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Error Analysis of a Fictitious Domain Method Applied to a Dirichlet Problem

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In this paper, we analyze the error of a fictitious domain method with a Lagrange multiplier. It is applied to solve a non homogeneous elliptic Dirichlet problem with conforming finite elements of degree one on a regular grid. The main point is the proof of a uniform inf-sup condition that holds provided the step size of the mesh on the actual boundary is sufficiently large compared to the size of the interior grid.

Dans cet article, nous étudions l'erreur d'une méthode de domaine fictif avec multiplicateur de Lagrange. Nous l'appliquons à la résolution d'un problème elliptique avec condition de Dirichlet non-homogène au bord par une méthode d'éléments finis conforme de degré un sur une grille uniforme. Ceci repose sur la démonstration d'une condition inf-sup uniforme qui est satisfaite lorsque le pas de la discrétisation sur la frontière du domaine d'origine est suffisamment grand comparé au pas de la grille intérieure.

Key words: Lagrange multiplier, finite elements, boundary mesh, uniform interior mesh, inf-sup condition, approximate boundary

1. Introduction

This paper follows two preceding articles of Glowinski, Pan and Périaux [11, 12] that describe a fictitious domain method and discuss its practical implementation when applied to several elliptic problems with non homogeneous Dirichlet boundary conditions. The principle of this method is to solve the problem in a larger domain (containing the domain of interest) with a very simple shape, the fictitious domain, and to impose the boundary condition by the introduction of a Lagrange multiplier on the boundary. Its advantage is that the problem in the fictitious domain can be discretized on a uniform mesh, independent of the boundary, thus skipping the time-consuming construction of a boundary-fitted mesh. This approach is discussed in [11, 12], where the problems considered are discretized with conforming standard finite elements of degree one on a uniform triangular grid in the fictitious domain and the Lagrange multiplier is discretized by piecewise constant functions on a regular grid on the boundary. The interesting point is that the two grids are chosen independently of each other, except that the boundary mesh size is larger than the mesh size in the domain. The purpose of the present paper is to derive error estimates of this method, provided that the ratio between the boundary mesh size and the mesh size in the domain is approximately two or three. The crucial step is the proof of a uniform discrete inf-sup condition via the construction of a suitable restriction operator.

The idea of imposing a boundary condition by means of a Lagrange multiplier is not new. It dates back to the work of Babuška [3], Aziz and Babuška [2], Babuška, Lee and Oden [4], which established error estimates when the ratio between the boundary mesh size and the mesh size in the domain is greater than some constant depending on the domain. Unfortunately, the constant can be large and its dependence on the domain is not straightforward. Their results were refined by Pitkäranta in [15] and recently by Agouzal in [1], but in both papers, the boundary mesh points are directly related to the mesh points of the interior grid, whereas in the present paper these mesh points are independent. In addition, we do not use the same argument to establish the discrete inf-sup condition. In the last two references, the negative boundary norm is replaced by the L^2 norm on the boundary through an inverse inequality and the discrete inf-sup condition is replaced by a sufficient condition that must be checked in the applications. In this paper, this negative boundary norm is eliminated by constructing an adequate restriction operator, the existence of which is equivalent to the discrete inf-sup condition.

This paper is organized as follows: Section 2 describes the fictitious domain formulation of the problem and it discusses its general approximation. Section 3 is devoted to the numerical analysis of one of the problems solved by Glowinski, Pan and Périaux [11], in the simplified case where the boundary is a polygon and the effect of approximating a curved boundary by a polygon is sketched in Section 4.

We end this section by recalling some Sobolev spaces that will be used in the sequel. For a domain Ω in $\mathbf{R}^n$, we shall mostly use the classical Sobolev spaces,

$$H^1(\Omega) = \{v \in L^2(\Omega); \frac{\partial v}{\partial x_i} \in L^2(\Omega), 1 \leq i \leq n\},$$

$$H^2(\Omega) = \{v \in H^1(\Omega); \frac{\partial^2 v}{\partial x_i \partial x_j} \in L^2(\Omega), 1 \leq i,\, j \leq n\},$$

both equipped with their graph norms denoted respectively by $\| \cdot \|_{H^1(\Omega)}$ and $\| \cdot \|_{H^2(\Omega)}$. We shall also use their seminorms

$$|v|_{H^1(\Omega)} = \left(\sum_{i=1}^{n} \| \frac{\partial v}{\partial x_i} \|^2_{L^2(\Omega)} \right)^{1/2},$$

$$|v|_{H^2(\Omega)} = \left(\sum_{i,j=1}^{n} \| \frac{\partial^2 v}{\partial x_i \partial x_j} \|^2_{L^2(\Omega)} \right)^{1/2}.$$

Finally, we shall also use the fractional Sobolev spaces $H^{1/2}(\Omega)$ and $H^{3/2}(\Omega)$, obtained respectively by interpolating between $L^2(\Omega)$ and $H^1(\Omega)$ and between $H^1(\Omega)$ and $H^2(\Omega)$. The reader can refer to Lions and Magenes [14] for properties of these fractional spaces.

2. A Fictitious Domain Formulation and Its Abstract Discretization

Let ω be a bounded domain of $\mathbf{R}^2$ with a Lipschitz-continuous boundary γ. We want to solve the following model problem:

For f given in $L^2(\omega)$ and g given in $H^{1/2}(\gamma)$, find u in $H^1(\omega)$ such that

$$\alpha u - \nu \Delta u = f \quad \text{in } \omega, \tag{2.1}$$

$$u = g \quad \text{on } \gamma, \tag{2.2}$$

where $\alpha \geq 0$ and $\nu > 0$ are two given constants. It is well-known that this problem has a unique solution u.

The fictitious domain formulation of problem (2.1), (2.2) is obtained by including ω in a larger square or rectangular domain Ω, whose boundary Γ has sides parallel to the axes, such that $\omega \Subset \Omega$. Next, we let $\widetilde{f}$ denote an extension of f in $L^2(\Omega)$, we choose a convenient closed subspace X of $H^1(\Omega)$, equipped with the norm of $H^1(\Omega)$, and we define on $X \times X$ the following bilinear form:

$$\forall v \in X,\ \forall w \in X, \quad a_\Omega(v, w) = \int_\Omega (\alpha v w + \nu \nabla v \cdot \nabla w) d\boldsymbol{x}.$$

Then we consider the following mixed problem:

Find a pair $(\widetilde{u}, \lambda)$ in $X \times H^{-1/2}(\gamma)$ such that

$$\forall v \in X, \quad a_\Omega(\widetilde{u}, v) = \int_\Omega \widetilde{f} v d\boldsymbol{x} + \langle v, \lambda \rangle_\gamma, \tag{2.3}$$

$$\forall \mu \in H^{-1/2}(\gamma), \quad \langle \widetilde{u}, \mu \rangle_\gamma = \langle g, \mu \rangle_\gamma, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle_\gamma$ denotes the duality pairing between $H^{1/2}(\gamma)$ and its dual space $H^{-1/2}(\gamma)$.

By applying the Babuška-Brezzi's Theorem (cf. Babuška [3] or Brezzi [5]), it is easy to prove that Problem (2.3), (2.4) is well-posed. Indeed, define the bilinear form

$$\forall v \in H^{1/2}(\gamma), \quad \forall \mu \in H^{-1/2}(\gamma), \quad b(v, \mu) = -\langle v, \mu \rangle_\gamma$$

and let V denote the kernel of b:

$$V = \{v \in X;\ v = 0 \text{ on } \gamma\}.$$

Then Problem (2.3), (2.4) is well-posed if a_Ω is elliptic on V: there exists a constant $\kappa > 0$ such that

$$\forall v \in V, \quad a_\Omega(v, v) \geq \kappa \| v \|^2_{H^1(\Omega)}, \tag{2.5}$$

and if b satisfies the inf-sup condition: there exists a constant $\beta > 0$ such that

$$\forall \mu \in H^{-1/2}(\gamma), \quad \sup_{v \in X} \frac{b(v, \mu)}{\| v \|_{H^1(\Omega)}} \geq \beta \| \mu \|_{H^{-1/2}(\gamma)}. \tag{2.6}$$

On one hand, owing to the Poincaré Inequality, the bilinear form a_Ω is elliptic on V for all values of nonnegative α and strictly positive ν. On the other hand, the inf-sup condition (2.6) (with $\beta \geq 1/\sqrt{2}$) is an easy consequence of the fact that on $H^{-1/2}(\gamma)$, the mapping

$$\mu \mapsto \sup_{v \in X} \frac{\langle \mu, v\rangle_\gamma}{\| v \|_{H^1(\Omega)}}$$

is a norm equivalent to the norm

$$\| \mu \|_{H^{-1/2}(\gamma)} = \sup_{\theta \in H^{1/2}(\gamma)} \frac{\langle \mu, \theta\rangle_\gamma}{\| \theta \|_{H^{1/2}(\gamma)}}.$$

Hence, Problem (2.3), (2.4) has a unique solution pair $(\widetilde{u}, \lambda)$. It is easy to check that

$$\alpha\widetilde{u} - \nu\Delta\widetilde{u} = \widetilde{f} \text{ in } \omega \text{ and in } \Omega\backslash\omega,$$

and

$$\widetilde{u} = g \text{ on } \gamma.$$

Thus the restriction of $\widetilde{u}$ to ω is the (unique) solution of (2.1), (2.2). Furthermore, the Lagrange multiplier λ satisfies

$$\lambda = -\nu \left[\frac{\partial \widetilde{u}}{\partial \boldsymbol{n}}\right]_\gamma, \tag{2.7}$$

where $\left[\frac{\partial \widetilde{u}}{\partial \boldsymbol{n}}\right]_\gamma$ denotes the jump of $\frac{\partial \widetilde{u}}{\partial \boldsymbol{n}}$ across γ, *i.e.*:

$$\left[\frac{\partial \widetilde{u}}{\partial \boldsymbol{n}}\right]_\gamma = \left.\frac{\partial \widetilde{u}}{\partial \boldsymbol{n}}\right|_\omega - \left.\frac{\partial \widetilde{u}}{\partial \boldsymbol{n}}\right|_{\Omega\backslash\omega}, \tag{2.8}$$

and $\boldsymbol{n}$ denotes the unit normal to γ exterior to ω.

Depending upon the extension $\widetilde{f}$ of f, the regularity of γ and the boundary conditions imposed on the functions of X, the solution $\widetilde{u}$ of (2.3), (2.4) may or may not belong to $H^2(\Omega)$. Nevertheless, if γ is of class $\mathcal{C}^{1,1}$ and if $X = H^1(\Omega)$, $H_0^1(\Omega)$ or has periodic boundary conditions (that are the three most common choices), then $\widetilde{u}$ restricted to ω (*resp.* $\Omega\backslash\omega$) belongs to $H^2(\omega)$ (*resp.* $H^2(\Omega\backslash\omega)$) and λ belongs to $H^{1/2}(\gamma)$.

To discretize Problem (2.3), (2.4), we introduce two parameters $h > 0$ and $\eta > 0$ that will tend to zero and two families of finite-dimensional spaces $X_h \subset X$ and $M_\eta \subset H^{-1/2}(\gamma)$; it is convenient to assume that M_η contains the constant functions. Consider the discrete problem:

Find a pair (u_h, λ_η) *in* $X_h \times M_\eta$ *such that*

$$\forall v_h \in X_h, \quad a_\Omega(u_h, v_h) = \int_\Omega \widetilde{f} v_h d\boldsymbol{x} + \langle v_h, \lambda_\eta\rangle_\gamma, \tag{2.9}$$

$$\forall \mu_\eta \in M_\eta, \quad \langle u_h, \mu_\eta\rangle_\gamma = \langle g, \mu_\eta\rangle_\gamma. \tag{2.10}$$

It follows from the abstract discretization theory of mixed problems (cf. for instance Girault and Raviart [10] or Brezzi and Fortin [6]) that good error estimates can be established for the solution of Problem (2.9), (2.10) if the bilinear form a_Ω satisfies an ellipticity condition and b satisfies an inf-sup condition, both uniform with respect to h and η. More precisely, let

$$V_h = \{v_h \in X_h; \langle v_h, \mu_\eta \rangle_\gamma = 0, \quad \forall \mu_\eta \in M_\eta\}.$$

Then a_Ω is uniformly elliptic on V_h if there exists a constant $\kappa^* > 0$, independent of h and η, such that

$$\forall v_h \in V_h, \quad a_\Omega(v_h, v_h) \geq \kappa^* \| v_h \|^2_{H^1(\Omega)}, \tag{2.11}$$

and b satisfies a uniform inf-sup condition if there exists a constant $\beta^* > 0$, independent of h and η, such that

$$\forall \mu_\eta \in M_\eta, \quad \sup_{v_h \in X_h} \frac{b(v_h, \mu_\eta)}{\| v_h \|_{H^1(\Omega)}} \geq \beta^* \| \mu_\eta \|_{H^{-1/2}(\gamma)}. \tag{2.12}$$

On one hand, the ellipticity condition (2.11), for all nonnegative α and strictly positive ν, follows readily from the assumption that M_η contains the constant functions. Indeed we have

$$V_h \subset \{v \in X; \int_\gamma v d\sigma = 0\},$$

and it is easy to prove that for the space in the right-hand side, the mapping $v \mapsto |v|_{H^1(\Omega)}$ is a norm equivalent to $\| v \|_{H^1(\Omega)}$.

But on the other hand, the inf-sup condition (2.12), which is a compatibility condition between the spaces X_h and M_η, will not necessarily hold for every choice of spaces. Besides, it is usually not easy to establish in practical examples, in particular because it involves the norm of $H^{-1/2}(\gamma)$, and this norm is hard to handle. The following result proved by Fortin [9] in an abstract situation, allows to eliminate this norm.

LEMMA 1. *Assume that b satisfies the inf-sup condition* (2.6). *Then the discrete inf-sup condition* (2.12) *holds if and only if there exists a restriction operator $\Pi_h \in \mathcal{L}(X; X_h)$ with the two properties*:

$$\forall v \in X, \quad \| \Pi_h(v) \|_{H^1(\Omega)} \leq C \| v \|_{H^1(\Omega)}, \tag{2.13}$$

where $C > 0$ is a constant independent of h and η, and

$$\forall v \in X, \quad \forall \mu_\eta \in M_\eta, \quad b(\Pi_h(v) - v, \ \mu_\eta) = 0. \tag{2.14}$$

The next two paragraphs will be mainly devoted to the construction, for a particular choice of spaces, of an operator Π_h satisfying (2.13) and (2.14).

3. An Example: the Case of a Polygonal Boundary

To simplify the discussion, we assume on one hand that the boundary γ is polygonal, with the restriction that its angles at corners are not too small, and on the other hand that $X = H^1(\Omega)$. The finite element spaces chosen here are the same as in Glowinski, Pan and Périaux [11]. Namely, we subdivide Ω by a uniform square grid and we divide each square (along the same diagonal) into two triangles, as in Figure 1. Let h denote the length of the longest side of these triangles (*i.e.* the diagonal) and let $\mathcal{T}_h$ denote the corresponding triangulation of $\overline{\Omega}$. We take

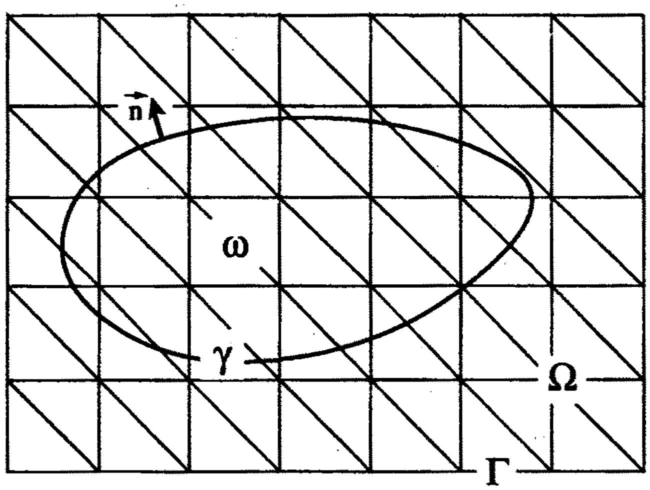

Figure 1.

$$X_h = \{v_h \in \mathcal{C}^0(\overline{\Omega}); \forall T \in \mathcal{T}_h, \quad v_h|_T \in \boldsymbol{P}_1\}, \tag{3.1}$$

where $\boldsymbol{P}_1$ denotes the space of polynomials, in two variables, of degree less than or equal to one. As far as M_η is concerned, we divide each side of γ into straight line segments S, not necessarily with equal length, but with length not less than $3h$ and not more than Lh, where L is fixed once and for all. Let η be the maximum length of these line segments and denote by $\mathcal{S}_\eta$ the corresponding subdivision of γ. Then, we set

$$M_\eta = \{\mu_\eta; \forall S \in \mathcal{S}_\eta, \ \mu_\eta|_S \in \boldsymbol{P}_0\}, \tag{3.2}$$

which indeed contains the constant functions.

Although $\mathcal{S}_\eta$ and $\mathcal{T}_h$ are constructed independently of each other, the fact that the length of each segment of $\mathcal{S}_\eta$ is not less than $3h$ and the assumption that the angular points of γ are not too sharp, imply that for each S, we can find a node a_S of $\mathcal{T}_h$ such that the macro-element Δ_S consisting of the six triangles of $\mathcal{T}_h$ with common vertex a_S satisfies the following properties:

(i) S intersects at least one interior segment of Δ_S at a distance from a_S that is not larger than half the length of this segment; in other words, a_S is the nearest end point of this segment to S;
(ii) the end points of S do not belong to the interior of Δ_S;
(iii) if S and S' are any two segments of $\mathcal{S}_\eta$, $\Delta_S \cap \Delta'_S$ is either empty or reduced to a node or a segment of $\mathcal{T}_h$; in other words, the macro-elements related to $\mathcal{S}_\eta$ do not overlap.

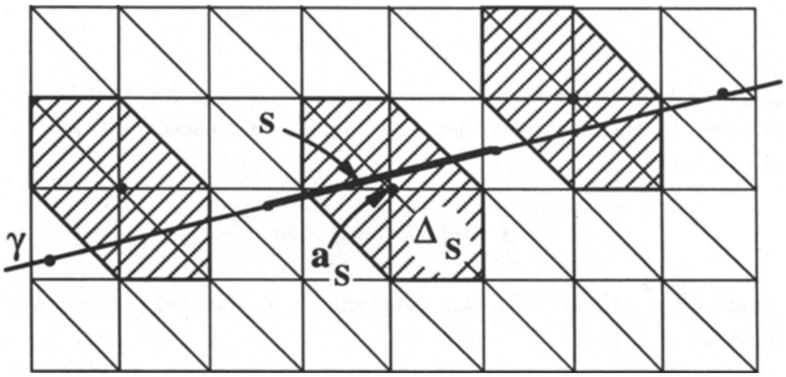

Figure 2.

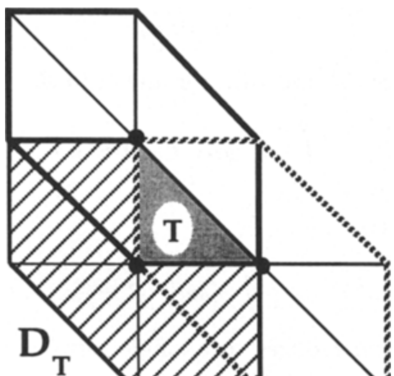

Figure 3.

As it is not necessarily unique, let us choose one such node a_S for each segment S of $\mathcal{S}_\eta$. Figure 2 shows an example of the intersection of a segment S and its macro-element Δ_S.

Now, let R_h be the regularizing operator, associated with X_h, introduced by Clément in [8]. Recall that for any v in $H^1(\Omega)$, $R_h(v)$ belongs to X_h, and R_h satisfies the following local error estimates for any T in $\mathcal{T}_h$, for $m = 1$ or 2, and for all v in $H^m(D_T)$, where D_T denotes the union of the triangles of $\mathcal{T}_h$ that share a vertex or a side with T (D_T consists of 13 triangles when T is far from the boundary

Γ, as in Figure 3):

$$\| R_h(v) - v \|_{L^2(T)} \leq C_1 h^m |v|_{H^m(D_T)}, \tag{3.3}$$

$$|R_h(v) - v|_{H^1(T)} \leq C_2 h^{m-1} |v|_{H^m(D_T)}. \tag{3.4}$$

Then for any v in $H^1(\Omega)$, we propose the following restriction $\Pi_h(v)$:

$$\Pi_h(v) = R_h(v) + \sum_{S \in \mathcal{S}_\eta} c_S \varphi_{a_S}, \tag{3.5}$$

where φ_{a_S} denotes the basis function of X_h, with support Δ_S, that takes the value 1 at the node a_S and 0 at all other nodes of $\mathcal{T}_h$, and each constant c_S is chosen (hopefully) so that

$$\int_S \Pi_h(v) d\sigma = \int_S v d\sigma. \tag{3.6}$$

Owing that the functions of M_η are constant on S, this last equality implies that $\Pi_h(v)$ satisfies (2.14).

It remains to show that such constants c_S exist and to establish the stability inequality (2.13). First, by substituting (3.5) into (3.6), condition (3.6) reads:

$$\int_S (R_h(v) - v) d\sigma + \sum_{U \in \mathcal{S}_\eta} c_U \int_S \varphi_{a_U} d\sigma = 0.$$

But, owing to properties (ii) and (iii), for any U in $\mathcal{S}_\eta$

$$\int_S \varphi_{a_U} d\sigma = 0 \quad \text{if } S \neq U,$$

and owing to property (i)

$$\int_S \varphi_{a_S} d\sigma > 0.$$

Therefore, the above sum reduces to a single term and it is easy to explicit the expression of the constant c_S:

$$c_S = -\frac{1}{\int_S \varphi_{a_S} d\sigma} \int_S (R_h(v) - v) d\sigma. \tag{3.7}$$

To derive an upper bound for the numerator of (3.7), we require the next two lemmas.

LEMMA 2. *Let $\widehat{T}$ denote the reference unit triangle and let $\widehat{\ell}$ be any straight line segment that intersects $\widehat{T}$. Then, there exists a constant $\widehat{C}$, independent of $\widehat{\ell}$, such that*

$$\forall \widehat{w} \in H^1(\widehat{T}), \quad \| \widehat{w} \|_{L^2(\widehat{\ell})} \leq \widehat{C} \| \widehat{w} \|_{H^1(\widehat{T})}. \tag{3.8}$$

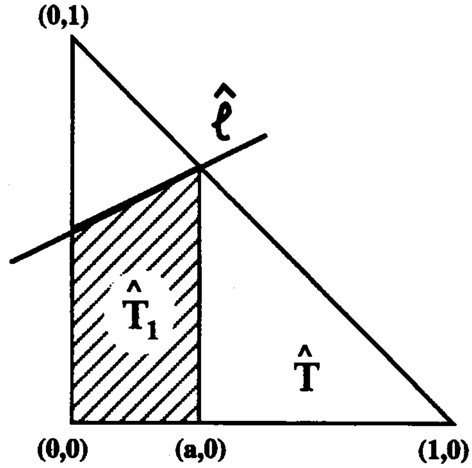

Figure 4.

Proof. By the trace theorem, we know that there exists a constant, say $\widehat{K}$, such that if $\widehat{\ell}$ is any side of $\widehat{T}$, (3.8) holds with the constant $\widehat{K}$. Therefore, we can assume that $\widehat{\ell}$ does not coincide with any side of $\widehat{T}$. Moreover, there is no loss of generality in supposing that, in $\widehat{T}$, $\widehat{\ell}$ has the parametric representation:

$$\widehat{y} = \alpha\widehat{x} + \beta \text{ for } \widehat{x} \in [0, a], \quad \text{with } |\alpha| \leq 1,$$

otherwise, we interchange $\widehat{x}$ and $\widehat{y}$. Then for any smooth function $\widehat{w}$, we have

$$\| \widehat{w} \|^2_{L^2(\widehat{\ell})} = (\alpha^2 + 1)^{1/2} \int_0^a \widehat{w}(\widehat{x}, \alpha\widehat{x} + \beta)^2 d\widehat{x}.$$

But

$$\widehat{w}(\widehat{x}, \alpha\widehat{x} + \beta)^2 = \widehat{w}(\widehat{x}, 0)^2 + \int_0^{\alpha\widehat{x}+\beta} \frac{\partial}{\partial t}(\widehat{w}(\widehat{x}, t)^2) dt,$$

and

$$\left| \frac{\partial}{\partial t}(\widehat{w}(\widehat{x}, t)^2) \right| \leq |\widehat{w}(\widehat{x}, t)|^2 + \left| \frac{\partial}{\partial t}\widehat{w}(\widehat{x}, t) \right|^2.$$

Hence

$$\| \widehat{w} \|^2_{L^2(\widehat{\ell})} \leq (\alpha^2 + 1)^{1/2} \{ \| \widehat{w} \|^2_{L^2(\widehat{\ell}_1)} + \| \widehat{w} \|^2_{H^1(\widehat{T}_1)} \},$$

where $\widehat{\ell}_1$ denotes the orthogonal projection of $\widehat{\ell}$ on the side $\widehat{y} = 0$ of $\widehat{T}$ and $\widehat{T}_1$ denotes the trapezoidal region of $\widehat{T}$ bounded by $\widehat{\ell}$ and $\widehat{\ell}_1$, as in Figure 4. Thus applying the trace theorem and using the fact that $|\alpha| \leq 1$, we derive for all smooth functions $\widehat{w}$:

$$\| \widehat{w} \|_{L^2(\widehat{\ell})} \leq 2^{1/4}(\widehat{K}^2 + 1)^{1/2} \| \widehat{w} \|_{H^1(\widehat{T})}.$$

By density this inequality carries over, with the same constant, to all functions $\widehat{w}$ in $H^1(\widehat{T})$. ◇

Note that the statement of Lemma 2 still holds when $\widehat{\ell}$ intersects partially $\widehat{T}$, *i.e.* its end points need not lie on the boundary of $\widehat{T}$.

LEMMA 3. *Let ℓ be a straight line segment that intersects a non degenerate triangle T and let $\widehat{\ell}$ be its image on the reference unit triangle $\widehat{T}$ by the affine transformation that maps $\widehat{T}$ onto T. Let B_T denote the matrix of this transformation and let $\| B_T \|$ be its Euclidean norm. Then,*

$$\frac{|\ell|}{|\widehat{\ell}|} \leq \| B_T \|. \tag{3.9}$$

We skip the proof, because it is straightforward. The above remark concerning the statement of Lemma 2 is also valid for Lemma 3, namely the end points of ℓ need not lie on the boundary of T. The next lemma derives a lower bound for the denominator of (3.7). This lower bound is not optimal but it is sufficient for our purpose and it has a simple proof.

LEMMA 4. *We always have:*

$$\int_S \varphi_{a_S} d\sigma \geq \frac{1}{4\sqrt{2}} h. \tag{3.10}$$

Proof. Previously, we have assumed that S is a straight line segment, but this proof can be easily derived in the more general case where S is a broken line segment with end points on the sides of the triangles of Δ_S; this will be useful in the next section. Then, using property (i), there is no loss of generality in assuming that the intersection point of S nearest to a_S is either on an oblique side as in Figure 5 or a vertical side as in Figure 6. Consider the case of Figure 5; the argument is similar but somewhat more intricate in the case of Figure 6. Let T_1 and T_2 denote the two triangles sharing the oblique side, let ℓ_1 and ℓ_2 denote the portions of S intersecting respectively T_1 and T_2 and let (δ, ε) denote the coordinates of the intersection of S

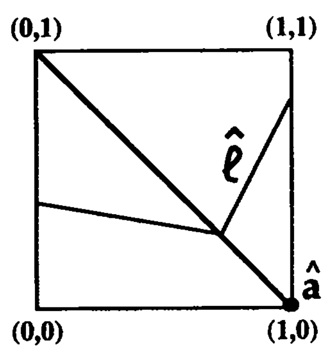

Figure 5.

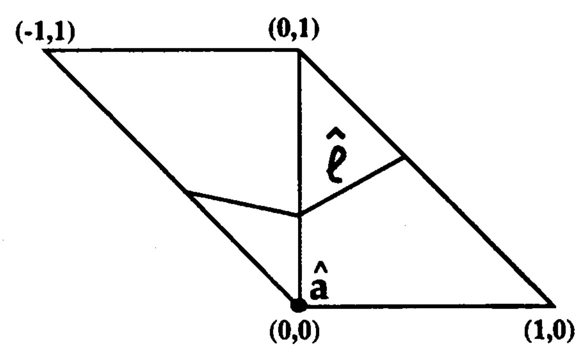

Figure 6.

with the oblique side. First, since $\varphi_{a_S} \geq 0$, we have

$$\int_S \varphi_{a_S} d\sigma \geq \int_{\ell_1 \cup \ell_2} \varphi_{a_S} d\sigma.$$

Next, switching to the reference element, we obtain

$$\int_{\ell_1 \cup \ell_2} \varphi_{a_S} d\sigma = \frac{|\ell_1|}{|\widehat{\ell}_1|} \int_{\hat{\ell}_1} \widehat{\varphi}_{a_S} d\widehat{\sigma} + \frac{|\ell_2|}{|\widehat{\ell}_2|} \int_{\hat{\ell}_2} \widehat{\varphi}_{a_S} d\widehat{\sigma}. \tag{3.11}$$

Then, using parametric representations of $\widehat{\ell}_1$ and $\widehat{\ell}_2$, we easily derive that for $i = 1$ and 2,

$$\int_{\hat{\ell}_i} \widehat{\varphi}_{a_S} d\widehat{\sigma} \geq \frac{1}{2} \widehat{\delta} |\widehat{\ell}_i|,$$

where $\widehat{\delta} \geq 1/2$, since a_S is the segment's end point nearest to the intersection of S. Thus

$$\int_{\hat{\ell}_i} \widehat{\varphi}_{a_S} d\widehat{\sigma} \geq \frac{1}{4} |\widehat{\ell}_i|,$$

and (3.10) follows by substituting this lower bound into (3.11). ◇

Now, we are in a position to establish the inf-sup condition (2.12) for the pair of spaces (3.1) and (3.2).

THEOREM 5. *Assume that the length of the segments of S_η is not less than $3h$ and that $\eta \leq Lh$. Then, there exists a constant $\beta^* > 0$, independent of h and η, such that* (2.12) *holds.*

Proof. Let us show that the operator Π_h defined by (3.5) satisfies the stability estimate (2.13) with a constant C independent of h and η; (we have already checked that it satisfies (2.14)). We can write the proof in the more general case where the segments S are broken line segments, as in the proof of Lemma 4. For any v in $H^1(\Omega)$, we have

$$\| \Pi_h(v) \|_{H^1(\Omega)} \leq \| R_h(v) \|_{H^1(\Omega)} + \| \sum_{S \in S_\eta} c_S \varphi_{a_S} \|_{H^1(\Omega)}.$$

As each φ_{a_S} has support Δ_S and these supports are all disjoint, the above sum reduces to

$$\| \sum_{S \in S_\eta} c_S \varphi_{a_S} \|_{H^1(\Omega)} = \left(\sum_{S \in S_\eta} |c_S|^2 \| \varphi_{a_S} \|^2_{H^1(\Delta_S)} \right)^{1/2}.$$

Let T be any triangle in Δ_S and, as in Lemma 3, let B_T be the matrix of the affine transformation that maps $\widehat{T}$ onto T. Then

$$\| \varphi_{a_S} \|_{L^2(T)} = |\det(B_T)|^{1/2} \| \widehat{\varphi}_{a_S} \|_{L^2(\widehat{T})} \leq \widehat{C}_1 h,$$

where $\widehat{C}_1$ is a constant that depends only on $\widehat{T}$. Similarly,

$$|\varphi_{a_S}|_{H^1(T)} \leq |\det(B_T)|^{1/2} \| B_T^{-1} \| \, |\widehat{\varphi}_{a_S}|_{H^1(\widehat{T})} \leq \widehat{C}_2,$$

where $\widehat{C}_2$ is also a constant that depends only on $\widehat{T}$. Hence there exists a constant $\widehat{C}_3$, independent of S, h and η, such that

$$\| \varphi_{a_S} \|_{H^1(\Delta_S)} \leq \widehat{C}_3. \tag{3.12}$$

Next, let us find a bound for c_S. From (3.7) and (3.10), we have

$$|c_S| \leq \frac{4\sqrt{2}}{h} \left| \int_S (R_h(v) - v) d\sigma \right|.$$

Let ℓ_i denote the straight line segments of S and T_i the element of $\mathcal{T}_h$ intersected by ℓ_i. Thus

$$|c_S| \leq \frac{4\sqrt{2}}{h} \sum_i \int_{\ell_i} \left| R_h(v) - v \right| d\sigma \leq \frac{4\sqrt{2}}{h} \sum_i |\ell_i|^{1/2} \| R_h(v) - v \|_{L^2(\ell_i)}.$$

Then switching to the reference element and applying Lemmas 2 and 3, we obtain

$$|c_S| \leq \frac{4\sqrt{2}}{h} \widehat{C} \sum_i |\ell_i|^{1/2} \| B_{T_i} \|^{1/2} \| \widehat{R_h(v)} - \widehat{v} \|_{H^1(\widehat{T})},$$

where $\widehat{C}$ is the constant of Lemma 2. Now, switching back to T_i, we have

$$\| \widehat{R_h(v)} - \widehat{v} \|_{H^1(\widehat{T})}$$
$$\leq | \det(B_{T_i}) |^{-1/2} \left(\| R_h(v) - v \|^2_{L^2(T_i)} + \| B_{T_i} \|^2 | R_h(v) - v |^2_{H^1(T_i)} \right)^{1/2}$$

Hence

$$|c_S| \leq \frac{4\sqrt{2}}{h} \widehat{C} \sum_i |\ell_i|^{1/2} \| B_{T_i} \|^{1/2} | \det(B_{T_i}) |^{-1/2}$$
$$(\| R_h(v) - v \|^2_{L^2(T_i)} + \| B_{T_i} \|^2 |R_h(v) - v|^2_{H^1(T_i)})^{1/2}. \tag{3.13}$$

As the triangulation $\mathcal{T}_h$ is trivially regular (cf. Ciarlet [7]), (3.13) and (3.12) yield

$$\Big(\sum_{S \in \mathcal{S}_\eta} |c_S|^2 \| \varphi_{a_S} \|^2_{H^1(\Delta_S)} \Big)^{1/2}$$
$$\leq \frac{1}{h} \widehat{C}_4 \sqrt{L} \Big(\sum_T (\| R_h(v) - v \|^2_{L^2(T)} + h^2 |R_h(v) - v|^2_{H^1(T)}) \Big)^{1/2},$$

where in the above sum, T runs over all the triangles of $\mathcal{T}_h$ intersected by γ and $\widehat{C}_4$ is another constant independent of h and η. Then the estimates (3.3) and (3.4) with $m = 1$ yield

$$\Big(\sum_{S\in\mathcal{S}_\eta} |c_S|^2 \| \varphi_{a_S} \|^2_{H^1(\Delta_S)}\Big)^{1/2} \le \widehat{C}_5 |v|_{H^1(\Omega)},$$

and (2.13) follows from this bound and another application of (3.2) and (3.3). ◇

The lower bound of the ratio between η and h in the first assumption of Theorem 5 ensures that the macroelements Δ_S do not overlap and this simplifies substantially the stability proof of Π_h. Indeed, when the macroelements overlap, the constants c_S are not defined explicitly by (3.7), but instead they satisfy a system of linear equations which does not easily yield a bound for $|c_S|$. But of course, this condition is only sufficient and good numerical results (cf. [11,12]) are obtained when this ratio is approximately 3/2.

Since a_Ω is uniformly elliptic and b satisfies the uniform inf-sup condition, we have immediately the following error bound.

PROPOSITION 6. *Under the assumptions of Theorem 5, problem* (2.9), (2.10) *has a unique solution* (u_h, λ_η) *and there exists a constant* C, *independent of* h *and* η, *such that*

$$\| \widetilde{u} - u_h \|_{H^1(\Omega)} + \| \lambda - \lambda_\eta \|_{H^{-1/2}(\gamma)}$$
$$\le C\big(\inf_{v_h\in X_h} \| \widetilde{u} - v_h \|_{H^1(\Omega)} + \inf_{\mu_\eta\in M_\eta} \| \lambda - \mu_\eta \|_{H^{-1/2}(\gamma)}\big).$$

Thus, the error estimates depend solely upon the regularity of the solution $(\widetilde{u}, \lambda)$. In the worst case, $\widetilde{u}$ belongs to $H^{3/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ and in the best case, $\widetilde{u}$ belongs to $H^2(\Omega)$. In either case, since the triangulation $\mathcal{T}_h$ is regular, the estimates for $\widetilde{u}$ are standard:

$$\inf_{v_h\in X_h} \| \widetilde{u} - v_h \|_{H^1(\Omega)} \le Ch^s \| \widetilde{u} \|_{H^{s+1}(\Omega)}, \tag{3.14}$$

where $s = 1/2 - \varepsilon$ or $s = 1$.

As far as the Lagrange multiplier is concerned, λ belongs at least to $L^2(\gamma)$, but since we have assumed that γ is a polygon, in the best case, λ does not belong to $H^{1/2}(\gamma)$; it belongs instead to $H^{1/2}(\gamma_i)$, for each straight line segment γ_i of γ. To derive an estimate for λ, we first prove the following auxiliary result.

LEMMA 7. *There exists a constant* C, *independent of* η *such that for all* λ *in* $L^2(\gamma)$,

$$\inf_{\mu_\eta\in M_\eta} \| \lambda - \mu_\eta \|_{H^{-1/2}(\gamma)} \le C\sqrt{\eta} \inf_{\mu_\eta\in M_\eta} \| \lambda - \mu_\eta \|_{L^2(\gamma)}. \tag{3.15}$$

Proof. On each segment S of $\mathcal{S}_\eta$ let us choose

$$\mu_\eta = p_S(\lambda) = \frac{1}{|S|}\int_S \lambda d\sigma,$$

the orthogonal projection of λ on the constant functions. Then, we prove (3.15) by a straightforward duality argument. For this choice of μ_η, we write

$$\| \lambda - \mu_\eta \|_{H^{-1/2}(\gamma)} = \sup_{\varphi \in H^{1/2}(\gamma)} \frac{\int_\gamma (\lambda - \mu_\eta)\varphi d\sigma}{\| \varphi \|_{H^{1/2}(\gamma)}}.$$

But

$$\int_S (\lambda - \mu_\eta)\varphi d\sigma = \int_S (\lambda - \mu_\eta)(\varphi - p_S(\varphi))d\sigma.$$

Obviously, for all φ in $L^2(S)$,

$$\| \varphi - p_S(\varphi) \|_{L^2(S)} \leq \| \varphi \|_{L^2(S)},$$

and an easy calculation yields for all φ in $H^1(S)$:

$$\| \varphi - p_S(\varphi) \|_{L^2(S)} \leq \widehat{C}_1 \eta |\varphi|_{H^1(S)},$$

with a constant $\widehat{C}_1$, independent of η. Now, take any segment γ_i of γ and let

$$L(\varphi) = \int_{\gamma_i} (\lambda - \mu_\eta)\varphi d\sigma.$$

Then on one hand

$$|L(\varphi)| \leq \| \lambda - \mu_\eta \|_{L^2(\gamma_i)} \| \varphi \|_{L^2(\gamma_i)},$$

and on the other hand

$$|L(\varphi)| \leq \widehat{C}_1 \eta \| \lambda - \mu_\eta \|_{L^2(\gamma_i)} \| \varphi \|_{H^1(\gamma_i)}.$$

Hence, by interpolating between these two results, we obtain

$$|L(\varphi)| \leq C_i \sqrt{\eta} \| \lambda - \mu_\eta \|_{L^2(\gamma_i)} \| \varphi \|_{H^{1/2}(\gamma_i)},$$

where the constant C_i is independent of η. Thus, summing over all segments γ_i of γ, we obtain

$$\left| \int_\gamma (\lambda - \mu_\eta)\varphi d\sigma \right| \leq \sup_i C_i \sqrt{\eta} \| \lambda - \mu_\eta \|_{L^2(\gamma)} \Big(\sum_i \| \varphi \|^2_{H^{1/2}(\gamma_i)} \Big)^{1/2},$$

and for φ in $H^{1/2}(\gamma)$, this gives

$$\left| \int_\gamma (\lambda - \mu_\eta)\varphi d\sigma \right| \leq \sup_i C_i \sqrt{\eta} \| \lambda - \mu_\eta \|_{L^2(\gamma)} \| \varphi \|_{H^{1/2}(\gamma)},$$

whence (3.15). $\diamond$

Obviously, when λ belongs to $L^2(\gamma)$, the above choice of μ_η yields

$$\| \lambda - \mu_\eta \|_{H^{-1/2}(\gamma_i)} \leq C\sqrt{\eta} \| \lambda \|_{L^2(\gamma)}.$$

and when λ belongs to $H^{1/2}(\gamma_i)$, for each straight line segment γ_i of γ, the same choice of μ_η and the argument of Lemma 7 yield

$$\| \lambda - \mu_\eta \|_{L^2(\gamma)} \leq \sup_i C_i \sqrt{\eta} \Big(\sum_i \| \lambda \|^2_{H^{1/2}(\gamma_i)} \Big)^{1/2}.$$

Therefore, in the worst case, when $\widetilde{u}$ belongs to $H^{3/2-\varepsilon}(\Omega)$ and λ belongs to $L^2(\gamma)$, we obtain

$$\begin{aligned} &\| \widetilde{u} - u_h \|_{H^1(\Omega)} + \| \lambda - \lambda_\eta \|_{H^{-1/2}(\gamma)} \\ &\leq C\Big(h^{1/2-\varepsilon} \| \widetilde{u} \|_{H^{3/2-\varepsilon}(\Omega)} + \sqrt{\eta} \| \lambda \|_{L^2(\gamma)} \Big). \end{aligned} \tag{3.16}$$

But, as mentioned in Section 2, it may frequently occur that the restrictions of $\widetilde{u}$ to ω and $\Omega \backslash \omega$ are both smooth, even though $\widetilde{u}$ does not belong to $H^2(\Omega)$. In this case, λ belongs to $H^{1/2}(\gamma_i)$ and Proposition 6 and Lemma 7 yield

$$\begin{aligned} &\| \widetilde{u} - u_h \|_{H^1(\Omega)} + \| \lambda - \lambda_\eta \|_{H^{-1/2}(\gamma)} \\ &\leq C\Big(h^{1/2-\varepsilon} \| \widetilde{u} \|_{H^{3/2-\varepsilon}(\Omega)} + \eta \Big(\sum_i \| \lambda \|^2_{H^{1/2}(\gamma_i)} \Big)^{1/2} \Big). \end{aligned}$$

Finally, when $\widetilde{u}$ belongs to $H^2(\Omega)$, the jump of its normal derivative vanishes across γ and $\lambda = 0$. In this case, Proposition 6 gives the estimate

$$\| \widetilde{u} - u_h \|_{H^1(\Omega)} + \| \lambda - \lambda_\eta \|_{H^{-1/2}(\gamma)} \leq Ch \| \widetilde{u} \|_{H^2(\Omega)}.$$

Except in this last case, Proposition 6 does not yield optimal estimates considering that the solution u of the original problem may belong to $H^2(\omega)$ while the extended solution $\widetilde{u}$ belongs only to $H^{3/2-\varepsilon}(\Omega)$. This remark is strongly supported by the numerical results obtained in [11], where the error of the discrete solution u_h restricted to the interior of ω is indeed of the order of h in the H^1 norm and h^2 in the L^∞ norm, while the normal derivative of the exact solution has a jump across the actual boundary γ. It is likely that in the interior of ω, u_h satisfies *local* error estimates that involve only the values of u in ω. This behaviour has been extensively studied and established by Schatz and Wahlbin (cf. Schatz and Wahlbin [16] and Wahlbin [17]) in the case of an elliptic problem. Of course the situation here is more complicated because we are dealing with a saddle-point problem, but it is conjectured that the arguments of [16] carry over to our problem. This would account for the good numerical results observed for this fictitious domain method.

4. The Case of a Curved Boundary

When γ is a curve, we must approximate it by an adequate polygonal line in order to apply the fictitious domain method described in the preceding sections. The error analysis of this method is somewhat long and technical and we shall only sketch the most salient results.

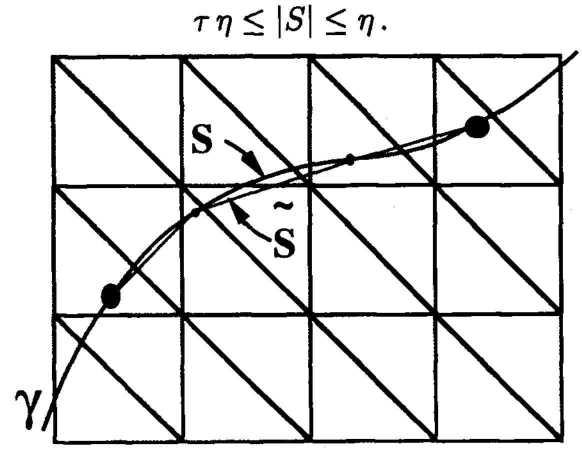

Figure 7.

Throughout this section, we assume that γ is at least of class $C^{1,1}$ (cf. Grisvard [13]). We take the same parameters h and η, satisfying the relation of Section 3. Let S_η be a partition of γ into curved line segments S such that for some fixed constant $\tau > 0$, independent of η,

$$\tau\eta \leq |S| \leq \eta.$$

We assume that η is sufficiently small for S to be parametrized either by

$$y = y_S(x) \text{ for } x \in [a_S, b_S] \quad \text{or} \quad x = x_S(y) \quad \text{for } y \in [a_S, b_S]. \tag{4.1}$$

To simplify the discussion, we shall only consider the first case, otherwise, we interchange x and y. As γ is $C^{1,1}$, each derivative y'_S is Lipschitz-continuous, with a Lipschitz constant k that can be bounded independently of S. As a consequence, there exists a constant $\mathcal{B}$, independent of S, such that

$$\forall x \in [a_S, b_S],\ |y'_S(x)| \leq \mathcal{B}, \tag{4.2}$$

and

$$b_S - a_S \leq |S| \leq (b_S - a_S)(1 + \mathcal{B}^2)^{1/2}.$$

Now, let us choose a segment S of S_η. We approximate S by a polygonal line $\widetilde{S}$ inscribed in S (possibly reduced to a chord), made of N_S straight line segments

ℓ_j for $1 \leq j \leq N_S$, as in Figure 7. In the present case, each segment ℓ_j has the parametric representation

$$\forall x \in [a_j, b_j],\ y = \alpha_j x + \beta_j.$$

(Strictly speaking, these four parameters also depend upon S, but we suppress the index S to simplify the notations.) As y'_S is Lipschitz-continuous, we can easily prove on one hand that

$$\forall x \in [a_j, b_j],\ |y_S(x) - (\alpha_j x + \beta_j)| \leq k\eta^2, \tag{4.3}$$

and on the other hand,

$$\forall x \in [a_j, b_j],\ |(1 + y'_S(x)^2)^{1/2} - (1 + \alpha_j^2)^{1/2}| \leq k\mathcal{B}\eta. \tag{4.4}$$

We denote by $\widetilde{S}_\eta$ the set of all such segments $\widetilde{S}$ and by $\widetilde{\gamma}$ the corresponding inscribed polygonal line that approximates γ.

REMARK 8. More accurately, we should assume that the length of ℓ_j is bounded by some constant δ; then (4.3) and (4.4) would hold with δ instead of η. However, to avoid a multiplicity of notation, we have preferred not to introduce this extra constant. $\diamond$

With the same notation, the curved segment S can be parametrized on the reference interval [0,1] by

$$F_S(t) = (x(t), y_S(x(t))) \quad \text{where } x(t) = (b_S - a_S)t + a_S.$$

The Jacobian of this transformation is

$$J_S(t) = (b_S - a_S)(1 + y'_S(x(t))^2)^{1/2},$$

and it satisfies

$$\forall t \in [0,1],\ (b_S - a_S) \leq J_S(t) \leq (b_S - a_S)(1 + \mathcal{B}^2)^{1/2}.$$

Similarly, each straight line segment ℓ_j of $\widetilde{S}$ is parametrized by

$$\widetilde{F}_S(t) = (x(t), \alpha_j x(t) + \beta_j).$$

Besides M_η, we define on γ the finite element space

$$\Theta_\eta = \{v_\eta \in \mathcal{C}^0(\gamma); \forall S \in S_\eta,\ v_\eta \circ F_S \in \boldsymbol{P}_1\},$$

where $\boldsymbol{P}_1$ denotes the space of polynomials (of one variable) of degree one. Similarly, we define on $\widetilde{\gamma}$ the finite element space

$$\widetilde{\Theta}_\eta = \{v_\eta \in \mathcal{C}^0(\widetilde{\gamma}); \forall \widetilde{S} \in \widetilde{S}_\eta,\ \forall \ell_j \subset \widetilde{S},\ v_\eta \circ \widetilde{F}_S \in \boldsymbol{P}_1\}. \tag{4.5}$$

$$p(\lambda) = p_S(\lambda) .$$

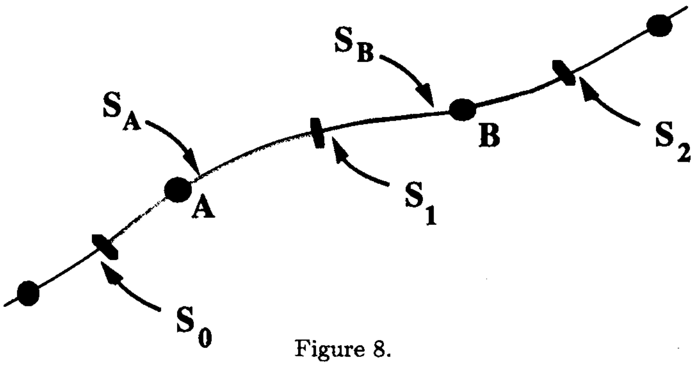

Figure 8.

Note that the functions of M_η have a natural extension on $\widetilde{\gamma}$: since they are constant on each segment S of S_η, we give them the same constant value on the inscribed segment $\widetilde{S}$. We also denote this space by M_η without distinction.

Recall the projection operator defined on M_η in the proof of Lemma 7: for all $S \in S_\eta$ and for all $\lambda \in L^1(S)$, we define $p_S(\lambda) \in \mathbf{R}$ by

$$p_S(\lambda) = \frac{1}{|S|} \int_S \lambda d\sigma, \tag{4.6}$$

and on each segment $S \in S_\eta$ we set

$$p(\lambda) = p_S(\lambda).$$

Next, on Θ_η we define the following regularizing operator analogous to the regularizing operator of Clément [8]. Let S_0, S_1 and S_2 be three consecutive segments of S_η, let A and B denote the end points of S_1 and let S_A and S_B denote the curved segments of γ "centered" respectively at A and B, *i.e.* S_A is the union of the portion of S_0 parametrized by $t \in [1/2, 1]$ and the portion of S_1 parametrized by $t \in [0, 1/2]$, as in Figure 8. Then for any function $g \in L^1(\gamma)$, we define $r_\eta(g)|_{S_1}$ as the restriction to S_1 of the function of Θ_η that interpolates the two values $p_{S_A}(g)$ and $p_{S_B}(g)$. Clearly, the function $r_\eta(g)$ with such restrictions belongs to Θ_η.

Finally, we define the following interpolation operator on $\widetilde{\Theta}_\eta$: for any function $g \in \mathcal{C}^0(\gamma)$, we define $\widetilde{I}_\eta(g) \in \widetilde{\Theta}_\eta$ by

$$\widetilde{I}_\eta(g)(a_j) = g(a_j),$$

for all end points a_j of ℓ_j, for all segments ℓ_j of $\widetilde{S}$ and for all $\widetilde{S} \in S_\eta$.

As noted in the proof of Lemma 7, p satisfies

$$\forall \lambda \in L^2(\gamma), \ \| p(\lambda) - \lambda \|_{L^2(\gamma)} \leq \| \lambda \|_{L^2(\gamma)},$$

and

$$\forall \lambda \in H^1(\gamma), \ \| p(\lambda) - \lambda \|_{L^2(\gamma)} \leq C_1 \eta |\lambda|_{H^1(\gamma)}. \tag{4.7}$$

Then by interpolating between these two bounds, we derive

$$\forall\lambda \in H^{1/2}(\gamma),\ \| p(\lambda) - \lambda \|_{L^2(\gamma)} \le C_2\sqrt{\eta}\| \lambda \|_{H^{1/2}(\gamma)}. \tag{4.8}$$

Next, it can be easily proved that r_η satisfies the local estimate

$$\forall g \in L^2(\gamma),\ \| r_\eta(g) \|_{L^2(S)} \le C_3\| g \|_{L^2(S_A \cup S_B)}, \tag{4.9}$$

and the global estimate

$$\forall g \in L^2(\gamma),\ \| r_\eta(g) \|_{L^2(\gamma)} \le C_4\| g \|_{L^2(\gamma)}. \tag{4.10}$$

Although this appears to be a trivial estimate, it serves to prove the following important inverse inequality.

THEOREM 9. *There exists a constant C, independent of η, such that*

$$\forall\mu_\eta \in M_\eta,\ \| \mu_\eta \|_{L^2(\gamma)} \le \frac{C}{\sqrt{\eta}}\| \mu_\eta \|_{H^{-1/2}(\gamma)}. \tag{4.11}$$

Proof. The negative norm in the right-hand side suggests to prove (4.11) by duality. Thus we write

$$\| \mu_\eta \|_{L^2(\gamma)} = \sup_{g\in L^2(\gamma)} \frac{\int_\gamma \mu_\eta g d\sigma}{\| g \|_{L^2(\gamma)}}. \tag{4.12}$$

Let us construct an operator Π_η defined on $L^2(\gamma)$, with values in a finite-dimensional subspace of $W^{1,\infty}(\gamma)$, such that on one hand, there exists a constant C_1, independent of η, with

$$\forall g \in L^2(\gamma),\ \| \Pi_\eta(g) \|_{L^2(\gamma)} \le C_1\| g \|_{L^2(\gamma)},$$

and on the other hand,

$$\forall\mu_\eta \in M_\eta,\ \int_\gamma \Pi_\eta(g)\mu_\eta d\sigma = \int_\gamma g\mu_\eta d\sigma,$$

i.e.

$$\forall S \in S_\eta,\ \int_S \Pi_\eta(g) d\sigma = \int_S g d\sigma.$$

This construction is similar to that of the operator Π_h of the preceding section. On $[0, 1]$ define the "bubble" function

$$\widehat{b} = 4t(1 - t),$$

and in each S, set $b_S = \widehat{b} \circ F_S^{-1}$. Then take

$$\Pi_\eta(g) = r_\eta(g) + \sum_S c_S b_S,$$

where each constant c_S is chosen so that

$$\int_S \Pi_\eta(g) d\sigma = \int_S g d\sigma.$$

This condition is fulfilled by

$$c_S = \frac{1}{\int_S b_S d\sigma} \int_S (g - r_\eta(g)) d\sigma.$$

Then an easy calculation yields

$$|c_S| \, \| b_S \|_{L^2(S)} \leq \frac{3}{2}(1 + \mathcal{B}^2)^{1/2} \| g - r_\eta(g) \|_{L^2(S)},$$

and with (4.10), this in turn implies

$$\| \Pi_\eta(g) \|_{L^2(\gamma)} \leq C_1 \| g \|_{L^2(\gamma)}.$$

On the other hand, an inverse inequality in each S gives

$$\| \frac{d}{ds} \Pi_\eta(g) \|_{L^2(S)} \leq \frac{\widehat{C}}{b_S - a_S} \| \Pi_\eta(g) \|_{L^2(S)} \leq \widehat{C} \frac{(1 + \mathcal{B}^2)^{1/2}}{\tau \eta} \| \Pi_\eta(g) \|_{L^2(S)},$$

where $\widehat{C}$ is an equivalence constant independent of S, and in turn, this implies

$$\| \frac{d}{ds} \Pi_\eta(g) \|_{L^2(\gamma)} \leq \frac{C_2}{\eta} \| g \|_{L^2(\gamma)}.$$

Thus, $\Pi_\eta \in \mathcal{L}(L^2(\gamma); L^2(\gamma)) \cap \mathcal{L}(L^2(\gamma); H^1(\gamma))$ and by interpolation between these two spaces, $\Pi_\eta \in \mathcal{L}(L^2(\gamma); H^{1/2}(\gamma))$ with

$$\| \Pi_\eta \|_{\mathcal{L}(L^2(\gamma); H^{1/2}(\gamma))} \leq \frac{C_3}{\sqrt{\eta}}.$$

Hence, we have the inverse estimate

$$\forall g \in L^2(\gamma), \; \| \Pi_\eta(g) \|_{H^{1/2}(\gamma)} \leq \frac{C_3}{\sqrt{\eta}} \| g \|_{L^2(\gamma)}.$$

Finally, going back to (4.12), we can write

$$\| \mu_\eta \|_{L^2(\gamma)} \leq \sup_{g \in L^2(\gamma)} \frac{C_3}{\sqrt{\eta}} \frac{\int_\gamma \mu_\eta \Pi_\eta(g) d\sigma}{\| \Pi_\eta(g) \|_{H^{1/2}(\gamma)}},$$

and from this, we easily deduce (4.11). $\diamond$

Now, let us discretize the surface integrals in (2.9), (2.10). For any $v_h \in X_h$ and $\lambda_\eta \in M_\eta$, we approximate $\int_\gamma v_h \lambda_\eta d\sigma$ by

$$\langle v_h, \lambda_\eta \rangle_{\gamma,\eta} = \sum_{\widetilde{S} \in \widetilde{S}_\eta} \int_{\widetilde{S}} v_h \lambda_\eta d\sigma.$$

Similarly, for any $g \in H^1(\gamma)$ and $\mu_\eta \in M_\eta$, we approximate $\int_\gamma \mu_\eta g d\sigma$ by

$$\langle \widetilde{I}_\eta(g), \mu_\eta \rangle_{\gamma,\eta} = \sum_{\widetilde{S} \in \widetilde{S}_\eta} \int_{\widetilde{S}} \widetilde{I}_\eta(g) \mu_\eta d\sigma.$$

Thus, we replace (2.9), (2.10) by:

Find a pair $(u_h, \lambda_\eta) \in X_h \times M_\eta$ *such that*

$$\forall v_h \in X_h,\ a_\Omega(u_h, v_h) = \int_\Omega \widetilde{f} v_h dx + \langle v_h, \lambda_\eta \rangle_{\gamma,\eta}, \tag{4.13}$$

$$\forall \mu_\eta \in M_\eta,\ \langle u_h, \mu_\eta \rangle_{\gamma,\eta} = \langle \widetilde{I}_\eta(g), \mu_\eta \rangle_{\gamma,\eta}. \tag{4.14}$$

The kernel of the constraint (4.14) is the space

$$\widetilde{V}_h = \{v_h \in X_h;\ \forall \mu_\eta \in M_\eta,\ \langle v_h, \mu_\eta \rangle_{\gamma,\eta} = 0\}.$$

Therefore, instead of (2.11), we must check the following discrete ellipticity condition: there exists a constant $\widetilde{\kappa} > 0$, independent of h and η, such that

$$\forall v_h \in \widetilde{V}_h,\ a_\Omega(v_h, v_h) \geq \widetilde{\kappa} \| v_h \|^2_{H^1(\Omega)}, \tag{4.15}$$

and instead of (2.12), we must check the following discrete inf-sup condition: there exists a constant $\widetilde{\beta} > 0$, independent of h and η, such that

$$\forall \mu_\eta \in M_\eta,\ \sup_{v_h \in X_h} \frac{\langle v_h, \mu_\eta \rangle_{\gamma,\eta}}{\| v_h \|_{H^1(\Omega)}} \geq \widetilde{\beta} \| \mu_\eta \|_{H^{-1/2}(\gamma)}. \tag{4.16}$$

In order to establish these two properties and adequate error estimates for the scheme (4.13), (4.14), we shall make the following assumption: there exists a constant B, independent of η such that

$$\forall S \in S_\eta,\ |S| - |\widetilde{S}| \leq B\eta^{5/2}. \tag{4.17}$$

REMARK 10. Assumption (4.17) is not very restrictive. Indeed, without this assumption, we always have

$$0 \leq |S| - |\widetilde{S}| \leq (b_S - a_S) k B \eta = O(\eta^2).$$

Moreover, when γ is a circle, $|S| - |\widetilde{S}| = O(\eta^3)$. Therefore, assumption (4.17) holds when the radius of curvature of γ is not too large, or also when $\widetilde{S}$ consists of sufficiently small straight line segments. $\diamond$

To derive some error estimates, we shall use the following lemma. Note that its proof does not require (4.17).

LEMMA 11. *Let T_γ denote the set of triangles of T_h intersected by γ and by $\widetilde{\gamma}$. There exists a constant C, independent of h and η, such that for any $\lambda \in H^{1/2}(\gamma)$, we have*

$$\forall v_h \in X_h, \ \left| \int_\gamma v_h \lambda d\sigma - \langle v_h, p(\lambda) \rangle_{\gamma,\eta} \right| \leq C(\eta \| v_h \|_{H^{1/2}(\gamma)} \| \lambda \|_{H^{1/2}(\gamma)} + \eta^{3/2} \| \nabla v_h \|_{L^2(T_\gamma)} \| \lambda \|_{L^2(\gamma)}). \tag{4.18}$$

Proof. Let us split the left-hand side of (4.18) into

$$\int_\gamma v_h \lambda d\sigma - \langle v_h, p(\lambda) \rangle_{\gamma,\eta} = \int_\gamma v_h(\lambda - p(\lambda)) d\sigma + \int_\gamma v_h p(\lambda) d\sigma - \langle v_h, p(\lambda) \rangle_{\gamma,\eta}.$$

Considering that p is a projection operator and applying (4.8), we easily derive an upper bound for the first term:

$$\left| \int_\gamma v_h(\lambda - p(\lambda)) d\sigma \right| \leq C_1 \eta \| v_h \|_{H^{1/2}(\gamma)} \| \lambda \|_{H^{1/2}(\gamma)}.$$

As far as the second term is concerned, its contribution to an arbitrary segment S of S_η has the form

$$\sum_j p_S(\lambda) \int_{a_j}^{b_j} \{ v_h(x, y_S(x))(1 + y_S'(x)^2)^{1/2} - v_h(x, \alpha_j x + \beta_j)(1 + \alpha_j^2)^{1/2} \} dx.$$

Each integral in this sum can in turn be split into

$$p_S(\lambda) \int_{a_j}^{b_j} v_h(x, y_S(x)) \{ (1 + y_S'(x)^2)^{1/2} - (1 + \alpha_j^2)^{1/2} \} dx$$
$$+ p_S(\lambda) \int_{a_j}^{b_j} \{ v_h(x, y_S(x)) - v_h(x, \alpha_j x + \beta_j) \} (1 + \alpha_j^2)^{1/2} dx.$$

Owing to (4.4), the sum of the first integral over the index j is bounded by

$$k \mathcal{B} \eta \| \lambda \|_{L^2(S)} \| v_h \|_{L^2(S)}.$$

To simplify the estimate of the second integral, we assume that for all $x \in [a_j, b_j]$, the points $(x, y_S(x))$ and $(x, \alpha_j x + \beta_j)$ belong both to the same triangle T (otherwise, we split the path between these two points). Then v_h is a polynomial of degree one:

$$v_h = v_0 + v_1 x + v_2 y \quad \text{where in particular } v_2 = \frac{\partial v_h}{\partial y},$$

and as v_2 is a constant, we can write

$$|v_2| = \frac{1}{|T|^{1/2}} \| \frac{\partial v_h}{\partial y} \|_{L^2(T)}.$$

Therefore, in view of (4.3), summing over j, we find

$$\left| p_S(\lambda) \sum_j \int_{a_j}^{b_j} \{v_h(x, y_S(x)) - v_h(x, \alpha_j x + \beta_j)\}(1+\alpha_j^2)^{1/2} dx \right|$$

$$\leq C_2 \eta^{3/2} \| \lambda \|_{L^2(S)} \| \frac{\partial v_h}{\partial y} \|_{L^2(T_S)},$$

where T_S denotes the union of all triangles of $\mathcal{T}_h$ intersected by S and $\widetilde{S}$. We easily derive (4.18) by collecting the above estimates. ◇

A variant of Lemma 11 permits to prove the discrete ellipticity condition (4.15).

COROLLARY 12. *There exists a constant* $\eta_0 > 0$ *such that* (4.15) *holds for all* $\eta \leq \eta_0$.

Proof. For all $v_h \in \widetilde{V}_h$, we have in particular

$$\int_\gamma v_h d\sigma = \int_\gamma v_h d\sigma - \langle v_h, 1 \rangle_{\gamma,\eta}.$$

Therefore, the argument of Lemma 11 with $\lambda = 1$ implies that

$$\left| \int_\gamma v_h d\sigma \right| \leq C_1(\eta \| v_h \|_{L^2(\gamma)} + \eta^{3/2} \| \nabla v_h \|_{L^2(T_\gamma)}) \leq C_2 \eta \| v_h \|_{H^1(\Omega)}.$$

Then (4.15) follows from this inequality and the fact that the mapping

$$v \mapsto \left(|v|^2_{H^1(\Omega)} + \left| \int_\gamma v d\sigma \right|^2 \right)^{1/2}$$

is a norm on $H^1(\Omega)$ equivalent to $\| \cdot \|_{H^1(\Omega)}$. ◇

As in the preceding section, the proof of the discrete inf-sup condition (4.16) can be achieved by constructing an adequate restriction operator $\widetilde{\Pi}_h$. In addition, it will be convenient to use $\widetilde{\Pi}_h$ for deriving error estimates. Therefore, instead of proving the analogue of (2.13), we shall prove a sharper error estimate for $\widetilde{\Pi}_h$. And of course, we shall construct $\widetilde{\Pi}_h$ so as to satisfy the analogue of (2.14):

$$\forall \mu_\eta \in M_\eta,\ \langle \widetilde{\Pi}_h(v), \mu_\eta \rangle_{\gamma,\eta} = \int_\gamma v \mu_\eta d\sigma,$$

i.e.

$$\forall \widetilde{S} \in \widetilde{S}_\eta, \int_{\widetilde{S}} \widetilde{\Pi}_h(v)d\sigma = \int_S v d\sigma. \tag{4.19}$$

This can be achieved by an operator of the form

$$\widetilde{\Pi}_h(v) = R_h(v) + \sum_{\widetilde{S}\in\widetilde{S}_\eta} c_{\widetilde{S}}\varphi_{a_S},$$

where the node a_S is chosen as in Section 3 and the constant $c_{\widetilde{S}}$ is chosen so that (4.19) holds, namely

$$c_{\widetilde{S}} = \frac{1}{\int_{\widetilde{S}} \varphi_{a_S} d\sigma}\left(\int_S v d\sigma - \int_{\widetilde{S}} R_h(v)d\sigma\right). \tag{4.20}$$

The following lemma gives a lower bound for the denominator in the above expression. Here again, this bound is not optimal but it has a simple proof.

LEMMA 13. *There exists a constant $\eta_1 > 0$, such that for all $\eta \le \eta_1$,*

$$\int_{\widetilde{S}} \varphi_{a_S} d\sigma \ge \frac{h}{8}. \tag{4.21}$$

Proof. This lower bound can be derived directly, but it is simpler to take advantage of Lemma 4. To this end, let $\overline{S}$ denote the polygonal line whose end points are the intersections of S and the triangles crossed by S. Then we can write

$$\int_{\widetilde{S}} \varphi_{a_S} d\sigma = \int_{\overline{S}} \varphi_{a_S} d\sigma + \left(\int_{\widetilde{S}} \varphi_{a_S} d\sigma - \int_{\overline{S}} \varphi_{a_S} d\sigma\right).$$

In view of (4.3) and (4.4), using the arguments of Lemma 11 and the fact that $0 \le \varphi_{a_S} \le 1$, it is easy to prove that

$$\left|\int_{\widetilde{S}} \varphi_{a_S} d\sigma - \int_{\overline{S}} \varphi_{a_S} d\sigma\right| \le C\eta^2.$$

Then (4.21) follows immediately from Lemma 4 and the fact that $\frac{\eta}{h} \le L$. ◇

As far as the numerator of (4.20) is concerned, let us prove the following auxiliary lemmas.

LEMMA 14. *There exists a constant C, independent of h and η, such that for all $v \in H^m(\Omega)$ with $m = 1$ or 2, we have the local estimate*

$$\| v - R_h(v) \|_{L^2(S)} \le C h^{m-1/2} \| v \|_{H^m(D_S)}, \tag{4.22}$$

where D_S denotes the set of all neighbouring triangles of T_S.

Proof. Let T_j be any triangle crossed by S. An easy variant of the argument used to prove Lemma 2 yields

$$\forall w \in H^1(T_j),\ \| w \|_{L^2(S \cap T_j)} \le h^{1/2} \left(\frac{1+\mathcal{B}^2}{2} \right)^{1/4} (1+\widehat{K}^2)^{1/2} \| \widehat{w} \|_{H^1(\widehat{T})},$$

where $\widehat{K}$ is the constant introduced in the proof of Lemma 2. Then (4.22) follows from the approximation properties (3.3) and (3.4) of R_h. ◇

LEMMA 15. *There exists a constant C, independent of h and η, such that for all $v \in H^m(\Omega)$ with $m = 1$ or 2, we have*

$$\begin{aligned} &\left| \int_S R_h(v) d\sigma - \int_{\widetilde{S}} R_h(v) d\sigma \right| \\ &\le C \left(\eta^2 \| v \|_{H^1(D_S)} + \eta^{m+1/2} \| \frac{d^{m-1}v}{ds^{m-1}} \|_{L^2(S)} \right). \end{aligned} \tag{4.23}$$

Proof. The argument is similar to but sharper than that of Lemma 11. We have

$$\begin{aligned} &\int_S R_h(v) d\sigma - \int_{\widetilde{S}} R_h(v) d\sigma \\ &= \sum_j \int_{a_j}^{b_j} (R_h(v)(x, y_S(x)) - R_h(v)(x, \alpha_j x + \beta_j))(1+\alpha_j^2)^{1/2} dx \\ &+ \sum_j \int_{a_j}^{b_j} (R_h(v) - p_S(v))(x, y_S(x)) \{(1+y_S'(x)^2)^{1/2} - (1+\alpha_j^2)^{1/2}\} dx \\ &+ \sum_j \int_{a_j}^{b_j} p_S(v) \{(1+y_S'(x)^2)^{1/2} - (1+\alpha_j^2)^{1/2}\} dx. \end{aligned}$$

In view of (4.6) and assumption (4.17), the last term has the bound

$$\left| \sum_j \int_{a_j}^{b_j} p_S(v) \{(1+y_S'(x)^2)^{1/2} - (1+\alpha_j^2)^{1/2}\} dx \right| \le \frac{\mathcal{B}}{\sqrt{\tau}} \eta^2 \| v \|_{L^2(S)}.$$

Owing to (4.4), (4.22) and (4.7), we can estimate the middle term

$$\begin{aligned} &\left| \sum_j \int_{a_j}^{b_j} (R_h(v) - p_S(v))(x, y_S(x)) \{(1+y_S'(x)^2)^{1/2} - (1+\alpha_j^2)^{1/2}\} dx \right| \\ &\le k\mathcal{B}\eta^{3/2} (\| v - R_h(v) \|_{L^2(S)} + \| v - p_S(v) \|_{L^2(S)}) \\ &\le C_1 \eta^{3/2} \Big(h^{m-1/2} \| v \|_{H^m(D_S)} + \eta^{m-1} \| \frac{d^{m-1}v}{ds^{m-1}} \|_{L^2(S)} \Big). \end{aligned}$$

Finally, the first term is estimated as in Lemma 11:

$$\Big|\sum_j \int_{a_j}^{b_j} \{R_h(v)(x, y_S(x)) - R_h(v)(x, \alpha_j x + \beta_j)\}(1+\alpha_j^2)^{1/2} dx\Big|$$

$$\leq C_2 \eta^2 \sqrt{L} \| \frac{\partial R_h(v)}{\partial y} \|_{L^2(T_S)},$$

and (4.23) follows by collecting these three inequalities. ◇

These two lemmas give the following upper bound for the numerator in (4.20).

LEMMA 16. *There exists a constant C, independent of h and η, such that for all $v \in H^m(\Omega)$ with $m = 1$ or 2, we have*

$$\Big|\int_S v d\sigma - \int_{\widetilde{S}} R_h(v) d\sigma\Big| \leq C\left(\eta^m \| v \|_{H^m(D_S)} + \eta^{m+1/2} \| \frac{d^{m-1}v}{ds^{m-1}} \|_{L^2(S)}\right). \tag{4.24}$$

Finally, combining (4.21), (4.24) and the approximation properties (3.3) and (3.4) of R_h, we derive the following error estimate for $\widetilde{\Pi}_h$.

THEOREM 17. *In addition to the hypotheses of Theorem 5, we suppose that (4.17) holds. Then there exists a constant C, independent of h and η, such that for all $\eta \leq \eta_1$, we have for all $v \in H^m(\Omega)$ with $m = 1$ or 2:*

$$\| v - \widetilde{\Pi}_h(v) \|_{H^1(\Omega)} \leq C\eta^{m-1} \| v \|_{H^m(\Omega)},$$

where $\eta_1 > 0$ is the constant of Lemma 13.

The next lemma derives an upper bound for the error on the boundary data. We skip the proof because it uses the same techniques as Lemma 15.

LEMMA 18. *Assume that (4.17) holds and that each segment S of S_η is of class C^2. Then there exists a constant C, independent of h and η, such that for all $g \in H^{3/2}(\gamma)$, we have*

$$\forall \mu_\eta \in M_\eta, \ \Big|\int_\gamma g\mu_\eta d\sigma - \langle \widetilde{I}_\eta(g), \mu_\eta \rangle_{\gamma,\eta}\Big| \leq C\eta^{3/2} \| g \|_{H^{3/2}(\gamma)} \| \mu_\eta \|_{L^2(\gamma)}. \tag{4.25}$$

The above results allow us to establish an error bound for the scheme (4.13), (4.14).

THEOREM 19. *Let $\eta_2 > 0$ denote the minimum of η_0 and η_1. In addition to the hypotheses of Theorem 5, assume that (4.17) holds and that each segment S of*

S_η is of class $\mathcal{C}^2$. Then there exists a constant C, independent of h and η, such that for all $\eta \le \eta_2$,

$$\begin{aligned} &\| u_h - \widetilde{u} \|_{H^1(\Omega)} + \| \lambda_\eta - \lambda \|_{H^{-1/2}(\gamma)} \\ &\quad \le C(\eta \| \lambda \|_{H^{1/2}(\gamma)} + \eta \| g \|_{H^{3/2}(\gamma)} + \| \widetilde{\Pi}_h(\widetilde{u}) - \widetilde{u} \|_{H^1(\Omega)}). \end{aligned} \tag{4.26}$$

Proof. For any $v_h \in X_h$ and any $\mu_\eta \in M_\eta$, we can write

$$\begin{aligned} a_\Omega(\widetilde{\Pi}_h(\widetilde{u}), v_h) &= a_\Omega(\widetilde{\Pi}_h(\widetilde{u}) - \widetilde{u}, v_h) + \int_\Omega \widetilde{f} v_h dx + \langle v_h, p(\lambda) \rangle_{\gamma,\eta} \\ &\quad + \left\{ \int_\gamma v_h \lambda d\sigma - \langle v_h, p(\lambda) \rangle_{\gamma,\eta} \right\}, \\ \langle \widetilde{\Pi}_h(\widetilde{u}), \mu_\eta \rangle_{\gamma,\eta} &= \langle \widetilde{I}_\eta(g), \mu_\eta \rangle_{\gamma,\eta} + \left\{ \int_\gamma g \mu_\eta d\sigma - \langle \widetilde{I}_\eta(g), \mu_\eta \rangle_{\gamma,\eta} \right\}. \end{aligned}$$

As the inf-sup condition (4.16) holds, there exists $z_h \in X_h$ such that

$$\langle z_h, \mu_\eta \rangle_{\gamma,\eta} = \int_\gamma g \mu_\eta d\sigma - \langle \widetilde{I}_\eta(g), \mu_\eta \rangle_{\gamma,\eta},$$

and

$$\| z_h \|_{H^1(\Omega)} \le \frac{1}{\widetilde{\widetilde{\beta}}} \sup_{\mu_\eta \in M_\eta} \frac{\int_\gamma g \mu_\eta d\sigma - \langle \widetilde{I}_\eta(g), \mu_\eta \rangle_{\gamma,\eta}}{\| \mu_\eta \|_{H^{-1/2}(\gamma)}} \le \frac{C_1}{\widetilde{\widetilde{\beta}}} \eta \| g \|_{H^{3/2}(\gamma)},$$

owing to (4.25) and the inverse inequality (4.11). In addition, $\widetilde{\Pi}_h(\widetilde{u}) - u_h - z_h$ belongs to $\widetilde{V}_h$ and satisfies for all $v_h \in X_h$

$$\begin{aligned} &a_\Omega(\widetilde{\Pi}_h(\widetilde{u}) - u_h - z_h, v_h) \\ &\quad = a_\Omega(\widetilde{\Pi}_h(\widetilde{u}) - \widetilde{u}, v_h) - a_\Omega(z_h, v_h) + \langle v_h, p(\lambda) \rangle_{\gamma,\eta} - \langle v_h, \lambda_\eta \rangle_{\gamma,\eta} \\ &\quad + \left\{ \int_\gamma v_h \lambda d\sigma - \langle v_h, p(\lambda) \rangle_{\gamma,\eta} \right\}. \end{aligned}$$

Let us choose $v_h = \widetilde{\Pi}_h(\widetilde{u}) - u_h - z_h$. Since a_Ω is elliptic on $\widetilde{V}_h$, we derive

$$\begin{aligned} \widetilde{\kappa} \| \widetilde{\Pi}_h(\widetilde{u}) - u_h - z_h \|_{H^1(\Omega)} &\le C_2(\| \widetilde{\Pi}_h(\widetilde{u}) - \widetilde{u} \|_{H^1(\Omega)} + \| z_h \|_{H^1(\Omega)}) \\ &\quad + \sup_{v_h \in X_h} \frac{\int_\gamma v_h \lambda d\sigma - \langle v_h, p(\lambda) \rangle_{\gamma,\eta}}{\| v_h \|_{H^1(\Omega)}}. \end{aligned}$$

Hence

$$\| u_h - \widetilde{u} \|_{H^1(\Omega)} \le C_3(\eta \| \lambda \|_{H^{1/2}(\gamma)} + \eta \| g \|_{H^{3/2}(\gamma)} + \| \widetilde{\Pi}_h(\widetilde{u}) - \widetilde{u} \|_{H^1(\Omega)}). \tag{4.27}$$

Finally, to upbound the error on λ, we write

$$\langle v_h, p(\lambda)\rangle_{\gamma,\eta} - \langle v_h, \lambda_\eta\rangle_{\gamma,\eta} = a_\Omega(\widetilde{u} - u_h, v_h) - \left\{ \int_\gamma v_h \lambda d\sigma - \langle v_h, p(\lambda)\rangle_{\gamma,\eta} \right\}.$$

Then (4.18) and the inf-sup condition (4.16) yield

$$\widetilde{\beta} \| p(\lambda) - \lambda_\eta \|_{H^{-1/2}(\gamma)} \leq C_2 \| u_h - \widetilde{u} \|_{H^1(\Omega)} + C_4 \eta \| \lambda \|_{H^{1/2}(\gamma)}. \tag{4.28}$$

Then (4.26) follows from (4.27), (4.28) and (3.15). ◇

Theorems 17 and 19 lead to the conclusion that the order of convergence of this fictitious domain method is not modified by approximating the curved boundary γ by an adequate polygonal line.

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