the global solution (ρ, u) *satisfies*

$$\int_0^1 u(x,t)^2 dx \longrightarrow 0,$$
$$\int_0^1 \rho_0(x) \left(\frac{1}{\rho(x,t)} - \frac{1}{\overline{\rho}(x)} \right)^2 \frac{dx}{(1-x)^{3/4}} \longrightarrow 0 \quad as \quad t \to +\infty.$$

As a corollary, $\rho(x,t) \to \overline{\rho}(x)$ a.e. x as $t \to +\infty$, and moreover, since $\rho \le C$, we know

$$\int_0^1 |\rho(x,t) - \overline{\rho}(x)|^q dx \longrightarrow 0, \quad (1 \le q < +\infty)$$

as $t \to +\infty$, for example. On the other hand, since $\frac{1}{\rho} \le \frac{B}{\overline{\rho}}$, we know

$$\int_0^1 \left| \frac{1}{\rho(x,t)} - \frac{1}{\overline{\rho}(x)} \right| dx \longrightarrow 0$$

and

$$R(t) = r(1,t) \longrightarrow \overline{r}$$

as $t \to +\infty$.

REMARK 1. We conjecture that the assumption $\gamma > \frac{4}{3}$ can be removed in the present case in which the effect of the self-gravitation is neglected. However if we take into account the effect of self-gravitation, namely if we replace M by

$$M + 4\pi \int_1^r \rho r^2 dr = M + x,$$

then the number of the equilibria is one when $\gamma \ge \frac{4}{3}$ and more than two when $\gamma < \frac{4}{3}$ (see W.-Ch. Kuan and S.-S. Lin, [1]). Therefore the assumption that $\gamma > \frac{4}{3}$ may be essential for the case with self-gravitation. We note that our proof can work with a slight modification for this case with self-gravitation.

REMARK 2. We have not yet been able to remove the assumption that a is sufficiently small, although this is a serious restriction of our result.

REMARK 3. It is difficult to describe our conclusion in terms of the Eulerian coordinates, since in the Eulerian coordinates the support of $\rho(\cdot, t)$, $[1, R(t)]$, which corresponds to the fixed interval $[0,1]$ in the Lagrangean coordinates, varies with time t.

References

[1] W.-C. Kuan and S.-S. Lin, Numbers of equilibria of self-gravitating isentropic gas surrounding a solid ball, preprint.

[2] M. Okada and T. Makino, Free boundary problem for the equation of spherically symmetric motion of viscous gas. Japan J. Indust. Appl. Math., **10** (1993), 219–235.
[3] Š. Matušů-Nečasová, M. Okada and T. Makino, Free boundary problem for the equation of spherically symmetric motion of viscous gas (II). Japan J. Indust. Appl. Math., **12** (1995), 195–203.
[4] I. Straškraba, Asymptotic development of vacuums for 1-D Navier-Stokes equations of compressible flow. Preprint (Matematicky ustav, **90** (1994), Akademie red ceske republiky).