Japan J. Indust. Appl. Math., 11 (1994), 289–342

Nonnegative Splitting Theory

Dedicated to my teacher and friend Professor Janusz R. Mika

Zbigniew I. Woźnicki

Institute of Atomic Energy,
05-400 Otwock-Swierk, Poland

Received February 6, 1992
Revised April 12, 1993

In this paper the nonnegative splitting theory, playing a fundamental role in the convergence analysis of iterative methods for solving large linear equation systems with monotone matrices and representing a broad class of physical and engineering problems, is formulated. As the main result of this theory, it is possible to make the comparison of spectral radii of iteration matrices in particular iterative methods.

Key words: linear equation systems, iterative methods, monotone matrices, eigenvalues, eigenvectors, (regular, nonnegative, weak nonnegative and weak) splittings, comparison theorems

1. Introduction

The nonnegative splitting theory provides many comparison theorems as useful tools in the convergence analysis of iterative methods for solving large linear equation systems with monotone matrices represented by different types of splittings. Many physical and engineering problems with monotone matrices are characterized by so called M-matrices and H-matrices which properties were studied by many authors. The results presented in the paper are related to monotone matrices representing a broader class of matrices.

The nonnegative splitting theory based on the Perron-Frobenius theory of nonnegative matrices is a generalization of the regular splitting theory originated by Varga [1] and improved later in developments of prefactorizationing methods known under the name the AGA two-sweep iterative methods [2, 3, 4, 5, 12, 13, 14]. In the next section the background material, well known in the theory of nonnegative matrices [1] and frequently used, is given for completeness of description. In Section 3 the main results of the nonnegative splitting theory proven under natural hypotheses easily verifiable in practice are presented. Section 4 demonstrates the application of some results of this theory in the convergence analysis of iterative methods. Section 5 deals with the author's comments on his earlier regular theory splitting results [2], finding a continuous interest in the literature [7, 8, 9, 10, 11, 15, 16], in relation to other developments [7, 15] as well as new extensions of nonnegative splitting theory results proven under weaker conditions but more cumbersome in its verification. In Section 6 further extentions of nonnegative splitting theory are presented. Reference [16] is one of not a few works which uses the earlier author's results in immediate applications.

The previous version of this paper (consisting basically of the material in

Sections 2, 3 and 4) prepared in 1989 was presented in internal seminars and recently in *The Second Conference of International Linear Algebra Society* (University of Lisbon, August 1992, Lisbon, Portugal), *The 1992 Shanghai International Numerical Algebra and its Applications Meeting* (Fudan University, October 1992, Shanghai, P.R. China) as well as in *International Workshop on Nonnegative Matrices, Applications and Generalizations* (Technion, 31 May–4 June, 1993, Haifa, Israel). Its preprint have been distributed among many people (among others Professors G. Alefeld, G.H. Golub, Erxiong Jiang, Keisuke Kobayashi, M. Neumann, G. de Oliveira, R.S. Varga). The author is indebted to people who by their criticism, comments and suggestions encouraged the author to further studies of nonnegative splitting theory. Referring to results of other authors [7, 15] discussed in Section 5 was also inspired by the suggestions of Professor G.H. Golub.

Thus, in last few years this work was under a continued development and the present paper is supplemented by many new comparison theorems included in Sections 3, 5 and 6, and the extension of Section 4.

2. Theoretical Background

Throughout the paper all matrices will be square with the *order of* n in $\mathbb{R}^{n\times n}$ where $\mathbb{R}^{n\times n}$ denotes the vector space of all $n \times n$ real matrices. For a matrix $\mathbf{A} = (\mathbf{a}_{ij}) \in \mathbb{R}^{n\times n}$ the set of indices $i, j = 1, 2, \ldots n$ will be denoted by S. In the assumed notation a matrix $\mathbf{A}$ is *nonnegative* or $\mathbf{A} \geq \mathbf{0}$ if $\mathbf{a}_{ij} \geq \mathbf{0}$ for each $i, j \in S$ and there exists at least one pair of indices $k, l \in S$ for which $\mathbf{a}_{kl} > \mathbf{0}$. If $\mathbf{a}_{ij} > \mathbf{0}$ for each $i, j \in S$, $\mathbf{A}$ *is positive or* $\mathbf{A} > \mathbf{0}$. Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices, then $\mathbf{A} \geq \mathbf{B}$ (or $\mathbf{A} > \mathbf{B}$) is equivalent to $\mathbf{A} - \mathbf{B} \geq \mathbf{0}$ (or $\mathbf{A} - \mathbf{B} > \mathbf{0}$). In the case of a vector $\mathbf{x} \in \mathbb{R}^n$ with n components, $\mathbb{R}^n$ denotes $\mathbb{R}^{n\times 1}$ because column vectors are $n \times 1$ matrices, the same notation and meaning for the terms nonnegative and positive are used. Thus, according to the above definitions of nonnegativity, with the notation $\mathbf{A} \geq \mathbf{B}$ and $\mathbf{x} \geq \mathbf{y}$ the equality is excluded.

However in the case of multiplication of a nonnegative and singular matrix $\mathbf{A}$ by other nonnegative and singular matrix $\mathbf{B}$ or by a nonnegative vector $\mathbf{x}$ it may occur that the product will be the null matrix or the null vector, respectively. Such a case is included by the following notation $\mathbf{AB} \geq (=) \mathbf{0}$ or $\mathbf{Ax} \geq (=) \mathbf{0}$ which means that $\mathbf{AB}$ ($\mathbf{Ax}$) may be either a nonnegative matrix (vector) or the null matrix (vector). Thus with the notation $\mathbf{A} \geq (=) \mathbf{B}$ and $\mathbf{x} \geq (=) \mathbf{y}$ the equality is not excluded.

Definition 2.1. A matrix $\mathbf{A}$ is *monotone* if $\mathbf{A}$ is nonsingular and $\mathbf{A}^{-1} \geq \mathbf{0}$.

Definition 2.2. A matrix $\mathbf{A}$ is *reducible* if there exists a nonvoid index set R, $R \subset S$ and $R \neq S$ such that $\mathbf{a}_{ij} = \mathbf{0}$ for $i \in R$ and $j \in S - R$, otherwise the matrix $\mathbf{A}$ is *irreducible*.

It is evident that each positive matrix is irreducible.

Definition 2.3. For a matrix $\mathbf{A}$ with eigenvalues λ_i, $i \in S$, the quantity

$$\rho(\mathbf{A}) = \max |\lambda_i| \text{ for all } i \in S$$

is the *spectral radius* of the matrix $\mathbf{A}$.

LEMMA 2.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two matrices. Then*

$$\rho(\mathbf{AB}) = \rho(\mathbf{BA}).$$

LEMMA 2.2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two matrices, with* $\mathbf{A} \geq |\mathbf{B}| \geq \mathbf{0}$. *Then*

$$\rho(\mathbf{A}) \geq \rho(\mathbf{B})$$

with the strict inequality sign when the matrix $\mathbf{A}$ *is irreducible.*

THEOREM 2.1. *If* $\mathbf{G}$ *is a matrix with* $\rho(\mathbf{G}) < \mathbf{1}$. *Then* $\mathbf{I} - \mathbf{G}$ *is nonsingular and*

$$(\mathbf{I} - \mathbf{G})^{-1} = \mathbf{I} + \mathbf{G} + \mathbf{G}^2 + \cdots \tag{2.1}$$

the series on the right hand side is converging. Conversely, if the series on the right hand side converges, then $\rho(\mathbf{G}) < \mathbf{1}$.

The Perron-Frobenius theory of nonnegative matrices provides many theorems concerning the eigenvalues and eigenvectors of nonnegative matrices. The most important results are contained in the two following theorems [1].

THEOREM 2.2. *If* $\mathbf{A} \geq \mathbf{0}$ *then*

1. $\mathbf{A}$ *has a nonnegative real eigenvalue equal to its spectral radius.*
2. *To* $\rho(\mathbf{A}) > \mathbf{0}$ *there corresponds an eigenvector* $\mathbf{x} \geq \mathbf{0}$.
3. $\rho(\mathbf{A})$ *does not decrease when any entry of* $\mathbf{A}$ *is increased.*

THEOREM 2.3. *Let* $\mathbf{A} \geq \mathbf{0}$ *be an irreducible matrix. Then*

1. $\mathbf{A}$ *has a positive real eigenvalue equal to its spectral radius.*
2. *To* $\rho(\mathbf{A})$ *there corresponds an eigenvector* $\mathbf{x} > \mathbf{0}$.
3. $\rho(\mathbf{A})$ *increases when any entry of* $\mathbf{A}$ *increases.*
4. $\rho(\mathbf{A})$ *is a simple eigenvalue of* $\mathbf{A}$.

3. Nonnegative Splitting Theory

All considerations are referred to the iterative solution of the following linear equation system

$$\mathbf{A}\phi = \mathbf{c} \tag{3.1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a given $n \times n$ nonsingular matrix, $\phi \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}^n$ are column vectors with n components, with unknown ϕ and $\mathbf{c}$ being given.

The iterative solution of Eq. (3.1) can be expressed in the following form

$$\mathbf{M}\phi^{(t+1)} = \mathbf{N}\phi^{(t)} + \mathbf{c}, \quad t \geq 0 \tag{3.2}$$

where $\phi^{(t)}$ denotes the successive iterates and

$$\mathbf{A} = \mathbf{M} - \mathbf{N} \tag{3.3}$$

represents the *single splitting* of $\mathbf{A}$ (iterative methods based on the double splitting of $\mathbf{A}$ are analyzed in [5]). The above iterative scheme is convergent to the unique solution

$$\phi = \mathbf{A}^{-1}\mathbf{c} \tag{3.4}$$

for each $\phi^{(0)}$ if and only if $\mathbf{M}$ is a nonsingular matrix and the corresponding *iteration matrix*

$$\mathcal{G} = \mathbf{M}^{-1}\mathbf{N} \tag{3.5}$$

has the spectral radius $\rho(\mathcal{G}) < \mathbf{1}$. Eq. (3.2) can be written in the following equivalent form

$$\phi^{(t+1)} = \mathcal{G}\phi^{(t)} + \mathbf{M}^{-1}\mathbf{c}, \quad t \geq 0. \tag{3.6}$$

The rapidity of convergence in a given iteration method is defined by the *(asymptotic) rate of convergence*

$$\mathbf{R}(\mathcal{G}) = -\mathbf{ln}\,\rho(\mathcal{G}). \tag{3.7}$$

The rate of convergence increases as the value of the spectral radius decreases. The reciprocal of $\mathbf{R}(\mathcal{G})$ can be used as a practical measure of the number of iterations required to reduce the norm of the error vector $\varepsilon^{(t)} = \phi^{(t)} - \phi$ (ϕ denotes the exact solution) by a factor $\mathbf{e}^{-1}$.

Thus, the spectral radius of an iteration matrix plays an important role in the comparison of the efficiency of different iterative methods.

DEFINITION 3.1. For matrices $\mathbf{A}$, $\mathbf{M}$ and $\mathbf{N}$ the following decomposition

$$\mathbf{A} = \mathbf{M} - \mathbf{N}$$

is called a *convergent splitting of* $\mathbf{A}$, if $\mathbf{A}$ and $\mathbf{M}$ are nonsingular matrices and $\rho(\mathbf{M}^{-1}\mathbf{N}) < \mathbf{1}$.

THEOREM 3.1. *Let* $\mathbf{A} = \mathbf{M} - \mathbf{N}$ *be a splitting of* $\mathbf{A}$. *If* $\mathbf{A}$ *and* $\mathbf{M}$ *are nonsingular matrices, then*

$$\mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1}$$

the matrices $\mathbf{M}^{-1}\mathbf{N}$ *and* $\mathbf{A}^{-1}\mathbf{N}$ *commute, and the matrices* $\mathbf{N}\mathbf{M}^{-1}$ *and* $\mathbf{N}\mathbf{A}^{-1}$ *commute too.*

Proof. From the definition of the splitting of $\mathbf{A}$, it follows that

$$\mathbf{M}^{-1} = (\mathbf{A} + \mathbf{N})^{-1} = \mathbf{A}^{-1}(\mathbf{I} + \mathbf{N}\mathbf{A}^{-1})^{-1} = (\mathbf{I} + \mathbf{A}^{-1}\mathbf{N})^{-1}\mathbf{A}^{-1} \tag{3.8}$$

or

$$\mathbf{A}^{-1} = \mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{M}^{-1} + \mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1} \tag{3.9}$$

which implies that

$$\mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1}. \tag{3.10}$$

Hence $\mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1}\mathbf{N} = \mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1}\mathbf{N}$ or $\mathbf{N}\mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{N}\mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1}$. ∎

The following concepts of the splittings of monotone matrices $\mathbf{A}$ will be central to the subsequent discussions.

DEFINITION 3.2. For matrices $\mathbf{A}$, $\mathbf{M}$ and $\mathbf{N}$ the following decomposition

$$\mathbf{A} = \mathbf{M} - \mathbf{N}$$

is called a *regular splitting of* $\mathbf{A}$, if $\mathbf{M}$ is a nonsingular matrix with $\mathbf{M}^{-1} \geq \mathbf{0}$ and $\mathbf{N} \geq \mathbf{0}$.

DEFINITION 3.3. For matrices $\mathbf{A}$, $\mathbf{M}$ and $\mathbf{N}$ the following decomposition

$$\mathbf{A} = \mathbf{M} - \mathbf{N}$$

is called a *nonnegative splitting of* $\mathbf{A}$, if $\mathbf{M}$ is a nonsingular matrix with $\mathbf{M}^{-1} \geq \mathbf{0}$, and $\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{0}$ and $\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}$.

DEFINITION 3.4. For matrices $\mathbf{A}$, $\mathbf{M}$ and $\mathbf{N}$ the following decomposition

$$\mathbf{A} = \mathbf{M} - \mathbf{N}$$

is called a *weak nonnegative splitting of* $\mathbf{A}$, if $\mathbf{M}$ is a nonsingular matrix with $\mathbf{M}^{-1} \geq \mathbf{0}$ and either $\mathbf{M}^{-1}\mathbf{N} = \mathcal{G} \geq \mathbf{0}$ (the *first type*) or $\mathbf{N}\mathbf{M}^{-1} = \mathcal{G}'' \geq \mathbf{0}$ (the *second type*).

The definition of the regular splitting has been introduced by Varga [1] and Definition 3.3 is equivalent to the definition of the weak regular splitting due to Ortega and Rheinboldt [6]. However it should be mentioned that $\mathbf{M}^{-1} \geq \mathbf{0}$ and only $\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{0}$ (without the condition $\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}$) is defined as the weak regular splitting of $\mathbf{A} = \mathbf{M} - \mathbf{N}$ by other authors, but in this case it is necessary to use additional assumptions in comparison theorems [10, 11].

It is evident that with the above definitions the following corollary holds.

COROLLARY 3.1. *Any regular splitting of a matrix* $\mathbf{A}$ *is a nonnegative splitting of* $\mathbf{A}$ *and any nonnegative splitting of* $\mathbf{A}$ *is a weak nonnegative splitting of* $\mathbf{A}$, *but the converse is not true.*

Thus the above corollary tells us that the properties of (weak) nonnegative splittings apply to regular splittings. The properties of weak nonnegative splittings are summarized in the following theorem.

THEOREM 3.2. *Let* $\mathbf{A} = \mathbf{M} - \mathbf{N}$ *be a weak nonnegative splitting of* $\mathbf{A}$. *If* $\mathbf{A}^{-1} \geq \mathbf{0}$, *then*

1. $\mathbf{A}^{-1} \geq \mathbf{M}^{-1}$
2. $\rho(\mathbf{M}^{-1}\mathbf{N}) = \rho(\mathbf{N}\mathbf{M}^{-1}) < 1$
3. *If* $\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{0}$, *then* $\mathbf{A}^{-1}\mathbf{N} \geq \mathbf{M}^{-1}\mathbf{N}$ *and if* $\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}$, *then* $\mathbf{N}\mathbf{A}^{-1} \geq \mathbf{N}\mathbf{M}^{-1}$

4. $$\rho(\mathbf{M}^{-1}\mathbf{N}) = \frac{\rho(\mathbf{A}^{-1}\mathbf{N})}{1+\rho(\mathbf{A}^{-1}\mathbf{N})} \tag{3.11}$$

5. *Conversely, if* $\rho(\mathbf{M}^{-1}\mathbf{N}) < \mathbf{1}$, *then* $\mathbf{A}^{-1} \geq \mathbf{0}$.

Proof.

(1) From Theorem 3.1 it follows

$$\mathbf{A}^{-1} = \mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{M}^{-1} + \mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1}$$

and since $\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{0}$ or $\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}$ by hypotheses then

$$\mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}$$

which gives us immediately that $\mathbf{A}^{-1} \geq \mathbf{M}^{-1}$.

(2) Let us assume that $\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{0}$. Then one can write

$$\begin{aligned}\mathbf{A}^{-1} &= \mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}(\mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1}) \\ &= [\mathbf{I} + \mathbf{M}^{-1}\mathbf{N}]\mathbf{M}^{-1} + (\mathbf{M}^{-1}\mathbf{N})^2\mathbf{A}^{-1} \\ &= [\mathbf{I} + \mathbf{M}^{-1}\mathbf{N} + (\mathbf{M}^{-1}\mathbf{N})^2]\mathbf{M}^{-1} + (\mathbf{M}^{-1}\mathbf{N})^3\mathbf{A}^{-1} \\ &= [\mathbf{I} + \mathbf{M}^{-1}\mathbf{N} + (\mathbf{M}^{-1}\mathbf{N})^2 + \cdots + (\mathbf{M}^{-1}\mathbf{N})^{k-1}]\mathbf{M}^{-1} \\ &\quad + (\mathbf{M}^{-1}\mathbf{N})^k\mathbf{A}^{-1}.\end{aligned} \tag{3.12}$$

The existence of nonnegative matrices $\mathbf{A}^{-1}$, $\mathbf{M}^{-1}$ and $\mathbf{M}^{-1}\mathbf{N}$ implies that the series

$$\mathbf{I} + \mathbf{M}^{-1}\mathbf{N} + (\mathbf{M}^{-1}\mathbf{N})^2 + \cdots$$

is convergent and by Theorem 2.1 $\rho(\mathbf{M}^{-1}\mathbf{N}) < \mathbf{1}$. Similarly, when only $\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}$ one obtains

$$\mathbf{I} + \mathbf{N}\mathbf{M}^{-1} + (\mathbf{N}\mathbf{M}^{-1})^2 + \cdots.$$

Using the result of Lemma 2.1 one obtains $\rho(\mathbf{M}^{-1}\mathbf{N}) = \rho(\mathbf{N}\mathbf{M}^{-1}) < \mathbf{1}$.

(3) In Eq. (3.12) for $k \to \infty$ $(\mathbf{M}^{-1}\mathbf{N})^k \to \mathbf{0}$ and

$$\mathbf{I} + \mathbf{M}^{-1}\mathbf{N} + (\mathbf{M}^{-1}\mathbf{N})^2 + \cdots = (\mathbf{I} - \mathbf{M}^{-1}\mathbf{N})^{-1} \geq \mathbf{I} \geq \mathbf{0} \tag{3.13}$$

and

$$\mathbf{A}^{-1} = (\mathbf{I} - \mathbf{M}^{-1}\mathbf{N})^{-1}\mathbf{M}^{-1} \tag{3.14}$$

or

$$\mathbf{A}^{-1}\mathbf{N} = (\mathbf{I} - \mathbf{M}^{-1}\mathbf{N})^{-1}\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{M}^{-1}\mathbf{N}. \tag{3.15}$$

In the case when $\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}$ it can be similarly shown that

$$\mathbf{N}\mathbf{A}^{-1} \geq \mathbf{N}\mathbf{M}^{-1}.$$

(4) Commutative properties of $\mathbf{M}^{-1}\mathbf{N}$ and $\mathbf{A}^{-1}\mathbf{N}$ imply that both matrices have the same eigenvectors, that is,

$$\mathbf{M}^{-1}\mathbf{N}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$
$$\mathbf{A}^{-1}\mathbf{N}\mathbf{x}_i = \tau_i \mathbf{x}_i$$

for all $i \in S$ and from Eq. (3.15) one can write

$$(\mathbf{I} - \mathbf{M}^{-1}\mathbf{N})^{-1}\mathbf{M}^{-1}\mathbf{N}\mathbf{x}_i = \tau_i \mathbf{x}_i$$

or

$$\mathbf{M}^{-1}\mathbf{N}\mathbf{x}_i = \frac{\tau_i}{1+\tau_i}\mathbf{x}_i$$

but this gives us the eigenvalue relationship

$$\lambda_i = \frac{\tau_i}{1+\tau_i}. \tag{3.16}$$

Since from the assumption $\mathbf{M}^{-1}\mathbf{N}$ and $\mathbf{A}^{-1}\mathbf{N}$ are nonnegative matrices, then by Theorem 2.2 one obtains the relationship for the spectral radii that is

$$\rho(\mathbf{M}^{-1}\mathbf{N}) = \frac{\rho(\mathbf{A}^{-1}\mathbf{N})}{1+\rho(\mathbf{A}^{-1}\mathbf{N})}.$$

(5) The nonnegative character of the matrix $(\mathbf{I}-\mathbf{M}^{-1}\mathbf{N})^{-1}$ when $\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{0}$ with $\rho(\mathbf{M}^{-1}\mathbf{N}) < \mathbf{1}$ (or $(\mathbf{I}-\mathbf{N}\mathbf{M}^{-1})^{-1} \geq \mathbf{0}$ when $\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}$ with $\rho(\mathbf{N}\mathbf{M}^{-1}) < \mathbf{1}$) implies that $\mathbf{A}^{-1} \geq \mathbf{0}$. ∎

The result of Eq. (3.11) has been proved by Varga [1] in the case of regular splitting of $\mathbf{A}$. As a consequence of this theorem the following corollary can be stated.

COROLLARY 3.2. *Each weak nonnegative splitting of a matrix* $\mathbf{A}$ *with* $\mathbf{A}^{-1} \geq \mathbf{0}$ *is a convergent splitting of* $\mathbf{A}$ *and conversely, for each convergent weak nonnegative splitting of* $\mathbf{A}$, $\mathbf{A}^{-1} \geq \mathbf{0}$.

Now the following theorems will be proven.

THEOREM 3.3. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$ *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If one of the following inequalities*

(a) $\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0}$ (*or* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$)

(b) $\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$ (*or* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$)

(c) $\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$ (*or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$)

(d) $\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0}$ (*or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$)

is satisfied, then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.17}$$

Proof. By Lemmas 2.1 and 2.2 and Theorem 3.2 where $\rho(\mathbf{M}^{-1}\mathbf{N})$ is monotone with respect to $\rho(\mathbf{A}^{-1}\mathbf{N})$ the result (3.17) follows immediately. In the case of inequalities given in parentheses the proof is obvious by Lemmas 2.1 and 2.2. ∎

THEOREM 3.4. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$ *of the same type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If* $\mathbf{N}_2 \geq \mathbf{N}_1$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.18}$$

Proof. The inequality $\mathbf{N}_2 \geq \mathbf{N}_1$ implies that either $\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0}$ or $\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$. Using the same argument as in the proof of Theorem 3.3 the result follows immediately. ∎

The same result of Theorem 3.4 has been proven by Varga [1] for the regular splitting of $\mathbf{A}$ and with the strict inequality in (3.18) when $\mathbf{A}^{-1} > \mathbf{0}$. It is evident that by Corollary 3.2 the case of nonnegative splittings of $\mathbf{A}$ is included in both the above theorems.

In iterative methods it is not always possible to compare matrices $\mathbf{N}$ (except the Jacobi and the Gauss-Seidel methods), but very often matrices $\mathbf{M}^{-1}$ can be compared. One might expect that the "closer" $\mathbf{M}$ is to $\mathbf{A}$, the faster the method will converge. Now the nonnegative splitting theory will be discussed from the viewpoint of the influence of $\mathbf{M}^{-1}$ on the behaviour of $\rho(\mathbf{M}^{-1}\mathbf{N})$.

The following theorems are generalizations of Theorem 3.4.

THEOREM 3.5. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two nonnegative splittings of* $\mathbf{A}$ *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If* $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.19}$$

Proof. By Theorems 2.2 and 3.2, and Lemma 2.1 one obtains $\lambda_1 = \rho(\mathbf{N}_1\mathbf{M}_1^{-1}) = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \mathbf{1}$ and $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}$ where the corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are nonnegative. Thus

$$\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \geq \mathbf{0} \tag{3.20}$$

and

$$\mathbf{x}_2^T\mathbf{M}_2^{-1}\mathbf{N}_2 = \lambda_2\mathbf{x}_2^T \geq \mathbf{0}. \tag{3.21}$$

Let us multiply Eq. (3.20) on the left by $\mathbf{A}^{-1}$ and Eq. (3.21) on the right by $\mathbf{A}^{-1}$, one obtains

$$\mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{x}_1 = \lambda_1\mathbf{A}^{-1}\mathbf{x}_1 \tag{3.22}$$

and

$$\mathbf{x}_2^T\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1} = \lambda_2\mathbf{x}_2^T\mathbf{A}^{-1}. \tag{3.23}$$

The relation

$$\mathbf{A}^{-1} = \mathbf{M}_2^{-1} + \mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1} = \mathbf{M}_1^{-1} + \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1} \tag{3.24}$$

implies that

$$\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1} = \mathbf{M}_1^{-1} - \mathbf{M}_2^{-1} \geq \mathbf{0} \tag{3.25}$$

or

$$\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1} \geq 0 \tag{3.26}$$

and according to the definition of nonnegativity given in Section 2

$$\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{x}_1 \geq (=)\ \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{x}_1 = \lambda_1\mathbf{A}^{-1}\mathbf{x}_1. \tag{3.27}$$

Multiplying Eq. (3.27) on the left by $\mathbf{x}_2^T$ and Eq. (3.23) on the right by $\mathbf{x}_1$ one obtains

$$\mathbf{x}_2^T\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{x}_1 \geq \lambda_1\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 \tag{3.28}$$

and

$$\mathbf{x}_2^T\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{x}_1 = \lambda_2\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 \tag{3.29}$$

hence

$$\lambda_1\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 \leq \lambda_2\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1. \tag{3.30}$$

As $\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 > \mathbf{0}$, it follows that $\lambda_1 \leq \lambda_2$ which corresponds to the inequality (3.19). The case when $\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 = \mathbf{0}$ is discussed in Remark given at the end of this section. ■

A somewhat stronger version of this theorem is given below.

THEOREM 3.6. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two nonnegative splittings of* $\mathbf{A}$ *where* $\mathbf{A}^{-1} > \mathbf{0}$. *If* $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1}$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{3.31}$$

Proof. It is easy to notice that the assumption $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1} \geq \mathbf{0}$ implies the strict inequality in (3.26), that is

$$\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0} \tag{3.32}$$

but this leads to the change of the non-strict inequality sign into a strict one in all remaining inequalities of the proof of Theorem 3.5 providing us the result $\lambda_1 < \lambda_2$ which with the notation $\lambda_1 = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1)$ and $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$ proves the theorem. ■

The importance of both the above theorems lies in extending the class of iterative methods in the analysis of convergence. Both of these theorems have been originally proven by the author [2, 3] in the case of regular splittings of $\mathbf{A}$, as a generalization of the results of Varga [1], by means of the Perron-Frobenius theory using only Theorems 2.2 and 2.3. The proofs from the work [2] are included in Section 5.

It is easy to verify that in the case of weak nonnegative splittings of a monotone matrix $\mathbf{A}$, the assumption $\mathbf{N}_2 \geq \mathbf{N}_1$ implies $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1} \geq \mathbf{0}$. From $\mathbf{N}_2 \geq \mathbf{N}_1$ it follows that

$$\mathbf{M}_2 - \mathbf{A} \geq \mathbf{M}_1 - \mathbf{A} \tag{3.33}$$

but this implies that $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1} \geq \mathbf{0}$. This result is included in the following lemma.

LEMMA 3.1. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$. *If* $\mathbf{N}_2 \geq \mathbf{N}_1$, *then* $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1} \geq \mathbf{0}$.

The converse statement need not be true [2, 3]. As will be shown in examples, $\mathbf{N}_1$ and $\mathbf{N}_2$ may have different locations of nonzero entries in spite of the fact that $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$. It should be mentioned that these observations were renewed later by other authors (see, for example [7, 8, 10, 11]). Thus the assumption $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$ is weaker than assumption $\mathbf{N}_2 \geq \mathbf{N}_1$, which motifies the generalization of Theorem 3.4, and in many cases is a verifiable condition only.

Now the case of weak nonnegative splittings of $\mathbf{A}$ will be considered.

THEOREM 3.7. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$ *but of different type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If* $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.34}$$

Proof. Assuming the case when $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} > \mathbf{0}$, one can write that

$$\mathbf{y}_1^T\mathbf{M}_1^{-1}\mathbf{N}_1 = \lambda_1\mathbf{y}_1^T \tag{3.35}$$

and

$$\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{y}_2 = \lambda_2\mathbf{y}_2 \tag{3.36}$$

where by Theorems 2.2 and 3.2, and Lemma 2.1 $\lambda_1 = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \mathbf{1}$ and $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) = \rho(\mathbf{N}_2\mathbf{M}_2^{-1}) < \mathbf{1}$ and the corresponding eigenvectors $\mathbf{y}_1$ and $\mathbf{y}_2$ are nonnegative. Multiplying Eq. (3.35) on the right by $\mathbf{A}^{-1}$ and Eq. (3.36) on the left by $\mathbf{A}^{-1}$ gives us

$$\mathbf{y}_1^T\mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1} = \lambda_1\mathbf{y}_1^T\mathbf{A}^{-1} \tag{3.37}$$

and

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{y}_2 = \lambda_2\mathbf{A}^{-1}\mathbf{y}_2. \tag{3.38}$$

From the relation (3.24), (3.10) and the assumption $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$, it follows that

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{3.39}$$

or

$$\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1} \geq (=)\, \mathbf{y}_1^T\mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1} = \lambda_1\mathbf{y}_1^T\mathbf{A}^{-1}. \tag{3.40}$$

Again multiplying Eq. (3.40) on the right by $\mathbf{y}_2$ and Eq. (3.38) on the left by $\mathbf{y}_1^T$, one obtains

$$\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{y}_2 \geq \lambda_1\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 \tag{3.41}$$

and

$$\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{y}_2 = \lambda_2\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 \tag{3.42}$$

and as $\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 > \mathbf{0}$, it follows that $\lambda_1 \leq \lambda_2$, which proves the inequality (3.34) for the case when $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2^{-1}\mathbf{M}_1 \geq \mathbf{0}$. The case when $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ is discussed in the proof of Theorem 3.5. ■

THEOREM 3.8. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$ *but of different type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} > \mathbf{0}$. *If* $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1}$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{3.43}$$

Proof. Similarly as in the proof of Theorem 3.7 it is evident that the assumption $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1} \geq \mathbf{0}$ implies the strict inequality in (3.39), that is,

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1} > \mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}. \tag{3.44}$$

The above inequality implies replacing the non-strict inequality sign to the strict one in the corresponding inequalities in the remaining part of the proof of Theorem 3.7, which proves the validity of the inequality (3.43). ■

It is easy to notice that the case of two mixed splittings of $\mathbf{A}$ (that is, when one of them is nonnegative and the second is weak nonnegative) is fulfilled by the assumption of Theorems 3.7 and 3.8. For completeness reasons this case is included in two following theorems.

THEOREM 3.9. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1$ *be a weak nonnegative splitting of* $\mathbf{A}$ *and* $\mathbf{A} = \mathbf{M}_2 - \mathbf{N}_2$ *be a nonnegative splitting of* $\mathbf{A}$ *or inversely. If* $\mathbf{A}^{-1} \geq \mathbf{0}$ *and if* $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{3.45}$$

THEOREM 3.10. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1$ *be a weak nonnegative splitting of* $\mathbf{A}$ *and* $\mathbf{A} = \mathbf{M}_2 - \mathbf{N}_2$ *be a nonnegative splitting of* $\mathbf{A}$ *or inversely. If* $\mathbf{A}^{-1} > \mathbf{0}$ *and if* $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1}$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{3.46}$$

Now the question arises whether the inequality $\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \le \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}$ is valid when both weak nonnegative splittings of $\mathbf{A}$ are the same type, that is, either $\mathbf{M}_1^{-1}\mathbf{N}_1 \ge \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \ge \mathbf{0}$ or $\mathbf{N}_1\mathbf{M}_1^{-1} \ge \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \ge \mathbf{0}$ with the assumption that $\mathbf{A}^{-1} \ge \mathbf{0}$ and $\mathbf{M}_1^{-1} \ge \mathbf{M}_2^{-1} \ge \mathbf{0}$. To give an answer to this question it will be interesting to consider some splittings of the monotone matrix $\mathbf{A}$ derived from the following example of 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix} \quad \text{where} \quad \mathbf{A}^{-1} = \begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}. \tag{3.47}$$

As can be seen in Table representing twelve splittings of the above matrix $\mathbf{A}$, the first four splittings are weak nonnegative of the first type ($\mathbf{M}^{-1}\mathbf{N} \ge \mathbf{0}$), the fifth is also weak nonnegative but of the second type ($\mathbf{N}\mathbf{M}^{-1} \ge \mathbf{0}$), the next three are nonnegative and the last four are regular. By the inspection of this table it follows that for $\mathbf{M}_1^{-1} \ge \mathbf{M}_3^{-1} > \mathbf{0}$, $\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_3^{-1}\mathbf{N}_3)$ but on the other hand for $\mathbf{M}_2^{-1} \ge \mathbf{M}_3^{-1} > \mathbf{0}$ there is $\rho(\mathbf{M}_2^{-1}\mathbf{N}_2) > \rho(\mathbf{M}_3^{-1}\mathbf{N}_3)$. A similar behaviour can be observed with strict inequalities, e.g. for $\mathbf{M}_1^{-1} > \mathbf{M}_4^{-1} > \mathbf{0}$ and $\mathbf{M}_3^{-1} > \mathbf{M}_4^{-1} > \mathbf{0}$ it can be seen that $\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_4^{-1}\mathbf{N}_4)$ and $\rho(\mathbf{M}_3^{-1}\mathbf{N}_3) < \rho(\mathbf{M}_4^{-1}\mathbf{N}_4)$ but for $\mathbf{M}_2^{-1} > \mathbf{M}_4^{-1} > \mathbf{0}$ there is $\rho(\mathbf{M}_2^{-1}\mathbf{N}_2) > \rho(\mathbf{M}_4^{-1}\mathbf{N}_4)$.

Moreover the inequalities $\mathbf{M}_{1,2,3,4,5}^{-1} > \mathbf{M}_6^{-1}$ and $\mathbf{M}_{1,2,3}^{-1} > \mathbf{M}_{9,10,11,12}^{-1}$ are an illustration for Theorem 3.10, $\mathbf{M}_{1,2,3}^{-1} > \mathbf{M}_5^{-1}$ for Theorem 3.8, $\mathbf{M}_6^{-1} > \mathbf{M}_{7,8}^{-1}$ for Theorem 3.6, and $\mathbf{M}_7^{-1} \ge \mathbf{M}_8^{-1}$ and $\mathbf{M}_{9,10,11}^{-1} \ge \mathbf{M}_{12}^{-1}$ for Theorem 3.5.

Thus, on the basis of the above examples one can conclude that in the case of weak nonnegative splittings of $\mathbf{A}$ which are of the same type with $\mathbf{A}^{-1} \ge \mathbf{0}$ (or $\mathbf{A}^{-1} > \mathbf{0}$) the assumption that $\mathbf{M}_1^{-1} \ge \mathbf{M}_2^{-1}$ (or $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1}$) is not a sufficient condition for proving that $\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \le \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$ (or $\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$).

However, as will be pointed out the assumption that $\mathbf{A}$ and at least one of $\mathbf{M}_1$ and $\mathbf{M}_2$ are symmetric matrices is a sufficient condition for proving the following theorems.

THEOREM 3.11. *Let $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ be two weak nonnegative splittings of a symmetric matrix $\mathbf{A}$, where $\mathbf{A}^{-1} \ge \mathbf{0}$. If $\mathbf{M}_1^{-1} \ge \mathbf{M}_2^{-1}$ and at least one of $\mathbf{M}_1$ and $\mathbf{M}_2$ is a symmetric matrix, then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \le \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.48}$$

Proof. Let us assume that both weak nonnegative splittings of $\mathbf{A}$ are the second type, that is, $\mathbf{N}_1\mathbf{M}_1^{-1} \ge \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \ge \mathbf{0}$. Then by Theorems 2.2 and 3.2, and Lemma 2.1 one obtains that $\lambda_1 = \rho(\mathbf{N}_1\mathbf{M}_1^{-1}) = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \mathbf{1}$ and $\lambda_2 = \rho(\mathbf{N}_2\mathbf{M}_2^{-1}) = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}$. Thus, one can write

$$\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \ge \mathbf{0} \tag{3.49}$$

and

$$\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{x}_2 = \lambda_2\mathbf{x}_2 \ge \mathbf{0} \tag{3.50}$$

Table. Splitting examples

i	$\mathbf{M}_1$	$\mathbf{N}_1$	$\mathbf{M}_1^{-1}$	$\mathbf{N}_1\mathbf{M}_1^{-1}$	$\mathbf{M}_1^{-1}\mathbf{N}_1$	$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1)$	$\mathbf{N}_1\mathbf{A}^{-1}$	$\mathbf{A}^{-1}\mathbf{N}_1$	$\rho(\mathbf{A}^{-1}\mathbf{N}_1)$	$\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}$
1	$\begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{12} & \frac{5}{6} & -1 \\ -\frac{5}{6} & \frac{1}{3} & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{12} & -\frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ \frac{2}{3} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{2}{3} & 1 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{12} & \frac{1}{6} & 0 \end{bmatrix}$	0	$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{2}{3} & 1 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{12} & \frac{1}{6} & 0 \end{bmatrix}$	0	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$
2	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{3}{2} \\ -1 & 0 & 3 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$	$\frac{1}{3}$	$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$\frac{1}{2}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
3	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & \frac{4}{5} & -\frac{6}{5} \\ -1 & \frac{2}{5} & \frac{12}{5} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ \frac{2}{3} & \frac{1}{2} & \frac{2}{3} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & -\frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$	$\frac{1}{6}$	$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{2}{5} & -\frac{3}{5} & -\frac{2}{5} \\ \frac{4}{5} & \frac{6}{5} & \frac{4}{5} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$	$\frac{1}{5}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & \frac{2}{5} \end{bmatrix}$
4	$\begin{bmatrix} 1 & -1 & 0 \\ -\frac{4}{23} & \frac{19}{23} & -\frac{20}{23} \\ -\frac{14}{23} & \frac{9}{23} & \frac{45}{23} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{4}{23} & -\frac{4}{23} & \frac{3}{23} \\ \frac{9}{23} & \frac{9}{23} & -\frac{1}{23} \end{bmatrix}$	$\begin{bmatrix} \frac{9}{5} & \frac{9}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{9}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{3}{5} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ 1 & \frac{7}{5} & \frac{3}{5} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & 0 \end{bmatrix}$	$-\frac{\sqrt{2}}{5}$	$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{9}{23} & -\frac{13}{23} & -\frac{5}{23} \\ \frac{26}{23} & \frac{35}{23} & \frac{17}{23} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{23} & \frac{1}{23} & \frac{5}{23} \\ \frac{1}{23} & \frac{1}{23} & \frac{5}{23} \\ \frac{5}{23} & \frac{5}{23} & \frac{2}{23} \end{bmatrix}$	$\frac{2+5\sqrt{2}}{23}$	$\begin{bmatrix} \frac{8}{23} & \frac{9}{23} & \frac{7}{23} \\ \frac{8}{23} & \frac{9}{23} & \frac{7}{23} \\ \frac{17}{23} & \frac{22}{23} & \frac{12}{23} \end{bmatrix}$
5	$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & -1 \\ -\frac{3}{2} & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{4}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$	$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{6} & 0 \end{bmatrix}$	$\frac{1}{2}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$	1	$\begin{bmatrix} 2 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \end{bmatrix}$
6	$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & -2 \\ -2 & 1 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} \frac{5}{6} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{6} & \frac{1}{2} \end{bmatrix}$	$\frac{2}{3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	2	$\begin{bmatrix} 3 & 4 & 2 \\ 2 & 4 & 2 \\ 2 & 3 & 2 \end{bmatrix}$

Table. Splitting examples (continued)

i	$\mathbf{M}_1$	$\mathbf{N}_1$	$\mathbf{M}_1^{-1}$	$\mathbf{N}_1\mathbf{M}_1^{-1}$	$\mathbf{M}_1^{-1}\mathbf{N}_1$	$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1)$	$\mathbf{N}_1\mathbf{A}^{-1}$	$\mathbf{A}^{-1}\mathbf{N}_1$	$\rho(\mathbf{A}^{-1}\mathbf{N}_1)$	$\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}$
7	$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & -2 \\ -2 & 1 & 5 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 3 \end{bmatrix}$	$\begin{bmatrix} \frac{3}{4} & \frac{5}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{3}{8} & \frac{3}{4} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{5}{8} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{5}{8} \end{bmatrix}$	$\frac{3}{4}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$	3	$\begin{bmatrix} 4 & 5 & 3 \\ 3 & 5 & 3 \\ 3 & 4 & 3 \end{bmatrix}$
8	$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & -2 \\ -1 & 1 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} \frac{5}{8} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & 0 & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{3}{8} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$	$\begin{bmatrix} \frac{5}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{5}{8} & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{2} \end{bmatrix}$	$\frac{3}{4}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 3 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	3	$\begin{bmatrix} 5 & 6 & 3 \\ 4 & 6 & 3 \\ 4 & 5 & 3 \end{bmatrix}$
9	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$	$\frac{1}{2}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	1	$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$
10	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$	$\frac{1}{2}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	1	$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$
11	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$	$\left(\frac{1}{2}\right)^{1/2}$	$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$	$1+\sqrt{2}$	$\begin{bmatrix} 4 & 6 & 4 \\ 3 & 4 & 3 \\ 2 & 3 & 2 \end{bmatrix}$
12	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$	$\left(\frac{1}{2}\right)^{1/3}$	$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$	$\frac{1}{\sqrt[3]{2}-1}$	$\begin{bmatrix} 6 & 8 & 5 \\ 5 & 6 & 4 \\ 4 & 5 & 3 \end{bmatrix}$

where $\mathbf{x}_1 \geq \mathbf{0}$ and $\mathbf{x}_2 \geq \mathbf{0}$. Multiplying both equations on the left by $\mathbf{A}^{-1}$, one obtains

$$\mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{x}_1 = \lambda_1\mathbf{A}^{-1}\mathbf{x}_1 \geq \mathbf{0} \tag{3.51}$$

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{x}_2 = \lambda_2\mathbf{A}^{-1}\mathbf{x}_2 \geq \mathbf{0}. \tag{3.52}$$

Let us suppose that $\mathbf{M}_2$ is a symmetric matrix. According to Eq. (3.9) one can write

$$\mathbf{A}^{-1} = \mathbf{M}_2^{-1} + \mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1} = \mathbf{M}_1^{-1} + \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1} \tag{3.53}$$

which by the hypotheses $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1} \geq \mathbf{0}$ implies that

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0} \tag{3.54}$$

and

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{x}_1 \geq (=)\, \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{x}_1 = \lambda_1\mathbf{A}^{-1}\mathbf{x}_1. \tag{3.55}$$

Now Eq. (3.52) can be written, as follows

$$\mathbf{x}_2^T(\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1})^T = \lambda_2\mathbf{x}_2^T(\mathbf{A}^{-1})^T. \tag{3.56}$$

From Eq. (3.53) it follows that $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1} = \mathbf{A}^{-1} - \mathbf{M}_2^{-1} \geq \mathbf{0}$ and as $\mathbf{A}^{-1}$ and $\mathbf{M}_2^{-1}$ are symmetric matrices by the hypotheses, one can conclude that the matrix $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}$ is also symmetric and hence Eq. (3.56) can be written, as follows

$$\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1} = \lambda_2\mathbf{x}_2^T\mathbf{A}^{-1}. \tag{3.57}$$

Multiplying Eq. (3.55) on the left by $\mathbf{x}_2^T$ and Eq. (3.57) on the right by $\mathbf{x}_1$, one obtains

$$\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{x}_1 \geq \lambda_1\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 \tag{3.58}$$

and

$$\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{x}_1 = \lambda_2\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 \tag{3.59}$$

but this allows us to conclude that $\lambda_1 \leq \lambda_2$, as $\mathbf{x}_1^T\mathbf{A}^{-1}\mathbf{x}_2 > \mathbf{0}$.

The assumption that only $\mathbf{M}_1$ is a symmetric matrix provides us with similar considerations for the following equations

$$\mathbf{x}_1^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{x}_2 = \lambda_1\mathbf{x}_1^T\mathbf{A}^{-1}\mathbf{x}_2 \tag{3.60}$$

and

$$\mathbf{x}_1^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{x}_2 \leq \lambda_2\mathbf{x}_1^T\mathbf{A}^{-1}\mathbf{x}_2 \tag{3.61}$$

which implies also that $\lambda_1 \leq \lambda_2$ as $\mathbf{x}_1^T\mathbf{A}^{-1}\mathbf{x}_2 > \mathbf{0}$.

In the case when $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, one can write

$$\mathbf{y}_1^T\mathbf{M}_1^{-1}\mathbf{N}_1 = \lambda_1\mathbf{y}_1^T \tag{3.62}$$

and

$$\mathbf{y}_2^T\mathbf{M}_2^{-1}\mathbf{N}_2 = \lambda_2\mathbf{y}_2^T. \tag{3.63}$$

Repeating the procedure of the proof described above, one obtains

$$\mathbf{y}_1^T\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y}_2 \geq \lambda_1\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 \tag{3.64}$$

and

$$\mathbf{y}_1^T\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y}_2 = \lambda_2\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 \tag{3.65}$$

when $\mathbf{M}_2$ is a symmetric matrix, or

$$\mathbf{y}_2^T\mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{y}_1 = \lambda_1\mathbf{y}_2^T\mathbf{A}^{-1}\mathbf{y}_1 \tag{3.66}$$

and

$$\mathbf{y}_2^T\mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{y}_1 \leq \lambda_2\mathbf{y}_2^T\mathbf{A}^{-1}\mathbf{y}_1 \tag{3.67}$$

when $\mathbf{M}_1$ is to be assumed a symmetric matrix, which consequently leads to proving that $\lambda_1 \leq \lambda_2$, as $\mathbf{y}_2^T\mathbf{A}^{-1}\mathbf{y}_1 > \mathbf{0}$.

The cases when both weak nonnegative splittings of $\mathbf{A}$ are a contrary type, that is, either $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ or $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, have been proven in Theorem 3.7, however without the assumption of symmetry. ■

THEOREM 3.12. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of a symmetric matrix* $\mathbf{A}$, *where* $\mathbf{A}^{-1} > \mathbf{0}$. *If* $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1}$ *and at least one of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *is a symmetric matrix, then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.68}$$

Proof. Using the same arguments as in the proof of Theorem 3.11, it can be seen that the inequality $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1}$ leads to the strict inequality in (3.54), that is

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0} \tag{3.69}$$

which implies the change of the inequality sign to the strict one in the corresponding inequalities in the proof of Theorem 3.11 and provides that $\lambda_1 < \lambda_2$, and with the notation that $\lambda_1 = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1)$ and $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$ proves the theorem. ■

The special case of conditions important in applications, discussed in the next sections, is considered in the following theorems.

THEOREM 3.13. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$ *of the same type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If* $\mathbf{N}_2^T \geq \mathbf{N}_1$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.70}$$

Proof. Assuming that $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, the inequality $\mathbf{N}_2^T \geq \mathbf{N}_1$ implies that

$$\mathbf{A}^{-1}\mathbf{N}_2^T(\mathbf{A}^{-1})^T \geq \mathbf{A}^{-1}\mathbf{N}_1(\mathbf{A}^{-1})^T \geq \mathbf{0}. \tag{3.71}$$

Let $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$ be such vectors that

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1 = \tau_1\mathbf{x}^T \tag{3.72}$$

and

$$\mathbf{y}^T\mathbf{A}^{-1}\mathbf{N}_2 = \tau_2\mathbf{y}^T$$

or

$$\mathbf{N}_2^T(\mathbf{A}^{-1})^T\mathbf{y} = \tau_2\mathbf{y} \tag{3.73}$$

where $\tau_1 = \rho(\mathbf{A}^{-1}\mathbf{N}_1)$ and $\tau_2 = \rho(\mathbf{A}^{-1}\mathbf{N}_2)$. Multiplying Eq. (3.72) on the right by $\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y}$ and Eq. (3.73) on the left by $\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T$, one obtains

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y} = \tau_1\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y} \tag{3.74}$$

and

$$\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{N}_2^T(\mathbf{A}^{-1})^T\mathbf{y} = \tau_2\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y}. \tag{3.75}$$

Both above equations combined with the inequality (3.71) provide us with

$$\begin{aligned} 2\tau_1\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y} \leq \mathbf{x}^T[&\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}(\mathbf{A}^{-1})^T \\ &+ \mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{N}_2^T(\mathbf{A}^{-1})^T]\mathbf{y} \end{aligned} \tag{3.76a}$$

$$\begin{aligned} \mathbf{x}^T[\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}(\mathbf{A}^{-1})^T + \mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{N}_2^T(\mathbf{A}^{-1})^T]\mathbf{y} \\ \leq 2\tau_2\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y} \end{aligned} \tag{3.76b}$$

hence as $\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y} > \mathbf{0}$, it follows that $\tau_1 \leq \tau_2$ which consequently by Theorem 3.2 proves the inequality (3.70). The case when $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ can be proven similarly by considering

$$(\mathbf{A}^{-1})^T\mathbf{N}_2^T\mathbf{A}^{-1} \geq (\mathbf{A}^{-1})^T\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$$

$\mathbf{N}_1\mathbf{A}^{-1}\mathbf{v} = \tau_1\mathbf{v}$ and $\mathbf{N}_2\mathbf{A}^{-1}\mathbf{w} = \tau_2\mathbf{w}$ which completes the theorem proof. ∎

It is evident that by Corollary 3.2 the case of nonnegative splittings of $\mathbf{A}$ is included in the above theorem. In particularity, however the following corollaries hold.

COROLLARY 3.3. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$ *of the same type, that is either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$.

(a) *If*

$$\mathbf{A}^{-1}\mathbf{N}_2^T(\mathbf{A}^{-1})^T \geq \mathbf{A}^{-1}\mathbf{N}_1(\mathbf{A}^{-1})^T \geq \mathbf{0} \tag{3.77}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \le \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) \le \mathbf{1}. \tag{3.78}$$

(b) *Moreover, if*

$$\mathbf{A}^{-1}\mathbf{N}_2^T(\mathbf{A}^{-1})^T > \mathbf{A}^{-1}\mathbf{N}_1(\mathbf{A}^{-1})^T \ge \mathbf{0} \tag{3.77a}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.78a}$$

Proof. It is evident that the proof of this corollary follows immediately from the proof of Theorem 3.13, where the inequality (3.77a) implies the change of the nonstrict inequality sign in (3.76a) and (3.76b) to the strict one providing the result (3.78a). ∎

COROLLARY 3.4. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two regular splittings of* $\mathbf{A}$, *where* $\mathbf{A}^{-1} > \mathbf{0}$. *If* $\mathbf{N}_2^T \ge \mathbf{N}_1$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.79}$$

Proof. The assumptions of this corollary imply that

$$\mathbf{A}^{-1}\mathbf{N}_2^T(\mathbf{A}^{-1})^T > \mathbf{A}^{-1}\mathbf{N}_1(\mathbf{A}^{-1})^T > \mathbf{0}$$

but this leads to the strict inequality sign in (3.76a) and (3.76b) providing the result (3.79). ∎

It is interesting to notice that in contradiction to the Lemma 3.1, $\mathbf{N}_2^T \ge \mathbf{N}_1$ need not imply that $(\mathbf{M}_1^{-1})^T \ge \mathbf{M}_2^{-1}$ when $\mathbf{A}^{-1} \ge \mathbf{0}$. Really from

$$\mathbf{N}_2^T \ge \mathbf{N}_1 \tag{3.80}$$

it follows that

$$\mathbf{M}_2^T - \mathbf{A}^T \ge \mathbf{M}_1 - \mathbf{A} \tag{3.81}$$

but this gives us

$$\mathbf{M}_1^{-1}[\mathbf{I} - \mathbf{A}^T(\mathbf{M}_2^{-1})^T] \ge [\mathbf{I} - \mathbf{M}_1^{-1}\mathbf{A}](\mathbf{M}_2^{-1})^T \tag{3.82}$$

hence

$$\mathbf{M}_1^{-1} - (\mathbf{M}_2^{-1})^T \ge \mathbf{M}_1^{-1}[\mathbf{A}^T - \mathbf{A}](\mathbf{M}_2^{-1})^T. \tag{3.83}$$

The equivalent condition

$$\mathbf{N}_2 \ge \mathbf{N}_1^T \tag{3.80a}$$

provides

$$(\mathbf{M}_1^{-1})^T - \mathbf{M}_2^{-1} \ge (\mathbf{M}_1^{-1})^T[\mathbf{A} - \mathbf{A}^T]\mathbf{M}_2^{-1}. \tag{3.83a}$$

It is not difficult to find examples of nonnegative splittings of a monotone matrix $\mathbf{A}$ such that $\mathbf{M}_1^{-1}[\mathbf{A}^T - \mathbf{A}](\mathbf{M}_2^{-1})^T$ may have positive and negative entries.

However the following theorems hold.

THEOREM 3.14. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$ *of the same type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If*

$$\mathbf{M}_1^{-1} \geq (\mathbf{M}_2^{-1})^T \geq \mathbf{0} \tag{3.84}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{3.85}$$

Proof. As can be seen for the hypothesis (3.84) the following inequality

$$(\mathbf{A}^{-1})^T - (\mathbf{M}_2^{-1})^T \geq \mathbf{A}^{-1} - \mathbf{M}_1^{-1} \geq \mathbf{0} \tag{3.86}$$

holds, which by the relation (3.9) gives us

$$(\mathbf{A}^{-1})^T\mathbf{N}_2^T(\mathbf{M}_2^{-1})^T = (\mathbf{M}_2^{-1})^T\mathbf{N}_2^T(\mathbf{A}^{-1})^T \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1} = \mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1}. \tag{3.87}$$

Assuming that $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, and let $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$ be such vectors that

$$\mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 = \lambda_1\mathbf{x}^T \tag{3.88}$$

and

$$\mathbf{y}^T\mathbf{M}_2^{-1}\mathbf{N}_2 = \lambda_2\mathbf{y}^T$$

or

$$\mathbf{N}_2^T(\mathbf{M}_2^{-1})^T\mathbf{y} = \lambda_2\mathbf{y} \tag{3.89}$$

where $\lambda_1 = \rho(\mathbf{M}^{-1}\mathbf{N}_1)$ and $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$. Multiplying Eq. (3.88) on the right by $\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y}$ and Eq. (3.89) on the left by $\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T$, one obtains

$$\mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y} = \lambda_1\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y} \tag{3.90}$$

and

$$\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{N}_2^T(\mathbf{M}_2^{-1})^T\mathbf{y} = \lambda_2\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y}. \tag{3.91}$$

Both above equations combined with the inequality (3.87) provide us with

$$\begin{aligned} 2\lambda_1\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y} \leq \mathbf{x}^T[&\mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1}(\mathbf{A}^{-1})^T \\ &+ \mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{N}_2^T(\mathbf{M}_2^{-1})^T]\mathbf{y} \end{aligned} \tag{3.92a}$$

$$\begin{aligned} \mathbf{x}^T[\mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1}(\mathbf{A}^{-1})^T + \mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{N}_2^T(\mathbf{M}_2^{-1})^T]\mathbf{y} \\ \leq 2\lambda_2\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y} \end{aligned} \tag{3.92b}$$

hence it follows that $\lambda_1 \leq \lambda_2$ with $\mathbf{x}^T\mathbf{A}^{-1}(\mathbf{A}^{-1})^T\mathbf{y} > 0$.

The case when $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ can be proven similarly. ∎

The tenth and eleventh splittings in Table illustrate the above theorem. A somewhat stronger version of this theorem is given below.

THEOREM 3.15. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$ *of the same type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$*, where* $\mathbf{A}^{-1} \geq \mathbf{0}$*. If*

$$\mathbf{M}_1^{-1} > (\mathbf{M}_2^{-1})^T \geq \mathbf{0} \tag{3.93}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.94}$$

Proof. It is evident that the condition (3.93) implies the change of the non-strict inequality sign in (3.87), (3.92a) and (3.92b) to the strict one providing the result (3.94). ∎

By an analogy to Theorem 3.3 the following result holds.

THEOREM 3.16. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$ *where* $\mathbf{A}^{-1} \geq \mathbf{0}$*. If one of the following inequalities*

(a) $\mathbf{A}^{-1}\mathbf{N}_2 \geq (\mathbf{A}^{-1}\mathbf{N}_1)^T \geq \mathbf{0}$ (*or* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq (\mathbf{M}_1^{-1}\mathbf{N}_1)^T \geq \mathbf{0}$)
(b) $\mathbf{A}^{-1}\mathbf{N}_2 \geq (\mathbf{N}_1\mathbf{A}^{-1})^T \geq \mathbf{0}$ (*or* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq (\mathbf{N}_1\mathbf{M}_1^{-1})^T \geq \mathbf{0}$)
(c) $\mathbf{N}_2\mathbf{A}^{-1} \geq (\mathbf{N}_1\mathbf{A}^{-1})^T \geq \mathbf{0}$ (*or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq (\mathbf{N}_1\mathbf{M}_1^{-1})^T \geq \mathbf{0}$)
(d) $\mathbf{N}_2\mathbf{A}^{-1} \geq (\mathbf{A}^{-1}\mathbf{N}_1)^T \geq \mathbf{0}$ (*or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq (\mathbf{M}_1^{-1}\mathbf{N}_1)^T \geq \mathbf{0}$)

is satisfied, then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.95}$$

Proof. Since matrices $\mathbf{A}^{-1}\mathbf{N}$ and $(\mathbf{A}^{-1}\mathbf{N})^T$ have the same eigenvalue spectrum, then by Lemmas 2.1 and 2.2 and Theorem 3.2 where $\rho(\mathbf{M}^{-1}\mathbf{N})$ is monotone with respect to $\rho(\mathbf{A}^{-1}\mathbf{N})$ the result (3.95) follows immediately. In the case of inequalities given in parentheses the proof is obvious by Lemma 2.2. ∎

It should be noticed that in the case of a monotone matrix $\mathbf{A}$, theorems based on the hypotheses of Lemma 3.1 allow us to compare weak nonnegative splittings of $\mathbf{A}$ which are different in type, whereas theorems based on the hypotheses (3.80), (3.84), and (3.93) allow us to compare weak nonnegative splittings of $\mathbf{A}$ which are same in type.

The following result generalizes the comparison theorems presented in this section.

THEOREM 3.17. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$ *where* $\mathbf{A}^{-1} \geq \mathbf{0}$*. Let* $\mathbf{x} \geq \mathbf{0}$*,* $\mathbf{y} \geq \mathbf{0}$ *and* $\mathbf{z} \geq \mathbf{0}$ *be vectors such that*

$\mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 = \lambda_1\mathbf{x}^T$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{y} = \lambda_2\mathbf{y}$ *or* $\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{z} = \lambda_2\mathbf{z}$ *when* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *respectively; and let* $\mathbf{u} \geq \mathbf{0}$, $\mathbf{v} \geq \mathbf{0}$ *and* $\mathbf{w} \geq \mathbf{0}$ *be vectors such that* $\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{u} = \lambda_1\mathbf{u}$ *and* $\mathbf{v}^T\mathbf{M}_2^{-1}\mathbf{N}_2 = \lambda_2\mathbf{v}^T$ *or* $\mathbf{w}^T\mathbf{N}_2\mathbf{M}_2^{-1} = \lambda_2\mathbf{w}^T$ *when* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *respectively; where* $\lambda_1 = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1)$ *and* $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$. *If one of the following inequalities*

$$\mathbf{x}^T\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0} \quad \textit{or} \quad \mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{y} \geq \mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{y} \geq \mathbf{0} \tag{3.96a}$$

$$\mathbf{x}^T\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0} \quad \textit{or} \quad \mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{z} \geq \mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{z} \geq \mathbf{0} \tag{3.96b}$$

$$\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{u} \geq \mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{u} \geq \mathbf{0} \quad \textit{or} \quad \mathbf{v}^T\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{v}^T\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0} \tag{3.96c}$$

$$\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{u} \geq \mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{u} \geq \mathbf{0} \quad \textit{or} \quad \mathbf{w}^T\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{w}^T\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0} \tag{3.96d}$$

or

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0} \quad \textit{or} \quad \mathbf{A}^{-1}\mathbf{N}_2\mathbf{y} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{y} \geq \mathbf{0} \tag{3.97a}$$

$$\mathbf{x}^T\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0} \quad \textit{or} \quad \mathbf{N}_2\mathbf{A}^{-1}\mathbf{z} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{z} \geq \mathbf{0} \tag{3.97b}$$

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{u} \geq \mathbf{N}_1\mathbf{A}^{-1}\mathbf{u} \geq \mathbf{0} \quad \textit{or} \quad \mathbf{v}^T\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{v}^T\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{3.97c}$$

$$\mathbf{N}_2\mathbf{A}^{-1}\mathbf{u} \geq \mathbf{N}_1\mathbf{A}^{-1}\mathbf{u} \geq \mathbf{0} \quad \textit{or} \quad \mathbf{w}^T\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{w}^T\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{3.97d}$$

is fulfilled, then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.98}$$

In particular, if the first non-strict inequality sign in the above inequalities is replaced by the strict one, then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.99}$$

Proof. Assuming the case when $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ for which the first inequality is fulfilled, then it is evident that the following inequality

$$\lambda_2\mathbf{x}^T\mathbf{y} = \mathbf{x}^T\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{y} \geq \mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{y} = \lambda_1\mathbf{x}^T\mathbf{y} \tag{3.100}$$

is satisfied, but this with $\mathbf{x}^T\mathbf{y} > 0$ implies the result (3.99). Other cases can be proven in an analogous way. The case of the strict inequality sign in (3.96) implies the result (3.100) with the strict inequality. Since the matrices $\mathbf{M}^{-1}\mathbf{N}$ and $\mathbf{A}^{-1}\mathbf{N}$ commute by Theorem 3.1 they have the same eigenvectors hence, e.g. in the case of (3.97a) one obtains

$$\tau_2\mathbf{x}^T\mathbf{y} = \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{y} \geq \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{y} = \tau_1\mathbf{x}^T\mathbf{y} \tag{3.101}$$

where $\tau_1 = \rho(\mathbf{A}^{-1}\mathbf{N}_1)$ and $\tau_2 = \rho(\mathbf{A}^{-1}\mathbf{N}_2)$, which provides $\tau_2 \geq \tau_1$ with $\mathbf{x}^T\mathbf{y} > 0$ and $\tau_2 > \tau_1$ when there is the strict inequality in (3.97a). Hence by Theorem 3.2

the results (3.98) and (3.99) follow immediately. The remaining cases of (3.97) can be proved similarly, which is completing the proof. ■

COROLLARY 3.5. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$ *of the same type, where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *Let* $\mathbf{x} \geq \mathbf{0}$ *and* $\mathbf{y} \geq \mathbf{0}$ *be vectors such that* $\mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 = \lambda_1\mathbf{x}^T$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{y} = \lambda_2\mathbf{y}$ *when* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$; *and let* $\mathbf{u} \geq \mathbf{0}$ *and* $\mathbf{v} \geq \mathbf{0}$ *be vectors such that* $\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{u} = \lambda_1\mathbf{u}$ *and* $\mathbf{v}^T\mathbf{N}_2\mathbf{M}_2^{-1} = \lambda_2\mathbf{v}^T$ *when* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$; *where* $\lambda_1 = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1)$ *and* $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$. *If one of the following inequalities*

$$\mathbf{x}^T\mathbf{N}_2 \geq \mathbf{x}^T\mathbf{N}_1 \quad \textit{or} \quad \mathbf{N}_2\mathbf{y} \geq \mathbf{N}_1\mathbf{y} \tag{3.102a}$$

$$\mathbf{N}_2\mathbf{u} \geq \mathbf{N}_1\mathbf{u} \quad \textit{or} \quad \mathbf{v}^T\mathbf{N}_2 \geq \mathbf{v}^T\mathbf{N}_1 \tag{3.102b}$$

holds, then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{3.103}$$

Proof. Multiplying the first inequality of (3.102a) on the right by $\mathbf{A}^{-1}\mathbf{y}$, one obtains

$$\tau_2\mathbf{x}^T\mathbf{y} = \mathbf{x}^T\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y} \geq \mathbf{x}^T\mathbf{N}_1\mathbf{A}^{-1}\mathbf{y} = \tau_1\mathbf{x}^T\mathbf{y}$$

where $\tau_1 = \rho(\mathbf{N}_1\mathbf{A}^{-1})$ and $\tau_2 = \rho(\mathbf{N}_2\mathbf{A}^{-1})$. As $\mathbf{x}^T\mathbf{y} > 0$, $\tau_2 \geq \tau_1$ which by Theorem 3.2 proves the result (3.103). For the remaining hypothesis inequalities, the result (3.103) can be proven in a similar way. ■

However in the case of applying this theorem or its corollary, it is necessary to know at least one eigenvector.

REMARK. Finally, it should be mentioned that some comparison theorems such as Theorems 3.5, 3.7, 3.9 and others with $\mathbf{A}^{-1} \geq \mathbf{0}$ as the hypothesis, have been proven with the assumption that $\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 > 0$ (or $\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 > 0$, as in Theorem 3.7) which however, may be not satisfied with $\mathbf{A}^{-1} \geq \mathbf{0}$. For instance, in the case of monotone triangular matrices $\mathbf{A}$, it is easy to find examples of not necessary (weak) nonnegative but also regular splittings of $\mathbf{A}$ for which $\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 = 0$ with $\mathbf{A}^{-1} \geq \mathbf{0}$, $\mathbf{x}_1 \geq \mathbf{0}$ and $\mathbf{x}_2 \geq \mathbf{0}$ (in the case when $\mathbf{A}^{-1} > \mathbf{0}$, $\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1$ is always positive).

Therefore, it seems to be natural to ask if the mentioned theorems are true when $\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 = 0$ is induced or when for $\lambda_1 = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1) = 0$ the corresponding eigenvector $\mathbf{x}_1 = \mathbf{0}$, as in the case of the first splitting given in Table. As can be shown in the example of the proof of Theorem 3.5, a simple modification allows us to avoid this apparent difficulty when $\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 = 0$.

Assuming a matrix $\mathbf{B} > \mathbf{0}$, then instead of Eqs. (3.20) and (3.21) in the proof of Theorem 3.5 the following equations may be considered

$$(\varepsilon\mathbf{B}\mathbf{A}^{-1} + \mathbf{N}_1\mathbf{M}_1^{-1})\widetilde{\mathbf{x}}_1 = \widetilde{\lambda}_1\widetilde{\mathbf{x}}_1 \tag{3.20a}$$

$$\widetilde{\mathbf{x}}_2^T(\varepsilon\mathbf{A}^{-1}\mathbf{B} + \mathbf{M}_2^{-1}\mathbf{N}_2) = \widetilde{\lambda}_2\widetilde{\mathbf{x}}_2^T. \tag{3.21a}$$

Since for $\varepsilon > 0$ both matrices $\varepsilon\mathbf{B}\mathbf{A}^{-1} + \mathbf{N}_1\mathbf{M}_1^{-1}$ and $\varepsilon\mathbf{A}^{-1}\mathbf{B} + \mathbf{M}_2^{-1}\mathbf{N}_2$ are irreducible, its eigenvalues $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ corresponding to spectral radii are strictly increasing functions of $\varepsilon \geq 0$ (by Theorem 2.3), and $\widetilde{\lambda}_1 = \lambda_1$ and $\widetilde{\lambda}_2 = \lambda_2$ with $\varepsilon = 0$. Multiplying Eq. (3.20a) on the left and Eq. (3.21a) on the right $\mathbf{A}^{-1}$, one obtains

$$(\varepsilon\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1})\widetilde{\mathbf{x}}_1 = \widetilde{\lambda}_1\mathbf{A}^{-1}\widetilde{\mathbf{x}}_1 \tag{3.22a}$$

$$\widetilde{\mathbf{x}}_2^T(\varepsilon\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} + \mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1}) = \widetilde{\lambda}_2\widetilde{\mathbf{x}}_2^T\mathbf{A}^{-1} \tag{3.23a}$$

and by using the inequality (3.25)

$$\begin{aligned}&(\varepsilon\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} + \mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1})\widetilde{\mathbf{x}}_1 \\ &\qquad \geq (\varepsilon\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{N}_1\mathbf{M}_1^{-1})\widetilde{\mathbf{x}}_1 = \widetilde{\lambda}_1\mathbf{A}^{-1}\widetilde{\mathbf{x}}_1.\end{aligned} \tag{3.27a}$$

Multiplying again Eq. (3.27a) on the left by $\widetilde{\mathbf{x}}_2^T$ and Eq. (3.23a) on the right by $\widetilde{\mathbf{x}}_1$, one obtains

$$\widetilde{\mathbf{x}}_2^T(\varepsilon\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} + \mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1})\widetilde{\mathbf{x}}_1 \geq \widetilde{\lambda}_1\widetilde{\mathbf{x}}_2^T\mathbf{A}^{-1}\widetilde{\mathbf{x}}_1 \tag{3.28a}$$

and

$$\widetilde{\mathbf{x}}_2^T(\varepsilon\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} + \mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{A}^{-1})\widetilde{\mathbf{x}}_1 = \widetilde{\lambda}_2\widetilde{\mathbf{x}}_2^T\mathbf{A}^{-1}\widetilde{\mathbf{x}}_1 \tag{3.29a}$$

hence

$$\widetilde{\lambda}_1\widetilde{\mathbf{x}}_2^T\mathbf{A}^{-1}\widetilde{\mathbf{x}}_1 \leq \widetilde{\lambda}_2\widetilde{\mathbf{x}}_2^T\mathbf{A}^{-1}\widetilde{\mathbf{x}}_1. \tag{3.30a}$$

Since for $\varepsilon > 0$ both vectors $\widetilde{\mathbf{x}}_1$ and $\widetilde{\mathbf{x}}_2$ are positive, it can be concluded that $\widetilde{\mathbf{x}}_2^T\mathbf{A}^{-1}\widetilde{\mathbf{x}}_1 > 0$, which implies $\widetilde{\lambda}_1 \leq \widetilde{\lambda}_2$ (in this case $\widetilde{\lambda}_1 < \widetilde{\lambda}_2$ for each $\varepsilon > 0$). Taking the limit for $\varepsilon \to 0$, it follows that $\widetilde{\lambda}_1 \to \lambda_1$ and $\widetilde{\lambda}_2 \to \lambda_2$ which allows us to conclude that $\lambda_1 \leq \lambda_2$ proving Theorem 3.5.

The same or similar modification can be used in the proofs of other theorems in which $\mathbf{x}_2^T\mathbf{A}^{-1}\mathbf{x}_1 = 0$ may be induced.

4. Prefactorization Iterative Methods

In this section the application of some results of the nonnegative splitting theory will be demonstrated in the convergence analysis of prefactorizationing iterative methods used for solving the following linear equation system

$$\mathbf{A}\phi = \mathbf{c}. \tag{4.1}$$

Assuming that $\mathbf{A} = \mathbf{M} - \mathbf{N}$ represents the splitting of $\mathbf{A}$, where $\mathbf{A}$ and $\mathbf{M}$ are nonsingular matrices, the iterative method for solving Eq. (4.1) can be expressed in the general form

$$\mathbf{M}\phi^{(t+1)} = \mathbf{N}\phi^{(t)} + \mathbf{c}, \quad t \geq 0$$

or equivalently

$$\phi^{(t+1)} = \mathcal{G}\phi^{(t)} + \mathbf{M}^{-1}\mathbf{c}, \quad t \geq 0. \tag{4.2}$$

As is well known the above iterative process is convergent to the unique solution

$$\phi = \mathbf{A}^{-1}\mathbf{c} \tag{4.3}$$

if and only if the spectral radius of the iteration matrix $\mathcal{G} = \mathbf{M}^{-1}\mathbf{N}$, $\rho(\mathcal{G}) < 1$.

In some sense the matrix $\mathbf{M}$ can be imagined as being an approximation to $\mathbf{A}$. Usually $\mathbf{M}$ is represented as a product of nonsingular matrices chosen in such a way that they are easy to obtain and relatively easy to invert, so that the matrix $\mathbf{M}$ can be considered as prefactorizationing $\mathbf{A}$ (so it is called the *prefactorizationer* of $\mathbf{A}$), and $\mathbf{M}^{-1}$ can be regarded as the preinvertioning matrix of $\mathbf{A}$ (so it is called the *preinvertioner* of $\mathbf{A}$ approximating $\mathbf{A}^{-1}$). Consequently, $\mathbf{N} = \mathbf{M} - \mathbf{A}$ can be regarded as the *residual matrix* obtained with the assumed prefactorizationing $\mathbf{A}$; and when $\mathbf{N}$ exists as a nonzero matrix, there is the *partial* (*incomplete*) *factorization of* $\mathbf{A}$, and the solution has the iterative nature. In the case when $\mathbf{N}$ becomes the null matrix, there is the *strict factorization of* $\mathbf{A}$ and the solution of Eq. (4.1) is obtained by means of the direct method equivalent to the Gaussian elimination.

Defining a given $n \times n$ nonsingular matrix $\mathbf{A}$ by the following decomposition

$$\mathbf{A} = \mathbf{K} - \mathbf{L} - \mathbf{U} \tag{4.4}$$

where $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{U}$ are nonsingular diagonal, strictly lower triangular and strictly upper triangular matrices respectively, and introducing additional strictly lower triangular and strictly upper triangular matrices $\mathbf{H}$ and $\mathbf{Q}$ respectively, then the following factorization can be used

$$\mathbf{M} = [\mathbf{I} - (\mathbf{L} + \mathbf{H})\mathbf{D}^{-1}]\mathbf{D}[\mathbf{I} - \mathbf{D}^{-1}(\mathbf{U} + \mathbf{Q})] \tag{4.5}$$

where $\mathbf{D}$ is assumed to be a nonsingular diagonal matrix defined by the following implicit relation

$$\mathbf{D} = \mathbf{K} - \operatorname{diag}\{(\mathbf{L} + \mathbf{H})\mathbf{D}^{-1}(\mathbf{U} + \mathbf{Q})\} \tag{4.6}$$

and as can be easily verified

$$\mathbf{N} = \operatorname{offdiag}\{(\mathbf{L} + \mathbf{H})\mathbf{D}^{-1}(\mathbf{U} + \mathbf{Q})\} - \mathbf{H} - \mathbf{Q} \tag{4.7}$$

where the notation $\operatorname{diag}\{\mathbf{B}\}$ denotes the diagonal matrix with diagonal entries identical with those of $\mathbf{B}$ and $\operatorname{offdiag}\{\mathbf{B}\} = \mathbf{B} - \operatorname{diag}\{\mathbf{B}\}$.

The iterative method can be written, as follows

$$\phi^{(t+1)} = \mathcal{F}\phi^{(t)} + \mathbf{M}^{-1}\mathbf{c}, \quad t \geq 0 \tag{4.8}$$

and

$$\mathcal{F} = [\mathbf{I} - \mathbf{D}^{-1}(\mathbf{U} + \mathbf{Q})]^{-1}\mathbf{D}^{-1}[\mathbf{I} - (\mathbf{L} + \mathbf{H})\mathbf{D}^{-1}]^{-1}\mathbf{N}. \tag{4.9}$$

Since $\mathbf{I} - (\mathbf{L} + \mathbf{H})\mathbf{D}^{-1}$ and $\mathbf{I} - \mathbf{D}^{-1}(\mathbf{U} + \mathbf{Q})$ are nonsingular lower and upper triangular matrices respectively, this method can be easily implemented by using the so-called *two-sweep* (*forward-backward*) *procedure*.

Let us multiply Eq. (4.8) on the left by $\mathbf{I} - \mathbf{D}^{-1}(\mathbf{U} + \mathbf{Q})$ and shift the term $\mathbf{D}^{-1}(\mathbf{U} + \mathbf{Q})\phi^{(t+1)}$ to the right side of the equation

$$\phi^{(t+1)} = \mathbf{D}^{-1}\{(\mathbf{U} + \mathbf{Q})\phi^{(t+1)} + [\mathbf{I} - (\mathbf{L} + \mathbf{H})\mathbf{D}^{-1}]^{-1}[\mathbf{N}\phi^{(t)} + \mathbf{c}]\}.$$

Denoting by

$$\beta^{(t+1)} = [\mathbf{I} - (\mathbf{L} + \mathbf{H})\mathbf{D}^{-1}]^{-1}[\mathbf{N}\phi^{(t)} + \mathbf{c}]$$

and multiplying again this expression on the left by $\mathbf{I} - (\mathbf{L} + \mathbf{H})\mathbf{D}^{-1}$, one finally obtains

$$\left.\begin{aligned} \beta^{(t+1)} &= (\mathbf{L} + \mathbf{H})\mathbf{D}^{-1}\beta^{(t+1)} + \mathbf{N}\phi^{(t)} + \mathbf{c} \\ \phi^{(t+1)} &= \mathbf{D}^{-1}[(\mathbf{U} + \mathbf{Q})\phi^{(t+1)} + \beta^{(t+1)}], \quad t \geq 0 \end{aligned}\right\}. \tag{4.10}$$

Since $(\mathbf{L} + \mathbf{H})\mathbf{D}^{-1}$ and $\mathbf{D}^{-1}(\mathbf{U} + \mathbf{Q})$ are strictly lower triangular and strictly upper triangular matrices respectively, the components of $\beta^{(t+1)}$ can be calculated recursively for increasing indices in the *forward elimination sweep*, and components of $\phi^{(t+1)}$ can be calculated recursively for decreasing indices in the *backward substitution sweep*. Equations (4.10) represent the general form of a broad class of prefactorizationing methods called the *two-sweep iterative methods*, and each of them is uniquely defined by the choice of the matrices $\mathbf{H}$ and $\mathbf{Q}$. The matrix $\mathcal{F}$ is called the *two-sweep iteration matrix*. The classification of the two-sweep iterative methods from the viewpoint of the choice of $\mathbf{H}$ and $\mathbf{Q}$ is given in [4].

Let us restrict our attention to the iterative schemes defined by the following choice of the matrices $\mathbf{H}$ and $\mathbf{Q}$ including also such classical schemes as the Jacobi and Gauss-Seidel methods.

1. The Jacobi method

$$\begin{aligned} &\mathbf{H} = -\mathbf{L}, \quad \mathbf{Q} = -\mathbf{U}, \quad \mathbf{D}_J = \mathbf{K} \\ &\mathbf{M}_J = \mathbf{K}, \quad \mathbf{N}_J = \mathbf{L} + \mathbf{U} \\ &\mathcal{B}_1 = \mathbf{M}_J^{-1}\mathbf{N}_J = \mathbf{K}^{-1}(\mathbf{L} + \mathbf{U}). \end{aligned} \tag{4.11}$$

2. The Gauss-Seidel method

backward order: $\mathbf{H} = -\mathbf{L}, \quad \mathbf{Q} = \mathbf{0}, \quad \mathbf{D}_G = \mathbf{K}$

$$\begin{aligned} &\mathbf{M}_G = \mathbf{K}(\mathbf{I} - \mathbf{K}^{-1}\mathbf{U}), \quad \mathbf{N}_G = \mathbf{L} \\ &\mathcal{L}_1 = \mathbf{M}_G^{-1}\mathbf{N}_G = (\mathbf{I} - \mathbf{K}^{-1}\mathbf{U})^{-1}\mathbf{K}^{-1}\mathbf{L}. \end{aligned} \tag{4.12a}$$

forward order: $\mathbf{H} = \mathbf{0}, \quad \mathbf{Q} = -\mathbf{U}, \quad \mathbf{D}_G = \mathbf{K}$

$$\begin{aligned} &\widetilde{\mathbf{M}}_G = \mathbf{K}(\mathbf{I} - \mathbf{K}^{-1}\mathbf{L}), \quad \widetilde{\mathbf{N}}_G = \mathbf{U} \\ &\widetilde{\mathcal{L}}_1 = \widetilde{\mathbf{M}}_G^{-1}\widetilde{\mathbf{N}}_G = (\mathbf{I} - \mathbf{K}^{-1}\mathbf{L})^{-1}\mathbf{K}^{-1}\mathbf{U}. \end{aligned} \tag{4.12b}$$

3. The EWA method

$$\begin{aligned} &\mathbf{H} = \mathbf{Q} = \mathbf{0}, \quad \mathbf{D}_E = \mathbf{K} - \operatorname{diag}\{\mathbf{L}\mathbf{D}_E^{-1}\mathbf{U}\} \\ &\mathbf{M}_E = (\mathbf{I} - \mathbf{L}\mathbf{D}_E^{-1})\mathbf{D}_E(\mathbf{I} - \mathbf{D}_E^{-1}\mathbf{U}), \quad \mathbf{N}_E = \text{off-diag}\{\mathbf{L}\mathbf{D}_E^{-1}\mathbf{U}\} \\ &\mathcal{E}_1 = \mathbf{M}_E^{-1}\mathbf{N}_E = (\mathbf{I} - \mathbf{D}_E^{-1}\mathbf{U})^{-1}\mathbf{D}_E^{-1}(\mathbf{I} - \mathbf{L}\mathbf{D}_E^{-1})\mathbf{N}_E. \end{aligned} \tag{4.13}$$

4. The AGA method

$$\begin{aligned} &\mathbf{H} = \mathbf{H}_A, \quad \mathbf{Q} = \mathbf{Q}_A, \quad \mathbf{D}_A = \mathbf{K} - \operatorname{diag}\{(\mathbf{L} + \mathbf{H}_A)\mathbf{D}_A^{-1}(\mathbf{U} + \mathbf{Q}_A)\} \\ &\mathbf{M}_A = [\mathbf{I} - (\mathbf{L} + \mathbf{H}_A)\mathbf{D}_A^{-1}]\mathbf{D}_A[\mathbf{I} - \mathbf{D}_A^{-1}(\mathbf{U} + \mathbf{Q}_A)] \\ &\mathbf{N}_A = \text{off-diag}\{(\mathbf{L} + \mathbf{H}_A)\mathbf{D}_A^{-1}(\mathbf{U} + \mathbf{Q}_A)\} - \mathbf{H}_A - \mathbf{Q}_A \\ &\mathcal{A}_1 = \mathbf{M}_A^{-1}\mathbf{N}_A = [\mathbf{I} - \mathbf{D}_A^{-1}(\mathbf{U} + \mathbf{Q}_A)]^{-1}\mathbf{D}_A^{-1}[\mathbf{I} - \mathbf{D}_A^{-1}(\mathbf{L} + \mathbf{H}_A)]^{-1}\mathbf{N}_A. \end{aligned} \tag{4.14}$$

The AGA method represents a broad class of algorithms and each of them is defined by the choice of the matrices $\mathbf{H}_A$ and $\mathbf{Q}_A$ such that $\mathbf{H}_A+\mathbf{Q}_A \neq \mathbf{0}$ (excluding the cases $\mathbf{H}_A = -\mathbf{L}$ and $\mathbf{Q}_A = -\mathbf{U}$) and with the assumption that the nonzero entries of $\mathbf{N}_A$ do not coincide in location with the nonzero entries of the matrix $\mathbf{L}+\mathbf{H}_A+\mathbf{U}+\mathbf{Q}_A$. The assumption that $\mathbf{H}_A$ and $\mathbf{Q}_A$ are strictly lower triangular and strictly upper triangular matrices respectively, allows us always to determine explicitly the values of nonzero entries in $\mathbf{H}_A$ and $\mathbf{Q}_A$ directly from the implicit form of $(\mathbf{L}+\mathbf{H}_A)\mathbf{D}_A^{-1}(\mathbf{U}+\mathbf{Q}_A)$, for an arbitrary assumed pattern of the location of nonzero entries in these matrices. In other words, with a postulated nonzero entry pattern in both $\mathbf{H}_A$ and $\mathbf{Q}_A$, all nonzero entries of $\mathbf{H}_A$, $\mathbf{Q}_A$ and consequently $\mathbf{D}_A$, and $\mathbf{N}_A$ can be computed immediately by equating them to the corresponding entries of the implicit matrix product $(\mathbf{L}+\mathbf{H}_A)\mathbf{D}_A^{-1}(\mathbf{U}+\mathbf{Q}_A)$.

Indeed, let S_H be the set of indices (i,j) for $\mathbf{H}_A$ such that $h_{i,j} \neq 0$ and S_Q, S_L, and S_U similar sets for matrices $\mathbf{Q}$, $\mathbf{L}$ and $\mathbf{U}$ respectively. Since $\mathbf{H}_A$ and $\mathbf{Q}_A$ are assumed to be strictly lower and strictly upper matrices respectively, then for a given matrix $\mathbf{A}=\mathbf{K}-\mathbf{L}-\mathbf{U}$ and defined sets S_H and S_Q, the entries of $\mathbf{D}_A$, $\mathbf{H}_A$, $\mathbf{Q}_A$ and $\mathbf{N}_A$ can be calculated in the simple algorithm by means of recursive formulae, simultaneously for each pair of increasing indices (i,j).

$$\left.\begin{aligned} & d_{1,1} = k_{1,1}; \\ & d_{i,i} = k_{i,i} - \sum_{s=1}^{i-1} \frac{(l_{i,s}+h_{i,s})(u_{s,i}+q_{s,i})}{d_{s,s}}, \quad i=2,\dots,n \end{aligned}\right\} \tag{4.15a}$$

$$\left.\begin{aligned} & h_{i,1} = 0, \quad i=1,\dots,n; \\ & h_{i,j} = \sum_{s=1}^{m-1} \frac{(l_{i,s}+h_{i,s})(u_{s,j}+q_{s,j})}{d_{s,s}}, \quad i>j>1, \quad m=\min(i,j) \\ & \qquad\qquad (i,j)\in S_H; \end{aligned}\right\} \tag{4.15b}$$

$$\left.\begin{aligned} & q_{1,j} = 0, \quad j=1,\dots,n; \\ & q_{i,j} = \sum_{s=1}^{m-1} \frac{(l_{i,s}+h_{i,s})(u_{s,j}+q_{s,j})}{d_{s,s}}, \quad 1<i<j, \quad m=\min(i,j) \\ & \qquad\qquad (i,j)\in S_Q; \end{aligned}\right\} \tag{4.15c}$$

$$\left.\begin{aligned} & n_{i,i} = 0, \quad n_{i,1}=n_{1,j}=0, \quad i,j=1,\dots,n; \\ & n_{i,j} = \sum_{s=1}^{m-1} \frac{(l_{i,s}+h_{i,s})(u_{s,j}+q_{s,j})}{d_{s,s}}, \quad i>1, \quad j>1, \quad i\neq j, \\ & \qquad\qquad m=\min(i,j) \\ & \qquad\qquad (i,j)\notin S_{L+H}, \quad (i,j)\notin S_{U+Q}. \end{aligned}\right\} \tag{4.15d}$$

Thus, the set S_N of indices (i,j) of the entries, $n_{i,j} \neq 0$ in the matrix $\mathbf{N}_A$ is complementatory to the set $S_H \cup S_Q \cup S_L \cup S_U$ for the nonzero entries, except $i=1$, $j=1$ and $i=j$, where nonzero entries of $\mathbf{H}_A + \mathbf{Q}_A$ may coincide in positions with those of $\mathbf{L}+\mathbf{U}$. It is worth noticing that, as can be concluded from the above formulae, the entries of $\mathbf{D}_A$, $\mathbf{H}_A$, $\mathbf{Q}_A$ and $\mathbf{N}_A$ for a given pair of indices (i,j) are computed only by means of the entries determined previously in the computation process. When S_N is the empty set, the above formulae define the direct AGA algorithm equivalent to the Gaussian elimination.

Now we will analyze the case when the matrix $\mathbf{A}$ defined in Eq. (4.4) is irreducibly diagonally dominant [1] where $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{U}$ are nonnegative matrices. As is well known such matrices, representing a broad class of physical and engineering problems, are monotone matrices, $\mathbf{A}^{-1} \geq \mathbf{0}$. For such matrices $\mathbf{A}$, the matrices $\mathbf{D}_E$ and $\mathbf{N}_E$, and $\mathbf{D}_A$, $\mathbf{H}_A$, $\mathbf{Q}_A$ and $\mathbf{N}_A$ are nonnegative and the irreducibly diagonal dominance of $\mathbf{A}$ implies that $\mathbf{D}_E$ and $\mathbf{D}_A$ are nonsingular matrices [3].

The comparison of spectral radii of iteration matrices arising in the methods defined above can be made by means of the following theorems.

Theorem 4.1 (Theorem 15 in [2]). *Let the Jacobi matrix $\mathcal{B}_1 = \mathbf{K}^{-1}(\mathbf{L}+\mathbf{U})$ be an $n \times n$ nonnegative matrix with $n > 2$ and zero diagonal entries such that $\rho(\mathcal{B}_1) < 1$. Further, let $\mathcal{L}_1 = \mathbf{M}_G^{-1}\mathbf{N}_G$ be the Gauss-Seidel matrix defined by (4.12), $\mathcal{E}_1 = \mathbf{M}_E^{-1}\mathbf{N}_E$ be the EWA matrix defined by (4.13) and $\mathcal{A}_1 = \mathbf{N}_A^{-1}\mathbf{N}_A$ be the AGA matrix defined by (4.14). Then all above matrices are convergent and*

$$\mathbf{0} \leq \rho(\mathcal{A}_1) \leq \rho(\mathcal{E}_1) \leq \rho(\mathcal{L}_1) \leq \rho(\mathcal{B}_1) < \mathbf{1}. \tag{4.16}$$

Proof. It is evident that when $\mathbf{K}$, $\mathbf{L}$, $\mathbf{U}$, $\mathbf{D}_E^{-1}$, $\mathbf{D}_A^{-1}$ and $\mathbf{H}_A$, $\mathbf{Q}_A$ are nonnegative matrices, all iterative methods defined above are based on the nonnegative splittings of $\mathbf{A}$ (exactly on the regular splittings of $\mathbf{A}$). Since $\rho(\mathcal{B}_1) < \mathbf{1}$ implies by Theorem 3.2 and $\mathbf{A}^{-1} \geq \mathbf{0}$ (where $\mathbf{A} = \mathbf{K} - \mathbf{L} - \mathbf{U}$), then by Corollary 3.2 they are convergent splittings of $\mathbf{A}$. Moreover, as can be verified by Theorem 2.1 the following inequalities are fulfilled:

$$\mathbf{M}_A^{-1} \geq \mathbf{M}_E^{-1} \geq \mathbf{M}_G^{-1} \geq \mathbf{M}_J^{-1} \geq \mathbf{0}. \tag{4.17}$$

Hence by Theorem 3.5 the inequality (4.16) is satisfied. ■

As can be noticed, the last four splittings given in Table for the matrix $\mathbf{A}$ defined by (3.47) are regular. The twelfth splitting corresponds to the Jacobi method where $\mathcal{B}_1 = \mathbf{M}_{12}^{-1}\mathbf{N}_{12}$. The eleventh splitting corresponds to the Gauss-Seidel method with the forward order where $\mathcal{L}_1 = \mathbf{M}_{11}^{-1}\mathbf{N}_{11}$ and $\mathbf{M}_{11}^{-1} \geq \mathbf{M}_{12}^{-1} \geq \mathbf{0}$. The tenth splitting corresponds also to the Gauss-Seidel method but with the backward order, where $\widetilde{\mathcal{L}}_1 = \mathbf{M}_{10}^{-1}\mathbf{N}_{10}$ and $\mathbf{M}_{10}^{-1} \geq \mathbf{M}_{12}^{-1} \geq \mathbf{0}$ and from Table, it follows that

$$\rho(\mathbf{M}_{10}^{-1}\mathbf{N}_{10}) < \rho(\mathbf{M}_{11}^{-1}\mathbf{N}_{11}) < \rho(\mathbf{M}_{12}^{-1}\mathbf{N}_{12}).$$

Both variants of the Gauss-Seidel method illustrate using Theorem 3.14 in the case of comparing splittings derived from nonsymmetric matrices $\mathbf{A}$. Since $\mathbf{M}_{10}^{-1} \geq$

$(\mathbf{M}_{11}^{-1})^T \geq \mathbf{0}$ then by Theorem 3.14, it follows that the tenth splitting representing the Gauss-Seidel method with the backward order is more efficient than that with the forward order and corresponding to the eleventh splitting. As can be easily verified by the formulae (4.15) the ninth splitting represents the algorithm of the EWA method where $\mathcal{E}_1 = \mathbf{M}_9^{-1}\mathbf{N}_9$, however with $\mathbf{D}_E \equiv \mathbf{K}$ because the product $\mathbf{L}\mathbf{D}_E^{-1}\mathbf{U}$ has only the form of $\mathbf{N}_9$. Since $\mathbf{M}_9^{-1} \geq \mathbf{M}_{10}^{-1}$ and $\mathbf{M}_9^{-1} \geq \mathbf{M}_{11}^{-1}$, then according to Theorem 3.5 and results given in Table

$$\rho(\mathbf{M}_9^{-1}\mathbf{N}_9) = \rho(\mathbf{M}_{10}^{-1}\mathbf{N}_{10}) \quad \text{and} \quad \rho(\mathbf{M}_9^{-1}\mathbf{N}_9) < \rho(\mathbf{M}_{11}^{-1}\mathbf{N}_{11}).$$

In the case of the above example of the matrix $\mathbf{A}$ defined by (3.47), there is only one algorithm of the AGA method which is simply the direct one. According to the formulae (4.15b) and (4.15c) defining nonzero entries of $\mathbf{H}_A$ and $\mathbf{Q}_A$ respectively, the nonzero entry location available for $\mathbf{H}_A$ is with $i = 3$ and $j = 2$ which coincides in position with only one nonzero entry of $\mathbf{N}_9$ and defines the matrix $\mathbf{N}_E$ in the EWA method; the nonzero entry location available for $\mathbf{Q}_A$ is with $i = 2$ and $j = 3$. Assuming the above pattern of nonzero entries of $\mathbf{H}_A$ and $\mathbf{Q}_A$, from computations by means of the formulae (4.15), one obtains that $h_{3,2} = 1$ and $d_{3,3} = 1$ (in the case of EWA $d_{3,3} = k_{3,3} = 2$), and $\mathbf{Q}_A$ and $\mathbf{N}_A$ are the null matrices. The product of the factors in (4.5) gives us that $\mathbf{M}_A = \mathbf{A}$.

A stronger result which generalizes the Stein-Rosenberg theorem (Theorem 3.3 in [1]) for the two-sweep iterative methods in an irreducible case is given in the following theorem.

THEOREM 4.2 (Theorem 14 in [2]). *Let the Jacobi matrix* $\mathcal{B}_1 = \mathbf{K}^{-1}(\mathbf{L} + \mathbf{U})$ *be an* $n \times n$ *nonnegative irreducible matrix with* $n > 2$ *and zero diagonal entries such that* $\mathbf{L}$ *has at least one positive entry in each column except the last one,* $\mathbf{U}$ *has at least one positive entry in each row except the last one and* $\mathbf{LU}$ *has some positive off-diagonal entries. Further let* $\mathcal{L}_1 = \mathbf{M}_G^{-1}\mathbf{N}_G$ *be the Gauss-Seidel matrix,* $\mathcal{E}_1 = \mathbf{M}_E^{-1}\mathbf{N}_E$ *be the EWA matrix and* $\mathcal{A}_1 = \mathbf{N}_A^{-1}\mathbf{N}_A$ *be the AGA matrix defined by* (4.12), (4.13) *and* (4.14), *respectively. If* $\rho(\mathcal{B}_1) < 1$, *then*

$$\mathbf{0} \leq \rho(\mathcal{A}_1) < \rho(\mathcal{E}_1) < \rho(\mathcal{L}_1) < \rho(\mathcal{B}_1) < \mathbf{1}. \tag{4.18}$$

Proof. The irreducibility of $\mathcal{B}_1$ with $\rho(\mathcal{B}_1) < 1$ implies that $\mathbf{A}^{-1} > \mathbf{0}$ [1]. Since all above splittings of $\mathbf{A}$ are nonnegative (exactly regular), they are convergent splittings. As is well known, in this case $\rho(\mathcal{L}_1) < \rho(\mathcal{B}_1)$ [1]. The assumption imposed on the matrices $\mathbf{L}$, $\mathbf{U}$ and $\mathbf{LU}$ ensures that at least $\mathcal{E}_1$ is a nonnegative matrix ($\mathcal{A}_1$ may be also the null matrix) and this by Theorem 2.1 implies that with $\mathbf{A}^{-1} > \mathbf{0}$ [2, 3]

$$\mathbf{M}_A^{-1} > \mathbf{M}_E^{-1} > \mathbf{M}_G^{-1} \geq \mathbf{0} \tag{4.19}$$

which gives us by Theorem 3.6 that $\mathbf{0} \leq \rho(\mathcal{A}_1) < \rho(\mathcal{E}_1) < \rho(\mathcal{L}_1)$ completing the proof. ∎

Before giving an example illustrating the above theorem, it seems to be useful to give some comments on the construction of AGA and EWA algorithms in actual practice.

Increasing the number of nonzero entries of $\mathbf{H}_A$ and $\mathbf{Q}_A$ affects the matrix $\mathbf{M}_A$, which suffers some fill-in and becomes closer to $\mathbf{A}$. This effect of fill-in is accompanied by decreasing the spectral radius of iteration matrix, but on the other hand increases storage requirements. In the majority of linear equation systems arising from the discretization of ordinary and partial differential equations, the matrices $\mathbf{A}$ have a sparse structure. The matrices $\mathbf{A}$ obtained with the standard finite difference (or element) discretization of multi-dimensional elliptic partial differential equations have a regular sparsity pattern. In the case of two-dimensional elliptic problems and with natural ordering, the matrices $\mathbf{A}$ obtained by means of the standard five-point differencing have the form of 5 nonzero diagonals and are symmetrically located with respect to the main diagonal. Assuming zero as the index for the main diagonal; $-n+1, -n+2, \ldots, -2, -1$ for the indices of the successive diagonals in the strictly lower part of $\mathbf{A}$ and $1, 2, \ldots, n-2, n-1$ for the indices of the successive diagonals in the strictly upper part of $\mathbf{A}$, where n is the order of $\mathbf{A}$, then $\mathbf{A}$ can be described by the following nonzero diagonal indices: $-s, -1, 0, 1, s$ where $1 < s < n$. Such matrices are also called *band matrices* with the *band width* equal to $2s+1$. For smaller values of s the direct AGA algorithm becomes efficient from the viewpoint of arithmetic effort. In this case the matrices $\mathbf{H}_A$ and $\mathbf{Q}_A$ fill-in the whole band region except for the positions on the main diagonal and both the upper most and the lower most nonzero diagonals, so that the band width of the matrix $(\mathbf{L}+\mathbf{H}_A)\mathbf{D}_A^{-1}(\mathbf{U}+\mathbf{Q}_A)$ is equal to $2s-1$.

When $s \gg 1$, the band has a sparse structure which provides a motivation for using an iterative solution. In such problems the choice of sparsity patterns for $\mathbf{H}_A$ and $\mathbf{Q}_A$ containing a few nonzero diagonals and closely related to the sparsity pattern of $\mathbf{A}$, allows us very often to reduce significantly the spectral radius of iteration matrix. The implementation of iterative algorithms of the AGA method is especially convenient in the mesh structure of discrete problems as is demonstrated in the works [12, 13]. The subsequent algorithms of the AGA method are created by involving the successive neighbouring mesh points to the recurrence formulas of both forward and backward sweep equations (4.10), where the coefficients of unknowns at these mesh points are interpreted as the entries of $\mathbf{H}_A$, $\mathbf{Q}_A$ and $\mathbf{N}_A$. The special graphic representation of the AGA method, useful with the construction of particular AGA iterative algorithms in different mesh structures, is given in [17]. There is a significant efficiency of AGA algorithms with solving linear equation systems with nonsymmetric matrices $\mathbf{A}$ which appear in discrete convection-diffusion problems.

For illustrating the result of Theorem 4.2, let us consider the example of the following 2-cyclic, consistently ordered matrix $\mathbf{A}$ representing the class of two-dimensional elliptic discrete problem [1].

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \frac{1}{15}\begin{bmatrix} 7 & 3 & 3 & 2 \\ 3 & 7 & 2 & 3 \\ 3 & 2 & 7 & 3 \\ 2 & 3 & 3 & 7 \end{bmatrix} \tag{4.20}$$

<u>The Jacobi method</u>

$$\mathbf{M}_J = \mathbf{K} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{N}_J = \mathbf{L} + \mathbf{U} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_J^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{B}_1 = \frac{1}{3}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \rho(\mathcal{B}_1) = \frac{2}{3}$$

<u>The Gauss-Seidel method</u>

$$\mathbf{M}_G = \mathbf{K}(\mathbf{I} - \mathbf{K}^{-1}\mathbf{U}) = \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{N}_G = \mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_G^{-1} = \frac{1}{27}\begin{bmatrix} 9 & 3 & 3 & 2 \\ 0 & 9 & 0 & 3 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 0 & 9 \end{bmatrix}, \quad \mathcal{L}_1 = \frac{1}{27}\begin{bmatrix} 6 & 2 & 2 & 0 \\ 9 & 3 & 3 & 0 \\ 9 & 3 & 3 & 0 \\ 0 & 9 & 9 & 0 \end{bmatrix} \quad \text{and} \quad \rho(\mathcal{L}_1) = \frac{4}{9}$$

<u>The EWA method</u>

$$\mathbf{D}_E = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & \frac{8}{3} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 0 \\ 0 & 0 & 0 & \frac{9}{4} \end{bmatrix}, \quad \mathbf{M}_E = [\mathbf{I} - \mathbf{L}\mathbf{D}_E^{-1}]\mathbf{D}_E[\mathbf{I} - \mathbf{D}_E^{-1}] = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & \frac{1}{3} & -1 \\ -1 & \frac{1}{3} & 3 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix},$$

$$\mathbf{N}_E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{M}_E^{-1} = \begin{bmatrix} \frac{4}{9} & \frac{1}{6} & \frac{1}{6} & \frac{1}{9} \\ \frac{1}{6} & \frac{7}{16} & \frac{1}{16} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{16} & \frac{7}{16} & \frac{1}{6} \\ \frac{1}{9} & \frac{1}{6} & \frac{1}{6} & \frac{4}{9} \end{bmatrix}, \quad \mathcal{E}_1 = \begin{bmatrix} 0 & \frac{1}{18} & \frac{1}{18} & 0 \\ 0 & \frac{1}{48} & \frac{7}{48} & 0 \\ 0 & \frac{7}{48} & \frac{1}{48} & 0 \\ 0 & \frac{1}{18} & \frac{1}{18} & 0 \end{bmatrix}$$

and $\rho(\mathcal{E}_1) = \frac{1}{6}$.

The AGA method

$$\mathbf{D}_A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & \frac{8}{3} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 0 \\ 0 & 0 & 0 & \frac{141}{64} \end{bmatrix}, \quad \mathbf{H}_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 \end{bmatrix}, \quad \mathbf{Q}_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{M}_A = [\mathbf{I} - (\mathbf{L} + \mathbf{H}_A)\mathbf{D}_A^{-1}]\mathbf{D}_A[\mathbf{I} - \mathbf{D}_A^{-1}(\mathbf{U} + \mathbf{Q}_A)] = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & \frac{1}{3} & 3 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix},$$

$$\mathbf{N}_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{M}_A^{-1} = \frac{1}{141} \begin{bmatrix} 64 & 24 & 27 & 17 \\ 27 & 63 & 18 & 27 \\ 24 & 9 & 63 & 24 \\ 17 & 24 & 27 & 64 \end{bmatrix},$$

$$\mathcal{A}_1 = \frac{1}{47} \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \rho(\mathcal{A}_1) = \frac{2}{47}.$$

The comparison of spectral radii obtained for particular methods is an illustration of the result (4.18).

Finally it should be mentioned that the nonnegative splitting theory is useful with the analysis of the successive overrelaxation processes applied to accelerating the convergence in the AGA(EWA) two-sweep iterative methods. Appearing iteration matrices $\mathcal{F}_\omega$ (dependent on the relaxation factor ω) represent a weak nonnegative splitting of $\mathbf{A}$ for the underrelaxation ($0 < \omega < 1$) [2, 3, 5]. In [18] is given a numerical analysis of the conjugate gradient method with different preconditionings (AGA prefactorizationings equivalent to the incomplete Cholesky decomposition) and in [19] is described an efficient subroutine for a priori estimate of the best relaxation factor ω_B in SOR methods.

5. Referring to Regular Splitting Results

The basic purpose of this section is to discuss some aspects related with earlier author's regular splitting results from 1973 and 1978 [2, 3] which find an interest in the current literature, as well as to show their interrelation with the developments presented in this paper and those given in works [7, 15]. The secondary objective of this section is to derive new comparison theorems generalizing some nonnegative splitting theory results.

Although these earlier results were not published by the author (author's attempts for publishing them in the known European mathematical journal did not succeed), none the less they are known in the literature as "little known results of Woźnicki" [7, 11].

These results as comparison theorems were obtained with the development of the AGA two-sweep iterative methods defined in Section 4 and implemented in producing programmes [12, 13] solving two- and three-dimensional neutron diffusion equations in nuclear engineering problems. The excellent convergence properties of AGA iterative algorithms observed in numerical experiments [14] encouraged us to seek its theoretical justification. The convenient matrix notation introduced with the description of the AGA methods allowed us only to verify the condition $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$ with comparing spectral radii. The regular splitting theory originated by Varga [1], restricted at this time to comparison theorems proven with the condition $\mathbf{N}_2 \geq \mathbf{N}_1 \geq \mathbf{0}$ (verifiable only with the comparison of the Jacobi and Gauss-Seidel methods), was useless in the case of the convergence analysis for the AGA iterative methods. Further investigations allowed us to extend the regular splitting theory by including new comparison theorems proven with the hypotheses $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1} \geq \mathbf{0}$ and $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1} \geq \mathbf{0}$; and its original proofs are given in the works [2 and 3], which are unfortunately not easily available.

From the time of the statement of these new comparison theorems, a renewed interest with the regular splitting theory is observed in the literature. It seems that just these theorems as well as their proofs were inspiratory for the work [7] in which the authors included both comparison theorems (collected in Theorem C) as the subsequent unpublished results of Woźnicki without, however, giving its original proofs. In other papers [8, 9, 10, 11] both theorems are quoted as the results of Woźnicki with a close connection to Csordas and Varga's results [7]. For instance in the wrok [10] the author's results are used but without indicating the author's reference. The inaccessibility of the proofs of these theorems as well as other results of the work [2] in the literature existing up to now may make difficulties in distinguishing the authorship. For example the authors of the work [11], extending the class of comparison theorems, report [11, p.388] that the items i) and ii) in Proposition 1.3 have been proven by Csordas and Varga. Whereas Csordas and Varga mentioned [7, p.25] that both assertions i) and ii) can be found in the author's work [2], and only for completeness did they include the proofs of these items.

Therefore by due respect to the reader, both earlier author's comparison theorems and its original proofs as they are given in the work [2] are presented below. In this way the methodology used in the proofs of these theorems can be compared with a methodology used elsewhere and moreover, it may be possible to verify how the nonnegative splitting theory results presented in this paper are related with the former author's theorems generalizing the results obtained by Varga [1] for regular splittings.

THEOREM 5.1 (Theorem 12 in [2]). *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two regular splittings of* $\mathbf{A}$ *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If* $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{5.1}$$

Proof. As can be concluded from Theorem 3.2, $\rho(\mathbf{M}^{-1}\mathbf{N}) < \mathbf{1}$ and is monotone with respect to $\rho(\mathbf{A}^{-1}\mathbf{N})$, and therefore it suffices to prove that $\rho(\mathbf{A}^{-1}\mathbf{N}_2) \geq \rho(\mathbf{A}^{-1}\mathbf{N}_1)$. The assumption $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$ can be written, as follows

$$(\mathbf{A}+\mathbf{N}_1)^{-1} \geq (\mathbf{A}+\mathbf{N}_2)^{-1}$$

or

$$(\mathbf{I}+\mathbf{A}^{-1}\mathbf{N}_1)^{-1}\mathbf{A}^{-1} \geq \mathbf{A}^{-1}(\mathbf{I}+\mathbf{N}_2\mathbf{A}^{-1})^{-1}.$$

Since the nonsingular matrices $\mathbf{I}+\mathbf{A}^{-1}\mathbf{N}_1$ and $\mathbf{I}+\mathbf{N}_2\mathbf{A}^{-1}$ are nonnegative, then

$$\mathbf{A}^{-1}(\mathbf{I}+\mathbf{N}_2\mathbf{A}^{-1}) \geq (\mathbf{I}+\mathbf{A}^{-1}\mathbf{N}_1)\mathbf{A}^{-1}$$

which is equivalent to

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}. \tag{5.2}$$

Since the nonnegative matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ may be singular, then with the assumed definitions of nonnegative matrices (Section 2), one obtains

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_2 = (\mathbf{A}^{-1}\mathbf{N}_2)^2 \geq (=)\ \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2 \tag{5.3}$$

and

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1 \geq (=)\ \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_1 = (\mathbf{A}^{-1}\mathbf{N}_1)^2. \tag{5.4}$$

As all matrices in the above expressions are nonnegative, then from Lemma 2.2, one obtains that

$$\rho((\mathbf{A}^{-1}\mathbf{N}_2)^2) = \rho^2(\mathbf{A}^{-1}\mathbf{N}_2) \geq \rho(\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2) \tag{5.5}$$

and

$$\rho(\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1) \geq \rho((\mathbf{A}^{-1}\mathbf{N}_1)^2) = \rho^2(\mathbf{A}^{-1}\mathbf{N}_1). \tag{5.6}$$

From Lemma 2.1 it follows that

$$\rho(\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2) = \rho(\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1) \tag{5.7}$$

so one can conclude that

$$\rho(\mathbf{A}^{-1}\mathbf{N}_2) \geq \rho(\mathbf{A}^{-1}\mathbf{N}_1) \tag{5.8}$$

which by Theorem 3.2 implies

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1$$

completing the proof of the theorem. ■

THEOREM 5.2 (Theorem 13 in [2]). *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two regular splittings of* $\mathbf{A}$ *where* $\mathbf{A}^{-1} > \mathbf{0}$. *If* $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1}$, *then*

$$\mathbf{0} < \rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{5.9}$$

Proof. $\mathbf{A}^{-1} > \mathbf{0}$ implies that when $\mathbf{N} \geq \mathbf{0}$, the matrix $\mathbf{A}^{-1}\mathbf{N}$ has at least one positive column, so that $\rho(\mathbf{A}^{-1}\mathbf{N}) > \mathbf{0}$. Hence by Theorems 3.2 and 5.1, one can conclude that

$$\mathbf{0} < \rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}.$$

Now it will be shown that $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} = \mathbf{A}^{-1}(\mathbf{N}_2 - \mathbf{N}_1)\mathbf{A}^{-1}$ is a positive matrix when $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1}$. The matrix $\mathbf{A}^{-1}(\mathbf{N}_2 - \mathbf{N}_1)\mathbf{A}^{-1}$ can be expressed, as follows

$$\begin{aligned}\mathbf{A}^{-1}(\mathbf{N}_2 - \mathbf{N}_1)\mathbf{A}^{-1} &= \mathbf{A}^{-1}(\mathbf{M}_2 - \mathbf{M}_1)\mathbf{A}^{-1} \\ &= \mathbf{A}^{-1}\mathbf{M}_1(\mathbf{M}_1^{-1} - \mathbf{M}_2^{-1})\mathbf{M}_2\mathbf{A}^{-1} \\ &= \mathbf{A}^{-1}(\mathbf{A} + \mathbf{N}_1)(\mathbf{M}_1^{-1} - \mathbf{M}_2^{-1})(\mathbf{A} + \mathbf{N}_2)\mathbf{A}^{-1} \\ &= (\mathbf{I} + \mathbf{A}^{-1}\mathbf{N}_1)(\mathbf{M}_1^{-1} - \mathbf{M}_2^{-1})(\mathbf{I} + \mathbf{N}_2\mathbf{A}^{-1})\end{aligned}$$

or

$$\begin{aligned}\mathbf{A}^{-1}(\mathbf{N}_2 - \mathbf{N}_1)\mathbf{A}^{-1} = (\mathbf{M}_1^{-1} - \mathbf{M}_2^{-1}) &+ \mathbf{A}^{-1}\mathbf{N}_1(\mathbf{M}_1^{-1} - \mathbf{M}_2^{-1}) \\ &+ (\mathbf{M}_1^{-1} - \mathbf{M}_2^{-1})\mathbf{N}_2\mathbf{A}^{-1} \\ &+ \mathbf{A}^{-1}\mathbf{N}_1(\mathbf{M}_1^{-1} - \mathbf{M}_2^{-1})\mathbf{N}_2\mathbf{A}^{-1}.\end{aligned} \tag{5.10}$$

Since by the hypothesis $\mathbf{M}_1^{-1} - \mathbf{M}_2^{-1} > \mathbf{0}$ one obtains $\mathbf{A}^{-1}(\mathbf{N}_2 - \mathbf{N}_1)\mathbf{A}^{-1} > \mathbf{0}$ or

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} > \mathbf{0}. \tag{5.11}$$

But the above inequality implies that in the inequality

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{0} \tag{5.12}$$

all positive entries of $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_2$ are greater than the corresponding entries of $\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2$ and in the inequality

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_1 \tag{5.13}$$

all positive entries of $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1$ are greater than the corresponding entries of $\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_1$, which leads to the conclusion that

$$\rho((\mathbf{A}^{-1}\mathbf{N}_2)^2) = \rho^2(\mathbf{A}^{-1}\mathbf{N}_2) > \rho(\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2) \tag{5.14}$$

and

$$\rho(\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1) > \rho((\mathbf{A}^{-1}\mathbf{N}_1)^2) = \rho^2(\mathbf{A}^{-1}\mathbf{N}_1). \tag{5.15}$$

Since $\rho(\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2) = \rho(\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1)$ by Lemma 2.1, hence one obtains

$$\rho(\mathbf{A}^{-1}\mathbf{N}_2) > \rho(\mathbf{A}^{-1}\mathbf{N}_1) \tag{5.16}$$

implying by Theorem 3.2 that

$$\mathbf{0} < \rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}$$

which completes the proof of the theorem. ■

As was already stated in Section 3 (Lemma 3.1) in the case of weak nonnegative splittings of a monotone matrix $\mathbf{A}$

$$\mathbf{N}_2 \geq \mathbf{N}_1$$

equivalent to

$$\mathbf{M}_2 - \mathbf{A} \geq \mathbf{M}_1 - \mathbf{A}$$

implies that

$$\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1} \geq \mathbf{0}$$

but this inequality gives us

$$(\mathbf{I} + \mathbf{A}^{-1}\mathbf{N}_1)^{-1}\mathbf{A}^{-1} \geq \mathbf{A}^{-1}(\mathbf{I} + \mathbf{N}_2\mathbf{A}^{-1})^{-1}$$

or

$$\mathbf{A}^{-1}(\mathbf{I} + \mathbf{N}_1\mathbf{A}^{-1})^{-1} \geq (\mathbf{I} + \mathbf{A}^{-1}\mathbf{N}_2)^{-1}\mathbf{A}^{-1}.$$

When both splittings are nonnegative or weak nonnegative but of different type, either $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ or $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ which by Theorem 3.2 implies either $\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{0}$ or $\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$ and $\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{0}$ respectively, one obtains

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$$

and

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$$

when $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1} \geq \mathbf{0}$, as shown in the proof of Theorem 5.2.

The above results are summarized in the following lemma.

LEMMA 5.1. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two splittings of* $\mathbf{A}$ *where* $\mathbf{A}^{-1} \geq \mathbf{0}$.

a) *If both splittings are weak nonnegative, then the inequality*

$$\mathbf{N}_2 \geq \mathbf{N}_1 \tag{5.17}$$

implies that

$$\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1} \geq \mathbf{0} \tag{5.18}$$

b) *if both splittings are nonnegative or weak nonnegative but of a different type (either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$*), then the inequality* (5.18) *implies that*

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{5.19}$$

c) *if* $\mathbf{M}_1 - \mathbf{N}_1$ *is a regular splitting, then*

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1 \geq (=)\ \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0} \tag{5.20a}$$

and

$$\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq (=)\ \mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{5.20b}$$

if $\mathbf{M}_2 - \mathbf{N}_2$ is a regular splitting

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_2 \geq (=)\ \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{0} \tag{5.21a}$$

and

$$\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq (=)\ \mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}. \tag{5.21b}$$

Moreover, when

$$\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1} \geq \mathbf{0}, \tag{5.18a}$$

d) if both splittings are nonnegative or weak nonnegative but of a different type (either $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ or $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$), then the inequality (5.18a) implies that

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{5.19a}$$

e) if $\mathbf{M}_1 - \mathbf{N}_1$ is a regular splitting, then

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0} \tag{5.20c}$$

and

$$\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{5.20d}$$

if $\mathbf{M}_2 - \mathbf{N}_2$ is a regular splitting

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{0} \tag{5.21c}$$

and

$$\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}. \tag{5.21d}$$

The conditions (5.17), (5.18), (5.19), (5.20) and (5.21) are progressively weaker but the converse may not be true as can be easily verified in the examples of splittings given in the Table. Each of these conditions can be assumed as a hypothesis in comparison theorems which may be proven by an analogy to the proofs of Theorems 5.1 and 5.2 or those given in Section 3.

In using the conditions (5.19) or (5.19a) as weaker hypotheses in the case of regular splittings of a monotone matrix $\mathbf{A}$, the proof of a comparison theorem follows immediately from the Theorem 5.1 or Theorem 5.2. It is natural to ask if for (weak) nonnegative splittings $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ of a monotone matrix $\mathbf{A}$, $\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}$ is satisfied, when

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}.$$

The answer to the above question is provided by the two following theorems generalizing the results of Theorem 5.1 and Theorem 5.2.

THEOREM 5.3. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two nonnegative splittings of* $\mathbf{A}$, *or two weak nonnegative splittings of* $\mathbf{A}$ *but of different type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If* $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{5.22}$$

Proof. Assuming the case when $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, one can write that

$$\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_1 = \tau_1\mathbf{y}_1^T \tag{5.23}$$

and

$$\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y}_2 = \tau_2\mathbf{y}_2 \tag{5.24}$$

where, by Theorems 3.2, 2.2, and Lemma 2.1, $\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, $\tau_1 = \rho(\mathbf{A}^{-1}\mathbf{N}_1)$ and $\tau_2 = \rho(\mathbf{A}^{-1}\mathbf{N}_2) = \rho(\mathbf{N}_2\mathbf{A}^{-1})$, and the corresponding eigenvectors $\mathbf{y}_1$ and $\mathbf{y}_2$ are nonnegative. Multiplying Eq. (5.23) on the right by $\mathbf{A}^{-1}$ and Eq. (5.24) on the left by $\mathbf{A}^{-1}$ gives us

$$\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} = \tau_1\mathbf{y}_1^T\mathbf{A}^{-1} \tag{5.25}$$

and

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y}_2 = \tau_2\mathbf{A}^{-1}\mathbf{y}_2. \tag{5.26}$$

By the hypothesis it follows that

$$\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq (=)\, \mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} = \tau_1\mathbf{y}_1^T\mathbf{A}^{-1}. \tag{5.27}$$

Again multiplying Eq. (5.27) on the right by $\mathbf{y}_2$ and Eq. (5.26) on the left by $\mathbf{y}_1^T$ one obtains

$$\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y}_2 \geq \tau_1\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 \tag{5.28}$$

and

$$\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y}_2 = \tau_2\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 \tag{5.29}$$

and hence, as $\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 > 0$, it follows that $\tau_1 \leq \tau_2$ which, by Theorem 3.2 proves the inequality (5.22) for the case when $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2^{-1}\mathbf{M}_1 \geq \mathbf{0}$. The case when $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ can be proven in a similar way. ∎

THEOREM 5.4. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two nonnegative splittings of* $\mathbf{A}$, *or two weak nonnegative splittings of* $\mathbf{A}$ *but of different type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} > \mathbf{0}$. *If* $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{5.30}$$

Proof. Similarly as in the proof of Theorem 5.3 it is evident that the assumption $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$ excludes equality in (5.27), which implies the strict inequality in (5.28) providing $\tau_1 < \tau_2$ and consequently by Theorem 3.2 the inequality (5.30). ■

To show that the above theorems may not be true in the case when both splittings are weak nonnegative of the same type, it is sufficient to consider the second, third and fourth splittings of the monotone matrix $\mathbf{A}$ given in Table for which $\mathbf{A}^{-1}\mathbf{N}_4\mathbf{A}^{-1} > \mathbf{A}^{-1}\mathbf{N}_3\mathbf{A}^{-1} \geq \mathbf{0}$ and $\mathbf{A}^{-1}\mathbf{N}_4\mathbf{A}^{-1} > \mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{0}$ satisfy the assumptions of both theorems. In the first case $\rho(\mathbf{M}_4^{-1}\mathbf{N}_4) < \rho(\mathbf{M}_3^{-1}\mathbf{N}_3)$ but in the second case $\rho(\mathbf{M}_4^{-1}\mathbf{N}_4) > \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$. This shows that both theorems fail when both splittings are weak nonnegative of the same type.

In the case of the conditions (5.20a) and (5.21a) or (5.20b) and (5.21b) as still weaker hypotheses, the following theorem generalizing the former results holds.

THEOREM 5.5. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two weak nonnegative splittings of* $\mathbf{A}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If either*

$$\begin{gathered}\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_2 \geq \left\{\begin{matrix}\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2\\ \textit{or}\\ \mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1\end{matrix}\right\} \geq \mathbf{0} \quad \textit{and}\\ \left\{\begin{matrix}\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2\\ \textit{or}\\ \mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1\end{matrix}\right\} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0}\end{gathered} \tag{5.31a}$$

or

$$\begin{gathered}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \left\{\begin{matrix}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\\ \textit{or}\\ \mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\end{matrix}\right\} \geq \mathbf{0} \quad \textit{and}\\ \left\{\begin{matrix}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\\ \textit{or}\\ \mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\end{matrix}\right\} \geq \mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}\end{gathered} \tag{5.31b}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{5.32}$$

Proof. It is evident that the proof follows from relations (5.5) to (5.7) given in the proof of Theorem 5.1. ■

This theorem allows us to compare weak nonnegative splittings of the same type. The second and third weak splittings given in Table for the monotone matrix $\mathbf{A}$ defined by (3.47), which can not be related by other comparison theorems, illustrate the application of Theorem 5.5. In this case one obtains:

$$(\mathbf{A}^{-1}\mathbf{N}_2)^2 - \mathbf{A}^{-1}\mathbf{N}_3\mathbf{A}^{-1}\mathbf{N}_2$$
$$= \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{3}{20}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \geq 0$$

and

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_3 - (\mathbf{A}^{-1}\mathbf{N}_3)^2$$
$$= \frac{1}{10}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} - \frac{1}{25}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \frac{3}{50}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \geq 0$$

so that the assumptions (5.31a) are satisfied and $\rho(\mathbf{M}_3^{-1}\mathbf{N}_3) = \frac{1}{5} < \frac{1}{2} = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$.

THEOREM 5.6. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two nonnegative splittings of* $\mathbf{A}$, *or two weak nonnegative splittings of* $\mathbf{A}$ *but of different type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *Let* $\mathbf{x} \geq \mathbf{0}$ *and* $\mathbf{y} \geq \mathbf{0}$ *be vectors such that* $\mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 = \lambda_1\mathbf{x}^T$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{y} = \lambda_2\mathbf{y}$ *when* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$; *and let* $\mathbf{v} \geq \mathbf{0}$ *and* $\mathbf{w} \geq \mathbf{0}$ *be vectors such that* $\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{v} = \lambda_1\mathbf{v}$ *and* $\mathbf{w}^T\mathbf{M}_2^{-1}\mathbf{N}_2 = \lambda_2\mathbf{w}^T$ *when* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, *where* $\lambda_1 = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1)$ *and* $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$. *If either*

$$\begin{gathered} \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq (=)\, \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \\ (\textit{or } \mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y} \geq (=)\, \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{y} \geq \mathbf{0}) \end{gathered} \tag{5.33a}$$

or

$$\begin{gathered} \mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{v} \geq (=)\, \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{v} \geq \mathbf{0} \\ (\textit{or } \mathbf{w}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq (=)\, \mathbf{w}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}) \end{gathered} \tag{5.33b}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{5.34}$$

Moreover, if either

$$\begin{gathered} \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} > \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \\ (\textit{or } \mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{y} \geq \mathbf{0}) \end{gathered} \tag{5.35a}$$

or

$$\begin{gathered} \mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{v} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{v} \geq \mathbf{0} \\ (\textit{or } \mathbf{w}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} > \mathbf{w}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}) \end{gathered} \tag{5.35b}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{5.36}$$

Proof. The matrices $\mathbf{M}^{-1}\mathbf{N}$ and $\mathbf{A}^{-1}\mathbf{N}$ have the same eigenvectors because they commute by Theorem 3.1 and their eigenvalues are related by Eq. (3.16).

Thus assuming the case when $\mathbf{M}_1^{-1}\mathbf{N} \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ for which $\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ by Theorem 3.2, one can write

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1 = \tau_1\mathbf{x}^T \tag{5.37}$$

and

$$\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y} = \tau_2\mathbf{y} \tag{5.38}$$

where $\tau_1 = \rho(\mathbf{A}^{-1}\mathbf{N}_1)$ and $\tau_2 = \rho(\mathbf{A}^{-1}\mathbf{N}_2) = \rho(\mathbf{N}_2\mathbf{A}^{-1})$. Multiplying (5.37) on the right by $\mathbf{A}^{-1}$ and (5.38) on the left by $\mathbf{A}^{-1}$ gives us

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} = \tau_1\mathbf{x}^T\mathbf{A}^{-1} \tag{5.39}$$

and

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y} = \tau_2\mathbf{A}^{-1}\mathbf{y}. \tag{5.40}$$

By the hypothesis it follows that

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq (=)\, \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} = \tau_1\mathbf{x}^T\mathbf{A}^{-1}. \tag{5.41}$$

Again multiplying (5.41) on the right by $\mathbf{y}$ and (5.40) on the left by $\mathbf{x}^T$ one obtains

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y} \geq \tau_1\mathbf{x}^T\mathbf{A}^{-1}\mathbf{y} \tag{5.42}$$

and

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y} = \tau_2\mathbf{x}^T\mathbf{A}^{-1}\mathbf{y} \tag{5.43}$$

and hence, as $\mathbf{x}^T\mathbf{A}^{-1}\mathbf{y} > 0$, it follows that $\tau_1 \leq \tau_2$ which, by Theorem 3.2, proves the inequality (5.34). It is evident that imposing the condition (5.35) implies strict inequality in (5.41) which leads to proving the inequality (5.36) for the case when $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2^{-1}\mathbf{M}_1 \geq \mathbf{0}$. The case when $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ can be proven in a similar way. ■

The importance of the above theorem relies on its generalization of comparison theorems presented in this paper in the case of nonnegative splittings and weak nonnegative splittings but of different types.

The application of this theorem can be illustrated by considering the fourth and fifth splittings given in Table for which none of the conditions of Lemma 5.1 is fulfilled. The left eigenvector of $\mathbf{N}_5\mathbf{M}_5^{-1}$, $\mathbf{x}^T = [1, 0, 0]$ and

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_5\mathbf{A}^{-1} = \left[2, 1, \frac{1}{2}\right] > \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_4\mathbf{A}^{-1} = \left[\frac{8}{23}, \frac{9}{23}, \frac{7}{23}\right] > \mathbf{0}$$

satisfies the assumption (5.35a), and $\rho(\mathbf{M}_4^{-1}\mathbf{N}_4) = \frac{\sqrt{2}}{5} < \frac{1}{2} = \rho(\mathbf{M}_5^{-1}\mathbf{N}_5)$ accordingly to the result (5.36).

REMARK. Theorems 5.3 and 5.6 have been proven with the assumption that $\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 > 0$ and $\mathbf{x}^T\mathbf{A}^{-1}\mathbf{y} > 0$ which, as was already mentioned in Remark given in Section 3, may not be satisfied when $\mathbf{A}^{-1} \geq \mathbf{0}$ is used as a hypothesis. However the use of the simple modification of the proof and described in Remark from Section 3 in the proofs of Theorems 5.3 and 5.6 allows us to easily avoid the case when $\mathbf{x}^T\mathbf{A}^{-1}\mathbf{y}$ or $\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2$ may be equal to zero.

Referring to Lemma 5.1, it is necessary to notice that the progressive weakening of the conditions of this lemma is accompanied by increasing the complexity of the successive conditions, which in consequence may be a burden to their verification. As was already demonstrated in the previous section the conditions (5.17) and (5.18), used in the comparison theorems of Section 3, are easily verifiable in immediate applications and therefore they may be considered in some sense as natural conditions. The next conditions of Lemma 5.1, when used as more compound hypotheses in comparison theorems, extend the class of applications; however, its verification may be more cumbersome in actual practice. In the case of Theorems 3.17 and 5.6 generalizing comparison theorems given in this paper, it is necessary to know additionally at least one eigenvector.

Thus imposing the successive conditions of Lemma 5.1 as hypotheses in comparison theorems leads to the successive generalizations which, however, may have more theoretical than practical significance.

The conditions (5.19) and (5.20), being the essence of the matter in the proofs of Theorems 5.1 and 5.2, became a basis for further developments by Beauwens [15] and Csordas and Varga [7] who consider their results as generalizations of the author's earlier results (Theorems 5.1 and 5.2). However, a close inspection of both works leads to some comments, in the case of work [15] to quite different conclusions. Since both works find an interest in the current literature (the work [7] in [8, 9, 10, 11] and the work [15] in [9] as well as in many of later Beauwens' papers), it seems that it is worth commenting both works.

(1) Beauwens results [15]

In Section 2.2 of [15] there is Theorem 2.3 which according to Beauwens' opinion generalizes Theorem 12 of Woźnicki [2] (Theorem 5.1 given here) and Corollary 1. Both Beauwens' results will be verified below.

THEOREM 2.3 (Beauwens [15, p.342]). *Let* $\mathbf{A} = \mathbf{M}_s - \mathbf{N}_t = \mathbf{M}_s - \mathbf{N}_t$ *be two splittings of* $\mathbf{A}$ *such that* $\mathbf{M}_s$ *and* $\mathbf{M}_t$ *are nonsingular,* $\mathbf{M}_s^{-1}\mathbf{N}_s$ *and* $\mathbf{M}_t^{-1}\mathbf{N}_t$ *are nonnegative and convergent. Then, any one of the four assumptions*

a) $(\mathbf{A}^{-1}\mathbf{N}_t - \mathbf{A}^{-1}\mathbf{N}_s)\mathbf{A}^{-1}\mathbf{N}_s \geq \mathbf{0}$

b) $(\mathbf{A}^{-1}\mathbf{N}_t - \mathbf{A}^{-1}\mathbf{N}_s)\mathbf{A}^{-1}\mathbf{N}_t \geq \mathbf{0}$

c) $\mathbf{A}^{-1}\mathbf{N}_s(\mathbf{A}^{-1}\mathbf{N}_t - \mathbf{A}^{-1}\mathbf{N}_s) \geq \mathbf{0}$

d) $\mathbf{A}^{-1}\mathbf{N}_t(\mathbf{A}^{-1}\mathbf{N}_t - \mathbf{A}^{-1}\mathbf{N}_s) \geq \mathbf{0}$

implies $\rho(\mathbf{M}_s^{-1}\mathbf{N}_s) \leq \rho(\mathbf{M}_t^{-1}\mathbf{N}_t)$.

It is evident that some splittings given in the Table for the matix $\mathbf{A}$ defined by (3.47) satisfy the assumptions on the matrices $\mathbf{M}$ and $\mathbf{M}^{-1}\mathbf{N}$ in the above theorem.

Taking in consideration splittings $s = 9$ and $t = 2$, one obtains

$$(\mathbf{A}^{-1}\mathbf{N}_t - \mathbf{A}^{-1}\mathbf{N}_s)\mathbf{A}^{-1}\mathbf{N}_t = (\mathbf{A}^{-1}\mathbf{N}_2 - \mathbf{A}^{-1}\mathbf{N}_9)\mathbf{A}^{-1}\mathbf{N}_2$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -2 & 0 \\ 0 & -2 & 0 \\ 0 & -2 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \geq 0$$

but from the results given in Table, it follows that

$$\rho(\mathbf{M}_s^{-1}\mathbf{N}_s) = \rho(\mathbf{M}_9^{-1}\mathbf{N}_9) = \frac{1}{2} > \frac{1}{3} = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) = \rho(\mathbf{M}_t^{-1}\mathbf{N}_t)$$

which shows that the above theorem fails.

The correct theorem of this kind which generalizes Theorem 5.1 is only Theorem 5.5.

COROLLARY 1 (Beauwens [15, p.342]). *Let* $\mathbf{A} = \mathbf{M}_s - \mathbf{N}_t = \mathbf{M}_s - \mathbf{N}_t$ *be two splittings of* $\mathbf{A}$ *such that* $\mathbf{M}_s$ *and* $\mathbf{M}_t$ *are nonsingular,* $\mathbf{M}_s^{-1}\mathbf{N}_s$ *and* $\mathbf{M}_t^{-1}\mathbf{N}_t$ *are nonnegative and convergent. Then, any one of the two assumptions*

a) $(\mathbf{M}_s^{-1} - \mathbf{M}_t^{-1})\mathbf{N}_s \geq \mathbf{0}$

b) $(\mathbf{M}_s^{-1} - \mathbf{M}_t^{-1})\mathbf{N}_t \geq \mathbf{0}$

implies $\rho(\mathbf{M}_s^{-1}\mathbf{N}_s) \leq \rho(\mathbf{M}_t^{-1}\mathbf{N}_t)$.

Taking in consideration now splittings $s = 2$ and $t = 4$ from Table, one obtains

$$(\mathbf{M}_s^{-1} - \mathbf{M}_t^{-1})\mathbf{N}_s = (\mathbf{M}_2^{-1} - \mathbf{M}_4^{-1})\mathbf{N}_2$$

$$= \frac{1}{15}\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 4 & 7 & 4 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \geq \mathbf{0}$$

and

$$(\mathbf{M}_s^{-1} - \mathbf{M}_t^{-1})\mathbf{N}_s = (\mathbf{M}_2^{-1} - \mathbf{M}_4^{-1})\mathbf{N}_4$$

$$= \frac{1}{15}\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 4 & 7 & 4 \end{bmatrix} \frac{1}{23}\begin{bmatrix} 0 & 0 & 0 \\ -4 & -4 & 3 \\ 9 & 9 & -1 \end{bmatrix} = \frac{1}{345}\begin{bmatrix} 15 & 15 & 6 \\ 15 & 15 & 6 \\ 8 & 8 & 17 \end{bmatrix} > \mathbf{0}.$$

In this case both assumptions a) and b) of Corollary 1 are satisfied, whereas

$$\rho(\mathbf{M}_s^{-1}\mathbf{N}_s) = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) = \frac{1}{3} > \frac{\sqrt{2}}{5} = \rho(\mathbf{M}_4^{-1}\mathbf{N}_4) = \rho(\mathbf{M}_t^{-1}\mathbf{N}_t)$$

shows that the corollary fails.

(2) Csordas and Varga results [7]

Csordas and Varga consider regular splittings of $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ with $\mathbf{A}^{-1} \geq \mathbf{0}$, for which

$$(\mathbf{A}^{-1}\mathbf{N}_2)^j\mathbf{A}^{-1} \geq (\mathbf{A}^{-1}\mathbf{N}_1)^j\mathbf{A}^{-1} \geq \mathbf{0} \tag{5.44}$$

for some positive integer $j > 1$, but for which $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}$ is not satisfied. Under this condition, they prove a theorem (Theorem 2 in [7]) generalizing the author's Theorem 5.1 by using the proof technique similar to that given in the second part of the proof of Theorem 5.1. That is, when the inequality (5.44) is satisfied for $j > 1$, with $\mathbf{A}^{-1} \geq \mathbf{0}$

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{5.45}$$

In the second theorem (Theorem 4 in [7]) they consider the case when

$$(\mathbf{A}^{-1}\mathbf{N}_2)^j\mathbf{A}^{-1} > (\mathbf{A}^{-1}\mathbf{N}_1)^j\mathbf{A}^{-1} \geq \mathbf{0}. \tag{5.44a}$$

Corollary 5 and its proof [7, p.27 and p.34] seem to be a part of the proof of Theorem 5.2.

It is evident that

$$(\mathbf{A}^{-1}\mathbf{N}_2)^j\mathbf{A}^{-1} = \mathbf{A}^{-1}(\mathbf{N}_2\mathbf{A}^{-1})^j \geq (\mathbf{A}^{-1}\mathbf{N}_1)^j\mathbf{A}^{-1} \geq \mathbf{0} \tag{5.46}$$

and assuming that to the eigenvectors $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{y}^T \geq \mathbf{0}$ correspond the eigenvalues $\tau_1 = \rho(\mathbf{A}^{-1}\mathbf{N}_1)$ and $\tau_2 = \rho(\mathbf{A}^{-1}\mathbf{N}_2) = \rho(\mathbf{N}_2\mathbf{A}^{-1})$ respectively, then

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1 = \tau_1\mathbf{x}^T$$

and

$$\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y} = \tau_2\mathbf{y}$$

or

$$\mathbf{x}^T(\mathbf{A}^{-1}\mathbf{N}_1)^j = (\tau_1)^j\mathbf{x}^T \tag{5.47}$$

and

$$(\mathbf{N}_2\mathbf{A}^{-1})^j\mathbf{y} = (\tau_2)^j\mathbf{y}. \tag{5.48}$$

Multiplying Eq. (5.47) on the right by $\mathbf{A}^{-1}$ and Eq. (5.48) on the left by $\mathbf{A}^{-1}$ gives us

$$\mathbf{x}^T(\mathbf{A}^{-1}\mathbf{N}_1)^j\mathbf{A}^{-1} = (\tau_1)^j\mathbf{x}^T\mathbf{A}^{-1} \tag{5.49}$$

and

$$\mathbf{A}^{-1}(\mathbf{N}_2\mathbf{A}^{-1})^j\mathbf{y} = (\tau_2)^j\mathbf{A}^{-1}\mathbf{y}. \tag{5.50}$$

From the assumption (5.46) it follows that

$$\begin{aligned}\mathbf{x}^T(\mathbf{A}^{-1}\mathbf{N}_2)^j\mathbf{A}^{-1} &= \mathbf{x}^T\mathbf{A}^{-1}(\mathbf{N}_2\mathbf{A}^{-1})^j \\ &\geq (=)\, \mathbf{x}^T(\mathbf{A}^{-1}\mathbf{N}_1)^j\mathbf{A}^{-1} = (\tau_1)^j\mathbf{x}^T\mathbf{A}^{-1}.\end{aligned} \tag{5.51}$$

Again multiplying Eq. (5.51) on the right by $\mathbf{y}$ and Eq. (5.50) on the left by $\mathbf{x}^T$, one obtains

$$\mathbf{x}^T(\mathbf{A}^{-1}\mathbf{N}_2)^j\mathbf{A}^{-1}\mathbf{y} \geq (\tau_1)^j\mathbf{x}^T\mathbf{A}^{-1}\mathbf{y} \tag{5.52}$$

and

$$\mathbf{x}^T(\mathbf{A}^{-1}\mathbf{N}_2)^j\mathbf{A}^{-1}\mathbf{y} = (\tau_2)^j\mathbf{x}^T\mathbf{A}^{-1}\mathbf{y} \tag{5.53}$$

and hence, as $\mathbf{x}^T\mathbf{A}^{-1}\mathbf{y} > 0$, it follows that $\tau_1 \le \tau_2$ which by Theorem 3.2 generalizes Csordas and Varga's results for the case when both splittings $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ are nonnegative or weak nonnegative but of different type, that is, when either $\mathbf{M}_1^{-1}\mathbf{N}_1 \ge \mathbf{0}$ and $\mathbf{N}_2^{-1}\mathbf{M}_1 \ge \mathbf{0}$ or $\mathbf{N}_1\mathbf{M}_1^{-1} \ge \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \ge \mathbf{0}$.

In conclusion, the inequalities (5.44) and (5.44a) may be satisfied for any $j \ge 1$, if and only if $\rho(\mathbf{A}^{-1}\mathbf{N}_1) \le \rho(\mathbf{A}^{-1}\mathbf{N}_2)$ and $\rho(\mathbf{A}^{-1}\mathbf{N}_1) < \rho(\mathbf{A}^{-1}\mathbf{N}_2)$ respectively.

Csordas and Varga give two examples of regular splittings of $\mathbf{A}$ with $\mathbf{A}^{-1} > \mathbf{0}$. One of them shows that (5.44) fails for each $j \ge 1$ with $\rho(\mathbf{A}^{-1}\mathbf{N}_1) = \rho(\mathbf{A}^{-1}\mathbf{N}_2) = \frac{1}{2}$ [7, pp.27 and 28]. In the second example [7, p.23] regular splittings with fixed and variable matrices are derived from the following matrix.

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \text{so that} \quad \mathbf{A}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{5.54}$$

The regular splitting with fixed matrices

$$\mathbf{M}_1 = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{N}_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_1^{-1} = \frac{1}{7}\begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix} \tag{5.55}$$

where

$$\mathbf{M}_1^{-1}\mathbf{N}_1 = \frac{1}{7}\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{N}_1\mathbf{M}_1^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{N}_1\mathbf{A}^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{A}^{-1}\mathbf{N}_1 = \frac{1}{6}\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} = \frac{1}{3}\left(\frac{1}{6}\right)\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \quad \text{and}$$

$$(\mathbf{A}^{-1}\mathbf{N}_1)^k\mathbf{A}^{-1} = \frac{1}{3}\left(\frac{1}{6}\right)^k\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

and with variable matrices

$$\begin{gathered} \mathbf{M}_2(\alpha) = \begin{bmatrix} 2 & -1 \\ -1+\alpha & 4 \end{bmatrix}, \quad \mathbf{N}_2(\alpha) = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} \quad \text{and} \\ [\mathbf{M}_2(\alpha)]^{-1} = \frac{1}{3+\alpha}\begin{bmatrix} 2 & 1 \\ 1-\alpha & 4 \end{bmatrix} \end{gathered} \tag{5.56}$$

where

$$[\mathbf{M}_2(\alpha)]^{-1}\mathbf{N}_2(\alpha) = \frac{\alpha}{3+\alpha}\begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}, \quad \mathbf{N}_2(\alpha)[\mathbf{M}_2(\alpha)]^{-1} = \frac{\alpha}{3+\alpha}\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix},$$

$$\mathbf{N}_2(\alpha)\mathbf{A}^{-1} = \frac{\alpha}{3}\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{A}^{-1}\mathbf{N}_2(\alpha) = \frac{\alpha}{3}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix},$$

$$\mathbf{A}^{-1}\mathbf{N}_2(\alpha)\mathbf{A}^{-1} = \frac{1}{3}\left(\frac{\alpha}{3}\right)\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad (\mathbf{A}^{-1}\mathbf{N}_2(\alpha))^k\mathbf{A}^{-1} = \frac{1}{3}\left(\frac{\alpha}{3}\right)^k\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}.$$

It is evident that $\mathbf{A} = \mathbf{M}_2(\alpha) - \mathbf{N}_2$ is the regular splitting of $\mathbf{A}$ if and only if $0 \leq \alpha \leq 1$ and

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho([\mathbf{M}_2(\alpha)]^{-1}\mathbf{N}_2(\alpha)) \tag{5.57}$$

is satisfied if and only if $\alpha \geq \frac{1}{2}$.

The hypothesis

$$(\mathbf{A}^{-1}\mathbf{N}_2(\alpha))^k\mathbf{A}^{-1} \geq (\mathbf{A}^{-1}\mathbf{N}_1)^k\mathbf{A}^{-1} \geq \mathbf{0} \tag{5.58}$$

is satisfied with $k > 1$ and if and only if

$$(2\alpha)^{k-1} < 4 \quad \text{and} \quad (2\alpha)^k \geq 4. \tag{5.59}$$

Considering the case of equality in the second part of (5.59), it is evident that setting

$$\alpha(k) \geq 4^{1/k}/2 \tag{5.60}$$

implies satisfying (5.58) for each positive integer $k \geq 2$, where $0.5 \leq \alpha(k) \leq 1$.

However, when α decreases to 0.5 the inequality (5.58) will be satisfied with increasing integers k. For instance, $k = 71$ for $\alpha = 0.51$, $k = 694$ for $\alpha = 0.501$ and, of course, in the extreme case k approaches infinity when $\alpha = 0.5$. Thus for the class of matrix problems represented by the above example determining the first positive integer for which (5.44) is satisfied may be too laborious. Csordas and Varga conclude similarly [7, p.28]. However it should be mentioned that the verification of (5.44) may not be only cumbersome but also impossible to apply in actual practice because the authors did not give any preliminary argument that the condition (5.44) holds at all.

As can be easily verified the vector $\mathbf{x}^T = [0,1]$ is the left eigenvector of $\mathbf{M}_1^{-1}\mathbf{N}_1$ and $\mathbf{A}^{-1}\mathbf{N}_1$ corresponding to $\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) = \frac{1}{7}$ and $\rho(\mathbf{A}^{-1}\mathbf{N}_1) = \frac{1}{6}$, and the following inequalities

$$\mathbf{x}^T\mathbf{N}_2(\alpha)[\mathbf{M}_2(\alpha)]^{-1} = \frac{\alpha}{3+\alpha}[2,1] \geq \frac{1}{7}[0,1] = \mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0} \tag{5.61}$$

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_2(\alpha)\mathbf{A}^{-1} = \frac{\alpha}{9}[4,2] \geq \frac{1}{18}[1,2] = \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{5.62}$$

are satisfied for $\alpha \geq \frac{1}{2}$, Hence, by Theorems 3.17 and 5.6 the result (5.57) follows immediately. Thus, in the application of Theorems 3.17 and 5.6 the hypothesis verification is always possible but it is necessary to compute an eigenvector additionally.

On the other hand both above examples of regular splittings of $\mathbf{A}$ defined by (5.54) can be compared by means of results given in Section 3. It can be seen that

$$[\mathbf{N}_2(\alpha)]^T \geq \mathbf{N}_1 \tag{5.63}$$

for $1 \geq \alpha > \frac{1}{2}$, hence the result (5.57) follows immediately by Theorem 3.13 and with the strict inequality by Corollary 3.3 because in this case $\mathbf{A}^{-1} > \mathbf{0}$. Moreover, it can be observed that with $1 \geq \alpha > \frac{1}{2}$

$$[\mathbf{M}_2(\alpha)]^{-1}\mathbf{N}_2(\alpha) \geq (\mathbf{N}_1\mathbf{M}_1^{-1})^T \quad \text{and} \quad \mathbf{N}_2(\alpha)[\mathbf{M}_2(\alpha)]^{-1} \geq (\mathbf{M}_1^{-1}\mathbf{N}_1)^T \tag{5.64}$$

and

$$\mathbf{A}^{-1}\mathbf{N}_2(\alpha) \geq (\mathbf{N}_1\mathbf{A}^{-1})^T \quad \text{and} \quad \mathbf{N}_2(\alpha)\mathbf{A}^{-1} \geq (\mathbf{A}^{-1}\mathbf{N}_1)^T \tag{5.65}$$

but the above inequalities correspond to the hypotheses of Theorem 3.16, which provides the result (5.57) as well.

Unfortunately, Csordas and Varga show only these two examples of regular splittings of $\mathbf{A}$, (5.55) and (5.56), satisfying the condition (5.44) as well as the inequalities (5.63), (5.64) and (5.65) corresponding to the hypotheses used in the theorems mentioned above. Therefore, it seems natural to ask if there is an equivalence between the condition (5.44) burdensome in practice and those natural ones used in Theorems 3.13, 3.14, 3.15 and 3.16 which on the other hand, as was already demonstrated in Section 4, are useful tools with the choice of forward or backward order in the Gauss-Seidel method as a more efficient splitting of a nonsymmetric monotone matrix $\mathbf{A}$. Perhaps, the answer to this matter remains an open question.

6. Further Extensions of the Nonnegative Splitting Theory

In this section further extensions of the nonnegative splitting theory are studied for the class of iterative methods represented by a weak splitting of a matrix $\mathbf{A}$ and defined, as follows.

DEFINITION 6.1. For matrices $\mathbf{A}$, $\mathbf{M}$ and $\mathbf{N}$ the following decomposition

$$\mathbf{A} = \mathbf{M} - \mathbf{N}$$

is called a *weak splitting of* $\mathbf{A}$, if either $\mathbf{M}^{-1}\mathbf{N} = \mathcal{G} \geq \mathbf{0}$ (the *first type*) or $\mathbf{N}\mathbf{M}^{-1} = \mathcal{G}'' \geq \mathbf{0}$ (the *second type*). In particular a given weak splitting can be both types.

The definition of the weak splitting of $\mathbf{A}$ for the first type case have been introduced by Marek and Szyld [11]. It is obvious that the following corollary holds.

COROLLARY 6.1. *Any (weak) nonnegative splitting of a matrix* $\mathbf{A}$ *is a weak splitting of* $\mathbf{A}$*, but the converse is not true.*

Since weak splittings used as hypotheses are weaker than in the case of (weak) nonnegative splittings, it seems interesting to study how and which results of the previous sections can be generalized.

As was already stated (Theorem 3.2), the assumption that $\mathbf{A}$ is a monotone matrix, that is $\mathbf{A}^{-1} \geq \mathbf{0}$, implies that each weak nonnegative splitting of $\mathbf{A}$ is convergent. However, in the case of weak splittings the assumption $\mathbf{A}^{-1} \geq \mathbf{0}$ is not a sufficient condition. As can be easily verified in examples, for a given weak splitting of $\mathbf{A} = \mathbf{M} - \mathbf{N}$, $\rho(\mathbf{M}^{-1}\mathbf{N})$ may be greater than unity with $\mathbf{A}^{-1} \geq \mathbf{0}$ (see e.g. [11]).

All results given in this section are presented for convergent weak splittings (Definition 3.1) with $\mathbf{A}^{-1} \geq \mathbf{0}$, and their properties are collected in the following theorem.

THEOREM 6.1. *Let* $\mathbf{A} = \mathbf{M} - \mathbf{N}$ *be a convergent weak splitting of* $\mathbf{A}$*. If* $\mathbf{A}^{-1} \geq \mathbf{0}$*, then*

1. $\mathbf{A}^{-1} \geq \mathbf{M}^{-1}$
2. $\rho(\mathbf{M}^{-1}\mathbf{N}) = \rho(\mathbf{N}\mathbf{M}^{-1})$
3. *if* $\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{0}$, *then* $\mathbf{A}^{-1}\mathbf{N} \geq \mathbf{M}^{-1}\mathbf{N}$ *and if* $\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}$, *then* $\mathbf{N}\mathbf{A}^{-1} \geq \mathbf{N}\mathbf{M}^{-1}$
4. $$\rho(\mathbf{M}^{-1}\mathbf{N}) = \frac{\rho(\mathbf{A}^{-1}\mathbf{N})}{1+\rho(\mathbf{A}^{-1}\mathbf{N})}. \tag{6.1}$$

Proof.

(1) From Theorem 3.1 it follows

$$\mathbf{A}^{-1} = \mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{M}^{-1} + \mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1} \tag{6.2}$$

and since $\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{0}$ or $\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}$ by hypotheses then

$$\mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0} \tag{6.3}$$

which gives us immediately that $\mathbf{A}^{-1} \geq \mathbf{M}^{-1}$.

(2) By using the result of Lemma 2.1, one obtains $\rho(\mathbf{M}^{-1}\mathbf{N}) = \rho(\mathbf{N}\mathbf{M}^{-1})$.

(3) Let us assume that $\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{0}$. Then one can write

$$\begin{aligned}\mathbf{A}^{-1} &= \mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}(\mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1})\\ &= [\mathbf{I} + \mathbf{M}^{-1}\mathbf{N}]\mathbf{M}^{-1} + (\mathbf{M}^{-1}\mathbf{N})^2\mathbf{A}^{-1}\\ &= [\mathbf{I} + \mathbf{M}^{-1}\mathbf{N} + (\mathbf{M}^{-1}\mathbf{N})^2]\mathbf{M}^{-1} + (\mathbf{M}^{-1}\mathbf{N})^3\mathbf{A}^{-1}\\ &= [\mathbf{I} + \mathbf{M}^{-1}\mathbf{N} + (\mathbf{M}^{-1}\mathbf{N})^2 + \cdots + (\mathbf{M}^{-1}\mathbf{N})^{k-1}]\mathbf{M}^{-1}\\ &\quad + (\mathbf{M}^{-1}\mathbf{N})^k\mathbf{A}^{-1}.\end{aligned} \tag{6.4}$$

Since $\rho(\mathbf{M}^{-1}\mathbf{N}) < \mathbf{1}$ by the hypothesis, then for $k \to \infty$ $(\mathbf{M}^{-1}\mathbf{N})^k \to \mathbf{0}$ the series

$$\mathbf{I} + \mathbf{M}^{-1}\mathbf{N} + (\mathbf{M}^{-1}\mathbf{N})^2 + \cdots$$

is convergent, and by Theorem 2.1 one obtains

$$\mathbf{I} + \mathbf{M}^{-1}\mathbf{N} + (\mathbf{M}^{-1}\mathbf{N})^2 + \cdots = (\mathbf{I} - \mathbf{M}^{-1}\mathbf{N})^{-1} \geq \mathbf{I} \geq \mathbf{0}. \tag{6.5}$$

Hence

$$\mathbf{A}^{-1} = (\mathbf{I} - \mathbf{M}^{-1}\mathbf{N})^{-1}\mathbf{M}^{-1} \geq \mathbf{0} \tag{6.6}$$

or

$$\mathbf{A}^{-1}\mathbf{N} = (\mathbf{I} - \mathbf{M}^{-1}\mathbf{N})^{-1}\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{M}^{-1}\mathbf{N} \geq \mathbf{0}. \tag{6.7}$$

In the case when $\mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}$ can be similarly shown that

$$\mathbf{N}\mathbf{A}^{-1} \geq \mathbf{N}\mathbf{M}^{-1} \geq \mathbf{0}. \tag{6.7a}$$

(4) Assuming the case $\mathbf{M}^{-1}\mathbf{N} \geq \mathbf{0}$ which implies that $\mathbf{A}^{-1}\mathbf{N} \geq \mathbf{0}$, the proof is identical with the item (4) given in the proof of Theorem 3.2. ■

Now the following theorems which can be proven by a close analogy to the proofs of the corresponding theorems given in Section 3 are presented.

THEOREM 6.2. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two convergent weak splittings of* $\mathbf{A}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If one of the following inequalities*

(a) $\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0}$ (*or* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$)

(b) $\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$ (*or* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$)

(c) $\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$ (*or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$)

(d) $\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0}$ (*or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$)

is satisfied, then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}.$$

Proof. The proof is the same as in the case of Theorem 3.3. ■

THEOREM 6.3. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two convergent weak splittings of* $\mathbf{A}$ *of the same type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If* $\mathbf{N}_2 \geq \mathbf{N}_1$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}.$$

Proof. The proof is the same as in the case of Theorem 3.4. ■

THEOREM 6.4. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two convergent weak splittings of* $\mathbf{A}$ *of different type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, *or one of them be a weak splitting of* $\mathbf{A}$ *but of both types, where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If* $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{6.8}$$

Proof. Assuming the case when $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, one can write that

$$\mathbf{y}_1^T\mathbf{M}_1^{-1}\mathbf{N}_1 = \lambda_1\mathbf{y}_1^T \tag{6.9}$$

and

$$\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{y}_2 = \lambda_2\mathbf{y}_2 \tag{6.10}$$

where by Theorems 2.2 and 6.1, and Lemma 2.1 $\lambda_1 = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \mathbf{1}$ and $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) = \rho(\mathbf{N}_2\mathbf{M}_2^{-1}) < \mathbf{1}$ and the corresponding eigenvectors $\mathbf{y}_1$ and $\mathbf{y}_2$ are nonnegative. Multiplying Eq. (6.9) on the right by $\mathbf{A}^{-1}$ and Eq. (6.10) on the left by $\mathbf{A}^{-1}$ gives us

$$\mathbf{y}_1^T\mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1} = \lambda_1\mathbf{y}_1^T\mathbf{A}^{-1} \tag{6.11}$$

and

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{y}_2 = \lambda_2\mathbf{A}^{-1}\mathbf{y}_2. \tag{6.12}$$

From the assumption $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$ by relation (6.2), it follows that

$$\mathbf{A}^{-1} - \mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}.$$

Since both matrices $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}$ and $\mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1}$ are nonnegative by the relation (6.3), hence

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{6.13}$$

or

$$\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1} \geq (=)\, \mathbf{y}_1^T\mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1} = \lambda_1\mathbf{y}_1^T\mathbf{A}^{-1}. \tag{6.14}$$

Again multiplying Eq. (6.14) on the right by $\mathbf{y}_2$ and Eq. (6.12) on the left by $\mathbf{y}_1^T$, one obtains

$$\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{y}_2 \geq \lambda_1\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 \tag{6.15}$$

and

$$\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{y}_2 = \lambda_2\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 \tag{6.16}$$

and as $\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 > \mathbf{0}$, it follows that $\lambda_1 \leq \lambda_2$, which proves the inequality (6.8) for the case when $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2^{-1}\mathbf{M}_1 \geq \mathbf{0}$. When $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ the proof is similar as in the proof of Theorem 3.5. The case when $\mathbf{y}_1^T\mathbf{A}^{-1}\mathbf{y}_2 = \mathbf{0}$ can be considered according to the modification described in Remark given at the end of Section 3. ■

THEOREM 6.5. *Let $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ be two convergent weak splittings of $\mathbf{A}$ of different type, that is, either $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ or $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, or one of them be a weak splitting of $\mathbf{A}$ but of both types, where $\mathbf{A}^{-1} > \mathbf{0}$. If $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1}$, then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{6.17}$$

Proof. Similarly as in the proof of Theorem 6.4 it is evident that the assumption $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1}$ implies the strict inequality in (6.13), that is

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{M}_2^{-1} > \mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}. \tag{6.18}$$

The above inequality implies replacing the non-strict inequality sign to the strict one in the corresponding inequalities in the remaining part of the proof of Theorem 6.4, which proves the validity of the inequality (6.17). ■

THEOREM 6.6. *Let $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ be two convergent weak splittings of a symmetric matrix $\mathbf{A}$, where $\mathbf{A}^{-1} \geq \mathbf{0}$. If $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$ and at least one of $\mathbf{M}_1$ and $\mathbf{M}_2$ is a symmetric matrix, then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1.$$

Proof. The same proof as in the case of Theorem 3.11.

THEOREM 6.7. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two convergent weak splittings of a symmetric matrix* $\mathbf{A}$, *where* $\mathbf{A}^{-1} > \mathbf{0}$. *If* $\mathbf{M}_1^{-1} > \mathbf{M}_2^{-1}$ *and at least one of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *is a symmetric matrix, then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1.$$

Proof. The same proof as in the case of Theorem 3.12.

THEOREM 6.8. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two convergent weak splittings of* $\mathbf{A}$ *of the same type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *or one of them be a weak splitting of* $\mathbf{A}$ *but of both types, where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If* $\mathbf{N}_2^T \geq \mathbf{N}_1$, *then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1.$$

Proof. The same proof as in the case of Theorem 3.13.

It is evident that by analogy to Corollary 3.2 and its proof the following corollary holds.

COROLLARY 6.2. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two convergent weak splittings of* $\mathbf{A}$ *of the same type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$.

(a) *If*

$$\mathbf{A}^{-1}\mathbf{N}_2^T(\mathbf{A}^{-1})^T \geq \mathbf{A}^{-1}\mathbf{N}_1(\mathbf{A}^{-1})^T \geq \mathbf{0} \tag{6.19}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) \leq 1. \tag{6.20}$$

(b) *Moreover, if*

$$\mathbf{A}^{-1}\mathbf{N}_2^T(\mathbf{A}^{-1})^T > \mathbf{A}^{-1}\mathbf{N}_1(\mathbf{A}^{-1})^T \geq \mathbf{0} \tag{6.19a}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{6.20a}$$

As can be easily verified Theorems 3.14, 3.15, 3.16, 3.17 and Corollary 3.5 as well as its proofs hold in the case of convergent weak splittings, and for completeness reasons they are reformulated below.

THEOREM 6.9. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two convergent weak splittings of* $\mathbf{A}$ *of the same type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If*

$$\mathbf{M}_1^{-1} \geq (\mathbf{M}_2^{-1})^T \tag{6.21}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{6.22}$$

THEOREM 6.10. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two convergent weak splittings of* $\mathbf{A}$ *of the same type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If*

$$\mathbf{M}_1^{-1} > (\mathbf{M}_2^{-1})^T \tag{6.23}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{6.24}$$

THEOREM 6.11. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two convergent weak splittings of* $\mathbf{A}$, *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *If one of the following inequalities*

(a) $\mathbf{A}^{-1}\mathbf{N}_2 \geq (\mathbf{A}^{-1}\mathbf{N}_1)^T \geq \mathbf{0}$ (*or* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq (\mathbf{M}_1^{-1}\mathbf{N}_1)^T \geq \mathbf{0}$)

(b) $\mathbf{A}^{-1}\mathbf{N}_2 \geq (\mathbf{N}_1\mathbf{A}^{-1})^T \geq \mathbf{0}$ (*or* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq (\mathbf{N}_1\mathbf{M}_1^{-1})^T \geq \mathbf{0}$)

(c) $\mathbf{N}_2\mathbf{A}^{-1} \geq (\mathbf{N}_1\mathbf{A}^{-1})^T \geq \mathbf{0}$ (*or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq (\mathbf{N}_1\mathbf{M}_1^{-1})^T \geq \mathbf{0}$)

(d) $\mathbf{N}_2\mathbf{A}^{-1} \geq (\mathbf{A}^{-1}\mathbf{N}_1)^T \geq \mathbf{0}$ (*or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq (\mathbf{M}_1^{-1}\mathbf{N}_1)^T \geq \mathbf{0}$)

is satisfied, then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{6.25}$$

THEOREM 6.12. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two convergent weak splittings of* $\mathbf{A}$ *where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *Let* $\mathbf{x} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$ *and* $\mathbf{z} \geq \mathbf{0}$ *be vectors such that* $\mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 = \lambda_1\mathbf{x}^T$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{y} = \lambda_2\mathbf{y}$ *or* $\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{z} = \lambda_2\mathbf{z}$ *when* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *respectively; and let* $\mathbf{u} \geq \mathbf{0}$, $\mathbf{v} \geq \mathbf{0}$ *and* $\mathbf{w} \geq \mathbf{0}$ *be vectors such that* $\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{u} = \lambda_1\mathbf{u}$ *and* $\mathbf{v}^T\mathbf{M}_2^{-1}\mathbf{N}_2 = \lambda_2\mathbf{v}^T$ *or* $\mathbf{w}^T\mathbf{N}_2\mathbf{M}_2^{-1} = \lambda_2\mathbf{w}^T$ *when* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$ *or* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$, *respectively; where* $\lambda_1 = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1)$ *and* $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$. *If one of the following inequalities*

$$\mathbf{x}^T\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0} \quad \textit{or} \quad \mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{y} \geq \mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{y} \geq \mathbf{0} \tag{6.26a}$$

$$\mathbf{x}^T\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0} \quad \textit{or} \quad \mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{z} \geq \mathbf{M}_1^{-1}\mathbf{N}_1\mathbf{z} \geq \mathbf{0} \tag{6.26b}$$

$$\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{u} \geq \mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{u} \geq \mathbf{0} \quad \textit{or} \quad \mathbf{v}^T\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{v}^T\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0} \tag{6.26c}$$

$$\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{u} \geq \mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{u} \geq \mathbf{0} \quad \textit{or} \quad \mathbf{w}^T\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{w}^T\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0} \tag{6.26d}$$

or

$$\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0} \quad \textit{or} \quad \mathbf{A}^{-1}\mathbf{N}_2\mathbf{y} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{y} \geq \mathbf{0} \tag{6.27a}$$

$$\mathbf{x}^T\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0} \quad \textit{or} \quad \mathbf{N}_2\mathbf{A}^{-1}\mathbf{z} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{z} \geq \mathbf{0} \tag{6.27b}$$

$$\mathbf{A}^{-1}\mathbf{N}_2\mathbf{u} \geq \mathbf{N}_1\mathbf{A}^{-1}\mathbf{u} \geq \mathbf{0} \quad \textit{or} \quad \mathbf{v}^T\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{v}^T\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{6.27c}$$

$$\mathbf{N}_2\mathbf{A}^{-1}\mathbf{u} \geq \mathbf{N}_1\mathbf{A}^{-1}\mathbf{u} \geq \mathbf{0} \quad \textit{or} \quad \mathbf{w}^T\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{w}^T\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{6.27d}$$

is fulfilled, then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{6.28}$$

In particular, if the first non-strict inequality sign in the above inequalities is replaced by the strict one, then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{6.29}$$

COROLLARY 6.3. *Let $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ be two convergent weak splittings of $\mathbf{A}$ of the same type, where $\mathbf{A}^{-1} \geq \mathbf{0}$. Let $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$ be vectors such that $\mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 = \lambda_1\mathbf{x}^T$ and $\mathbf{M}_2^{-1}\mathbf{N}_2\mathbf{y} = \lambda_2\mathbf{y}$ when $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$; and let $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$ be vectors such that $\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{u} = \lambda_1\mathbf{u}$ and $\mathbf{v}^T\mathbf{N}_2\mathbf{M}_2^{-1} = \lambda_2\mathbf{v}^T$ when $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$; where $\lambda_1 = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1)$ and $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$. If one of the following inequalities*

$$\mathbf{x}^T\mathbf{N}_2 \geq \mathbf{x}^T\mathbf{N}_1 \quad \textit{or} \quad \mathbf{N}_2\mathbf{y} \geq \mathbf{N}_1\mathbf{y} \tag{6.30a}$$

$$\mathbf{N}_2\mathbf{u} \geq \mathbf{N}_1\mathbf{u} \quad \textit{or} \quad \mathbf{v}^T\mathbf{N}_2 \geq \mathbf{v}^T\mathbf{N}_1 \tag{6.30b}$$

holds, then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{6.31}$$

Referring back to the results given in Section 5, it is easy to notice that Theorems 5.3 to 5.6 and their proofs can be generalized to the class of weak splittings and they are reformulated in the following four theorems.

THEOREM 6.13. *Let $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ be two convergent weak splittings of $\mathbf{A}$ of different type, that is, either $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ or $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, or one of them be a weak splitting of $\mathbf{A}$ but of both types, where $\mathbf{A}^{-1} \geq \mathbf{0}$. If $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$, then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{6.32}$$

THEOREM 6.14. *Let $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ be two convergent weak splittings of $\mathbf{A}$ of different type, that is, either $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ and $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ or $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ and $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, or one of them be a weak splitting of $\mathbf{A}$ but of both types, where $\mathbf{A}^{-1} > \mathbf{0}$. If $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}$, then*

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < \mathbf{1}. \tag{6.33}$$

THEOREM 6.15. *Let $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ be two convergent weak splittings of $\mathbf{A}$, where $\mathbf{A}^{-1} \geq \mathbf{0}$. If either*

$$\begin{gathered}\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_2 \geq \left\{\begin{matrix}\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2\\ \textit{or}\\ \mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1\end{matrix}\right\} \geq \mathbf{0} \quad \textit{and}\\ \left\{\begin{matrix}\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2\\ \textit{or}\\ \mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1\end{matrix}\right\} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{0}\end{gathered} \tag{6.34a}$$

or

$$\mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \left\{\begin{matrix} \mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \\ or \\ \mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \end{matrix}\right\} \geq \mathbf{0} \quad and$$
$$\left\{\begin{matrix} \mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \\ or \\ \mathbf{N}_2\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \end{matrix}\right\} \geq \mathbf{N}_1\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \tag{6.34b}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{6.35}$$

THEOREM 6.16. *Let* $\mathbf{A} = \mathbf{M}_1 - \mathbf{N}_1 = \mathbf{M}_2 - \mathbf{N}_2$ *be two convergent weak splittings of* $\mathbf{A}$ *of different type, that is, either* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$ *or* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq 0$, *or one of them be a weak splitting of* $\mathbf{A}$ *but of both types, where* $\mathbf{A}^{-1} \geq \mathbf{0}$. *Let* $\mathbf{x} \geq \mathbf{0}$ *and* $\mathbf{y} \geq \mathbf{0}$ *be vectors such that* $\mathbf{x}^T\mathbf{M}_1^{-1}\mathbf{N}_1 = \lambda_1\mathbf{x}^T$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1}\mathbf{y} = \lambda_2\mathbf{y}$ *when* $\mathbf{M}_1^{-1}\mathbf{N}_1 \geq \mathbf{0}$ *and* $\mathbf{N}_2\mathbf{M}_2^{-1} \geq \mathbf{0}$; *and let* $\mathbf{v} \geq \mathbf{0}$ *and* $\mathbf{w} \geq \mathbf{0}$ *be vectors such that* $\mathbf{N}_1\mathbf{M}_1^{-1}\mathbf{v} = \lambda_1\mathbf{v}$ *and* $\mathbf{w}^T\mathbf{M}_2^{-1}\mathbf{N}_2 = \lambda_2\mathbf{w}^T$ *when* $\mathbf{N}_1\mathbf{M}_1^{-1} \geq \mathbf{0}$ *and* $\mathbf{M}_2^{-1}\mathbf{N}_2 \geq \mathbf{0}$, *where* $\lambda_1 = \rho(\mathbf{M}_1^{-1}\mathbf{N}_1)$ *and* $\lambda_2 = \rho(\mathbf{M}_2^{-1}\mathbf{N}_2)$. *If either*

$$\begin{aligned} &\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq (=)\, \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \\ &(or\ \mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y} \geq (=)\, \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{y} \geq \mathbf{0}) \end{aligned} \tag{6.36a}$$

or

$$\begin{aligned} &\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{v} \geq (=)\, \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{v} \geq \mathbf{0} \\ &(or\ \mathbf{w}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq (=)\, \mathbf{w}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}) \end{aligned} \tag{6.36b}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) \leq \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{6.37}$$

Moreover, if either

$$\begin{aligned} &\mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} > \mathbf{x}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0} \\ &(or\ \mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{y} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{y} \geq \mathbf{0}) \end{aligned} \tag{6.38a}$$

or

$$\begin{aligned} &\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1}\mathbf{v} > \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1}\mathbf{v} \geq \mathbf{0} \\ &(or\ \mathbf{w}^T\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} > \mathbf{w}^T\mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{0}) \end{aligned} \tag{6.38b}$$

then

$$\rho(\mathbf{M}_1^{-1}\mathbf{N}_1) < \rho(\mathbf{M}_2^{-1}\mathbf{N}_2) < 1. \tag{6.39}$$

References

[1] R.S. Varga, Matrix Iterative Analysis. Prentice Hall, Englewood Cliffs, N.J., 1962.

[2] Z.I. Woźnicki, Two-sweep iterative methods for solving large linear systems and their application to the numerical solution of multi-group, multi-dimensional neutron diffusion equations. Doctoral Dissertation, Rep. No. 1447-CYFRONET-PM-A, Inst. Nuclear Res., Swierk-Otwock, Poland, 1973.

[3] Z.I. Woźnicki, AGA two-sweep iterative method and their application in critical reactor calculations. Nukleonika **9** (1978), 941–968.

[4] Z.I.Woźnicki, AGA two-sweep iterative methods and their application for the solution of linear equation systems. Proc. International Conference on Linear Algebra and Applications., Valencia, Spain, Sept. 28–30, 1987 (published in Linear Algebra Appl., **121** (1989), 702–710.

[5] Z.I. Woźnicki, Estimation of the optimum relaxation factors in the partial factorization iterative methods. Proc. International Conference on the Physics of Reactors: Operation, Design and Computation, Marseille, France, April 23–27, 1990, pp. P-IV-173-186. (To appear at beginning 1993 in SIAM J. Matrix Anal. Appl.)

[6] J.M. Ortega and W. Rheinboldt, Monotone iterations for nonlinear equations with applications to Gauss-Seidel methods. SIAM J. Numer.Anal., **4** (1967), 171–190.

[7] G. Csordas and R.S. Varga, Comparison of regular splittings of matrices. Numer. Math., **44** (1984), 23–35.

[8] G. Alefeld and P. Volkmann, Regular splittings and monotone iteration functions. Numer. Math., **46** (1985), 213–228.

[9] V.A. Miller and M. Neumann, A note on comparison theorems for nonnegative matrices. Numer. Math., **47** (1985), 427–434.

[10] L. Elsner, Comparisons of weak regular splittings and multisplitting methods. Numer. Math., **56** (1989), 283–289.

[11] I. Marek and D.B. Szyld, Comparison theorems for weak splittings of bounded operators. Numer. Math., **58** (1990), 389–397.

[12] Z.I. Woźnicki, HEXAGA-II-120, -60, -30 Two-dimensional multi-group neutron diffusion programmes for a uniform triangular mesh with arbitrary group scattering. Report KfK-2789, 1979.

[13] Z.I. Woźnicki, HEXAGA-III-120, -30 Three-dimensional multi-group neutron diffusion programmes for a uniform triangular mesh with arbitrary group scattering. Report KfK-3572, 1983.

[14] Z.I. Woźnicki, Two- and three-dimensional benchmark calculations for triangular geometry by means of HEXAGA programmes. Proc. International Meeting on Advances in Nuclear Engineering Computational Methods, Knoxville, Tennessee, April 9–11, 1985, pp. 147–156.

[15] R. Beauwens, Factorization iterative methods, M-operators and H-operators. Numer. Math., **31** (1979), 335–357.

[16] H.C. Elman and G.H. Golub, Line iterative methods for cyclically reduced discrete convection-diffusion problems. SIAM J. Sci. Statist. Comput., **13** (1992), 339–363.

[17] Z.I. Woźnicki, The graphic representation of the algorithms of the AGA two-sweep iterative method (under preparation).

[18] Z.I. Woźnicki, On numerical analysis of conjugate gradient method. Japan J. Indust. Appl. Math., **10** (1993), 487–519.

[19] Z.I. Woźnicki, The Sigma-SOR algorithm and the optimal strategy for the utilization of the SOR iterative method. Math. Comp., **62**, 206 (1994), 619–644.