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Structure Stability of Congestion in Traffic Dynamics

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In our previous paper, we proposed a dynamical model, whose equation of motion is expressed as a second order differential equation. This model generates traffic congestion spontaneously. In this paper we study the characteristic properties of the traffic congestion in our model, especially the organization process and the stability of the structure of congestion. It turns out that these phenomena are well described by plotting motions of vehicles in the phase space of velocity and headway. The most remarkable feature is the universality of "the hysterisis loop" in this phase space, which is observed in the final stage of the congestion organization. This loop is understood as a limit cycle of the dynamical system. This universality guarantees the stability of total cluster size.

Key words: traffic dynamics, congestion, structure stability, limit cycle, numerical simulation

1. Introduction

One of the most interesting problems of traffic dynamics is the generation of traffic congestion. We treat traffic problems as dynamics of a multi-body system of vehicles. Then congestion in the traffic flow is regarded as a sort of collective motion of vehicles. The generation of traffic congestion can be understood as a phase transition induced by the nonlinear effect of dynamical equations of motion. Our interest exists in this kind of spontaneous transition from flow without congestion to that with congestion. In order to study this phase transition we must discuss the structure of all the vehicles' motion.

In earlier works on traffic dynamics [1] [2] [3], the dynamical equations of most models have been essentially first order differential equations with respect to time. However, the solution of this differential equation shows slightly different character from the realistic behavior of vehicles. This is why the model never show instabilities unless the authors take into account the time lag of the driver's response.

In our previous paper [4] we presented a very simple but realistic model of traffic dynamics which induces spontaneous traffic congestion. Our model accounts for the effect of time lag through second order differential equations based on the equation of motion in physics. Because of this feature, we can make a perturbative analysis and discuss the stability easily. Also, the numerical simulation shows that traffic congestion evolves spontaneously with time in our simple model.

In this paper we study further the characteristic properties of traffic congestion in our model. We concentrate on two problems. One is the stability of the structure of congestion, and the other is the organization process of congestion.

For the purpose of investigating these problems, we analyze the movement of each vehicle by plotting in the plane of headway Δx_n and velocity $\dot{x}_n$. We shall hereafter call this plane $(\Delta x_n, \dot{x}_n)$ "phase space". The combination of these two variables present the detailed structure of the movement of the flow of traffic sensitively.

After making a quick review of our model of traffic dynamics in Section 2, we introduce the hysterisis loop in the phase space in Section 3. In Sections 4 and 5, we discuss the structure stability of congestion and the hysterisis loops as a limit cycle, respectively. Section 6 is devoted to the dynamical behavior of clusters of congestion. Summary and discussions are made in Section 7.

2. Model

Let us make a brief review of our dynamical model of traffic proposed in our previous paper.

We investigate the situation where vehicles move on a single lane circuit with no passing. Here, we ignore the length of vehicle and consider the case in which all the drivers react to a stimulus in the same way. Throughout this paper, we take the periodic boundary conditions for the vehicles: N vehicles move on the circuit with length L, and the $(N+1)$th vehicle is identical to the first vehicle.

So far as we consider the case where a sufficiently large number of vehicles, N, have a sufficient length, L, we can ignore the effect of the boundary condition, as will be discussed in Sections 4 and 7. So, the system under consideration can be represented by the system in which the vehicles move on a circuit.

Each driver controls the velocity in order to maintain safety and responds to a given stimulus, which is expressed in terms of the acceleration. In practice, a driver actually has direct control of this quantity [2]. Stimuli may be functions of velocity, headway or the relative velocity of two successive vehicles. We assume a driver responds to a stimulus only from the vehicle ahead of him. So generally the equation of motion is expressed as

$$\ddot{x}_n = F(\dot{x}_n, \Delta x_n, \Delta \dot{x}_n), \tag{1}$$

where

$$\Delta x_n = x_{n+1} - x_n, \tag{2}$$

for each vehicle $\sharp n$ $(n = 1, 2, \dots, N)$. N is the total number of vehicles, and x_n is the coordinate of the nth vehicle. The dots denote differentiation with respect to time t.

There are two major types of theories for regulations. The first type is based on the idea that each vehicle must maintain the legal safe distance behind the vehicle in front, which depends on the relative velocity of these two vehicles [2]. The other is that each vehicle moves at the legal velocity, which depends on the distance between the vehicles. Our model uses the latter idea. We define the legal velocity V: each driver must control acceleration in such a way that the velocity is maintained at the postulated value according to the head way. So, this legal velocity is assumed to be the function of the distance Δx between the vehicles, having the following properties:

1) a monotonically increasing function,
2) $|V(\Delta x)|$ has an upper bound. $V^{\max} \equiv V(\Delta x \to \infty)$.

Then, the driver must maintain this legal velocity according to the motion of preceding vehicle. The dynamical equation of the system is thus expressed as

$$\ddot{x}_n = a\left\{V(\Delta x_n) - \dot{x}_n\right\}, \tag{3}$$

where a is a constant representing the driver's sensitivity, which has been assumed to be independent of n.

The stability is investigated by analyzing the deviations from the steady state flow of the following solutions of equation (3).

$$x_n^{(0)} = bn + ct, \tag{4}$$

with

$$b = L/N, \quad c = V(b), \tag{5}$$

in which vehicles are uniformly distributed with identical car spacing b and move with the same constant velocity c.

Let y_n be a small deviation from the steady state flow $x_n^{(0)}$, i.e.,

$$x_n = x_n^{(0)} + y_n. \tag{6}$$

Then, the linearized equation is obtained as

$$\ddot{y}_n = a\left\{f \cdot \Delta y_n - \dot{y}_n\right\}, \tag{7}$$

where f is the derivative of V at b,

$$f = V'(b). \tag{8}$$

The stability criteria for this steady state have been obtained in the previous paper, and are summarized as follows.

i) $f < \frac{a}{2}$; the state is stable,

ii) $f = \frac{a}{2}$; the state is marginal,

iii) $f > \frac{a}{2}$; the state is unstable.

As a realistic model we take the following function for the legal velocity

$$V(\Delta x) = \tanh(\Delta x - 2) + \tanh 2, \tag{9}$$

in which a driver controls the vehicle gradually accelerating or breaking in such a way that it never passes the preceding vehicle. This model gives the expected behavior of traffic flow, and generates spontaneous congestion instead of accidents.

We will take $a = 1$ in the numerical analysis. This assumption, however, does not mean the restriction of our model because a can be absorbed by the redefinition of $V(\Delta x)$ and the rescale of time t. Therefore our model is specified by the choice of legal velocity function $V(\Delta x)$. The simulation was made by setting the initial small disturbance in such a way that vehicles move according to solution (4) except the one which shifts with 0.1 unit length ahead from this solution:

$$\begin{aligned} x_1(0) &= x_1^{(0)} + 0.1, \\ x_n(0) &= x_n^{(0)} \text{ for } n \neq 1, \\ \dot{x}_n(0) &= 0. \end{aligned} \tag{10}$$

Here we took the parameters $N = 100$ and $L = 200$. A typical result of traffic congestion induced by this model is demonstrated in Figure 1, where the positions of all vehicles on the circuit are indicated with time development; (x_n, t).

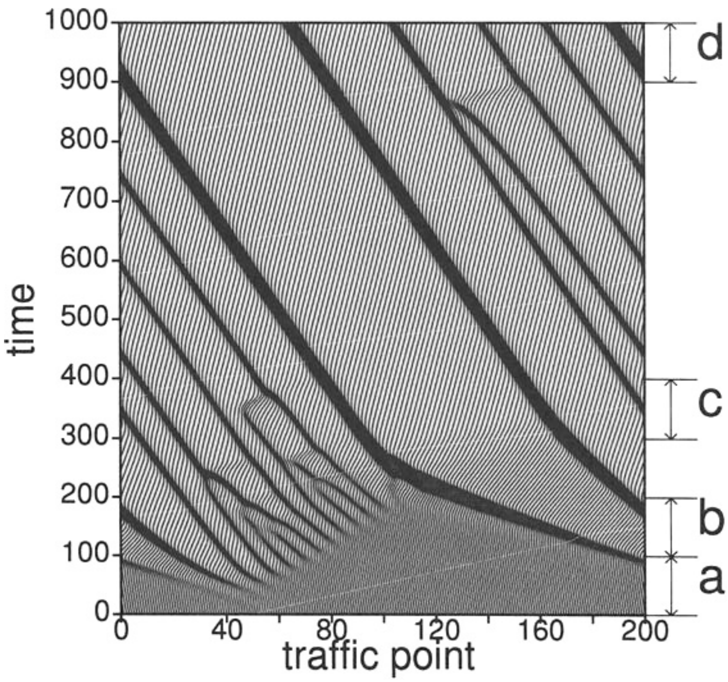

Fig. 1. Typical result of traffic congestion induced in our model. Plots of the positions of all vehicles on the circuit with time development (x_n, t). The parameters are taken as $N = 100$ and $L = 200$.

3. Hysterisis Loop

From Figure 1, we can observe the global feature of the growth of congestion. However, the mechanism and the process of the movement of each vehicle during the organization of congestion is not yet clearly understood. In order to see this, we plot the vehicle's movement in the phase space of the headway and velocity, $(\Delta x_n, \dot{x}_n)$ with time development. Figures 2 show the patterns of all the vehicle points in this phase space at every 100 steps of the time interval during the organization of congestion.

At $t = 0$, points of all the vehicles except one (initial tiny disturbance) are concentrated at the point $(\Delta x_n, \dot{x}_n) = (b, c)$, which is taken as $(200/100 = 2.00$, $\tanh 2 = 0.964)$, in the present case. As time goes on, the points of the vehicles begin

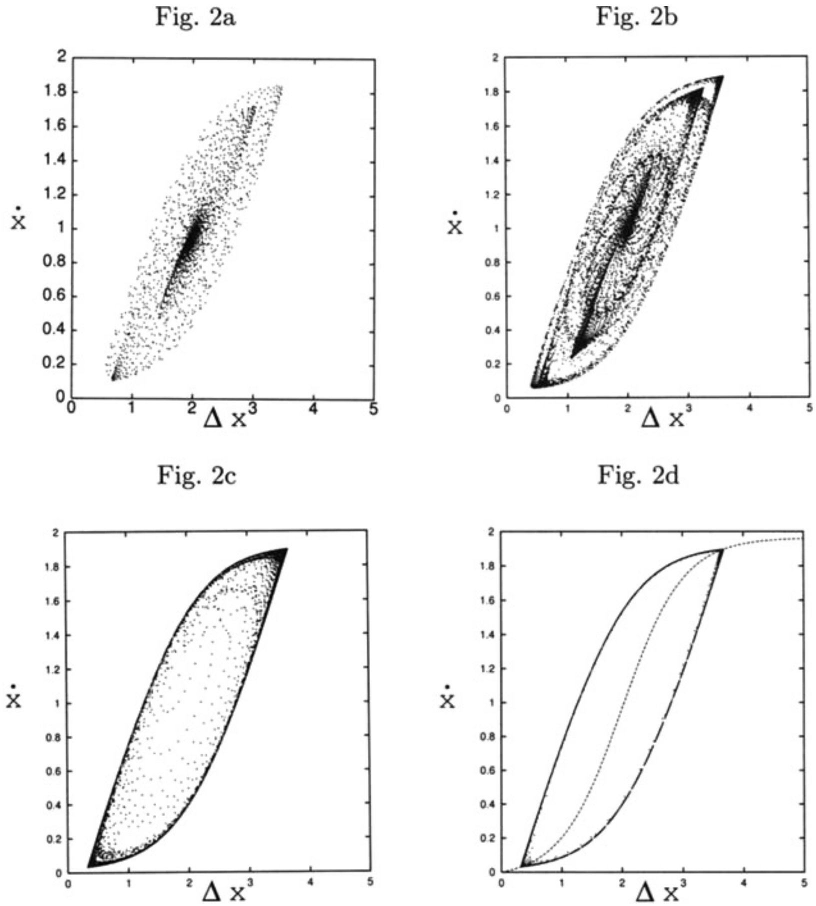

Fig. 2. The patterns of all the vehicle points in the phase space corresponding to Figure 1, during the organization of congestions.
Fig. 2a: All the vehicle points of first 100 steps,
Fig. 2b: those of 100th to 200th steps,
Fig. 2c: those of 200th to 300th steps,
Fig. 2d: those after 700th step.

to move away from this point. All the vehicle points of first 100 steps are plotted in Figure 2a, those of 100th to 200th steps in Figure 2b, 200th to 300th steps in Figure 2c, and finally (after the 700th step) they are located in a closed curve as shown in Figure 2d. The reader can recognize the process of the organization by referring to the corresponding pattern in Figure 1. Several closed curves are observed in the evolution of congestion. Each closed curve has two end points; one has large headway with large velocity (the upper point), and the other has small headway with small velocity (the lower point). The upper one corresponds to the region where vehicles are moving smoothly, and the lower one, to congestion. Closed curves recognized in Figures 2b, 2c represent several kinds of congestion with different car densities. For example, in the intermediate stage, $t = 200 \simeq 400$, Δx_n takes values $0.3, 0.5$ and 1.2 for the lower end points of quasi closed curves. These closed curves exist temporally. Figure 2d shows the final stage of the organization of congestion. After sufficient time, temporal loops disappear and the vehicle points approach the specific closed curve and at last all vehicles move along this closed curve.

We call this closed curve the "hysterisis loop" of congestion. The gate time that vehicles begin to move along this loop coincides with the time around 700 steps when the congestion formation is completed. Two cusp points of this hysterisis loop represent two areas in the circuit. The upper point $(\Delta x_n, \dot{x}_n) = (3.677, 1.896)$ represents the region of low concentration in the circuit, where vehicles move smoothly. The lower one $(\Delta x_n, \dot{x}_n) = (0.323, 0.032)$ corresponds to the congested region. Almost all vehicles exist at one of these points. Since these two points satisfy the legal velocity function (9), they exist right on the legal velocity curve (dashed line). Two curves between two cusp points represent the movement of vehicles in the transitional region between high and low concentrations. The upper curve shows the motion of vehicles entering into congestion from the smoothly moving area, that is, vehicles moving at the largest velocity are made to slow down to the smallest one along this curve. On the other hand, the lower curve corresponds to the motion of vehicles leaving a congested region and going into the smoothly moving area. Vehicles with the smallest velocity are gradually accelerated to catch up to the largest one along this curve. We note that vehicles are changing their velocity along these two curves, not along the legal velocity function (9). The decelerating vehicle has larger velocity than the corresponding legal velocity $V(\Delta x)$. On the contrary, the accelerating vehicle has smaller velocity than the legal one. This discrepancy indicates that the acceleration and deceleration are delayed. Even if a driver wants to change the velocity to the legal velocity for a given headway, the accelerating or breaking force cannot affect the velocity so quickly. This is the most characteristic feature of our model, which is easily understood from the fact that our model is formulated as a differential equation of the second order derivative with respect to time. This kind of equation is most familiar in the standard equation of motion in physics, which guarantees that vehicles must obey the law of inertia. This is the main reason that our model generates congestion spontaneously. In short, the relation of inertia and the sensitivity of drivers in the background of the postulated legal velocity curve, induces the congestion and makes it stable.

In Figure 1, we have observed 4 clusters of congestion in the final stage. The figure of this hysterisis loop (Figure 2) indicates each cluster of congestion has the same specific loop, independently of cluster size of congestion. It is interesting that a kind of universal property of congestion is observed in our model.

Most vehicles on the circuit in Figure 1 are located at the two cusp points of the hysterisis loop in Figure 2d, leaving the other few on the curves between the two cusp points. In other words, if one observes the movement of a specific vehicle, it is almost always around one of the two cusp points, and it moves very quickly between the congestion and free points along the hysterisis curve. After enough time, the congestion and smoothly moving regions have their specific uniform densities of vehicles, and the traffic flow always preserves these densities. This is a reflection of the balance between the number of vehicles moving into a congestion cluster from a smoothly moving flow and the number of vehicles moving into a smoothly moving area from a congestion cluster. The equation of motion guarantees to keep this balance, by forcing the drivers to control velocity in such manner as described by the hysterisis loop.

4. Structure Stability of Congestion

Let us study the characteristics of congestion. In this section, we discuss the stability of the "total cluster size" of congestion: the total number of vehicles located in the congested region. For the purpose of this study, we examine the organization of congestion with various initial conditions with the total number of vehicles $N = 100$ and the circuit length $L = 200$ unchanged. The results will imply that the total cluster size of congestion in the final stage is independent of the initial distribution of vehicles.

Here, we denote the total cluster size by N_C. N_C has been 50 with the headway having universal value 0.334 in the congested region, which can be seen in Figures 1 and 2. This is the typical case in the previous section.

First, we simulate the organization of congestion by taking various initial clustering patterns with the same total size of clusters $N_C = 50$. The results are shown in Figures 3 and 13.

In Figure 3, the congestion is initially set to be gathered to a single cluster with the total size, $N_C = 50$. In this case one cluster of congestion is stable, that is, preserving its total cluster size constant. The cluster never splits. On the other hand, Figure 13 corresponds to the condition in which congestions are initially scattered into several clusters. With time, we see the absorption and disappearance of clusters, but the total cluster size N_C in the final stage is still the same and stable as in the previous cases. Thus we may conclude that the total cluster size is stable, and does not depend upon initial conditions as long as the total number of vehicles N and the circuit length L are fixed.

Next, we examine the case for different values of the cluster size. We show the results in Figures 4 and 5.

In the case (Figure 4) where the initial cluster size is larger than 50 (say

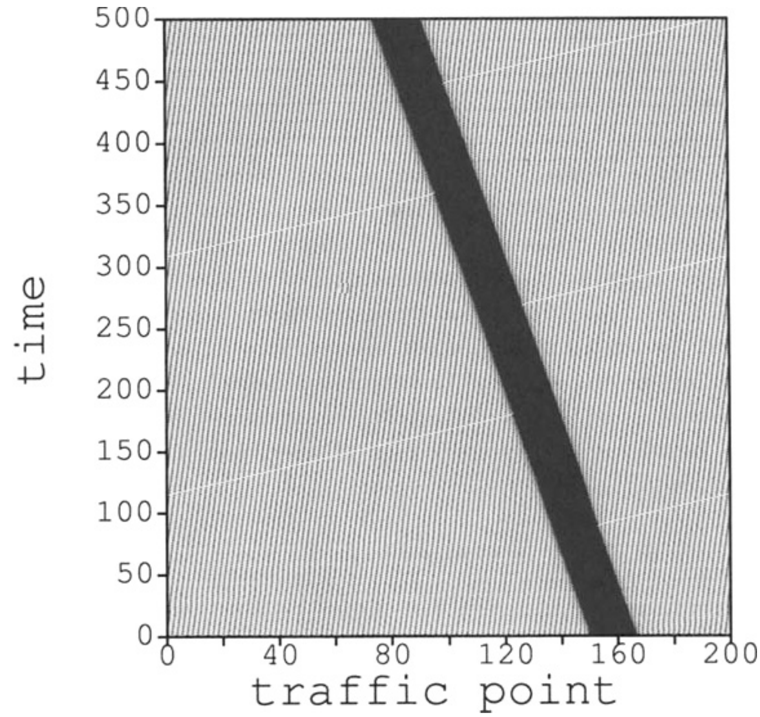

Fig. 3. Plot of the positions of all vehicles on the circuit with time development (x_n, t) for the case of $N = 100$ and $L = 200$. The organization of congestion by taking an initial clustering pattern gathered to a single cluster with the total size, $N_C = 50$. The simulation indicates that the total cluster size is stable and in the final stage becomes $N_C = 65$.

$N_C = 70$), the cluster first becomes thin, then grows fat and finally it reduces to 50, which is the same as the previous cases. The process is well described by the plots of all vehicles on the phase space $(\Delta x_n, \dot{x}_n)$ in Figure 6a (first 100 steps), Figure 6b (100th-200th steps) and Figure 6c (after 300th step: final stage). In the beginning, vehicles in congestion exist at the bottom end point $(\Delta x_n, \dot{x}_n) = (0.323, 0.032)$. On the other hand, the other vehicles are located at the top end point $(\Delta x_n, \dot{x}_n) = (5.000, 1.959)$, in which the headway is larger than that of the smooth moving region of the final state.

For the first stage (Figure 6a), the vehicles with velocity $\dot{x}_n = 1.959$ enter the congested region with larger velocity, tending to reach the legal velocity (the dashed line) which temporarily creates congestion with a higher density than the initial one. Then, this high density congestion gradually evaporates, creating a smooth moving region and a region of congestion with normal density grows (Figure 6b), approaching to the final stage (Figure 6c). This is easily understood from the fact

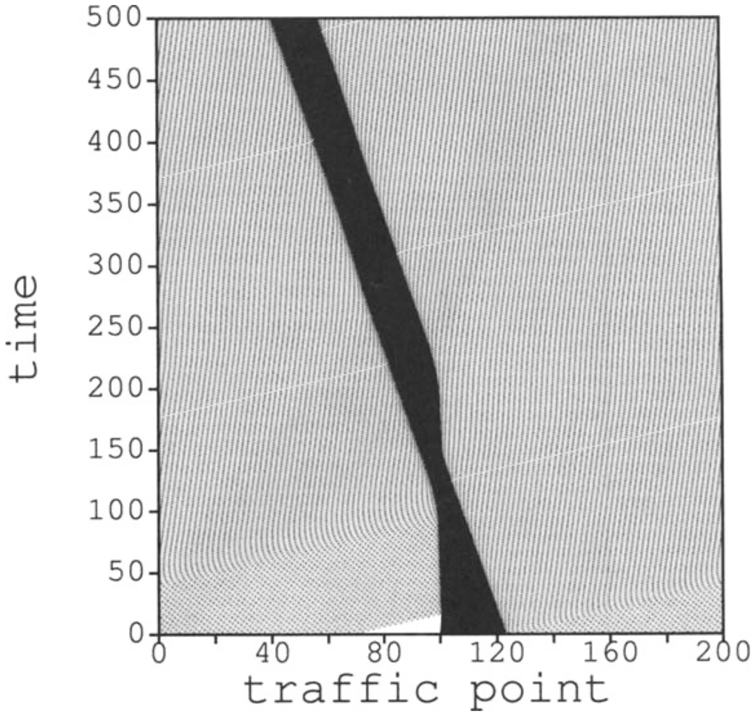

Fig. 4. Plot of the positions of all vehicles on the circuit with time development (x_n, t) for the case of $N = 100$ and $L = 200$. We take the initial clustering pattern gathered to a single cluster with the total size, $N_C = 70$. The simulation indicates that the total cluster in the final stage becomes $N_C = 50$.

that the curves in the initial stage, are outside of the final shape of the hysterisis loop (Figure 6d).

Let us consider the smaller N_C case (say $N_C = 30$, Figure 5). This time, the cluster size first becomes larger with lower density, and then becomes smaller. Finally, it approaches to the final cluster size $N_C = 50$, which is also the same size as the previous cases. We can observe this process in more detail in the distributions of all the vehicles plotted on the phase space in Figure 7. Initially, the vehicles in the cluster of congestion exist at the bottom end point $(\Delta x_n, \dot{x}_n) = (0.323, 0.032)$, with the other vehicles being at the middle point $(\Delta x_n, \dot{x}_n) = (2.600, 1.501)$, where the headway is smaller than that in the smooth moving region of the final state. During the time interval of $50 \sim 300$ steps, the headway in the congestion becomes larger (Figure 5). This temporary congestion is presented by the inner hysterisis curve of Figure 7b. After 300 steps, the headway of the congestion region is reduced to 0.032 and vehicles approach the final hysterisis loop (Figure 7c), forming exactly the same hysterisis loop as in the previous cases.

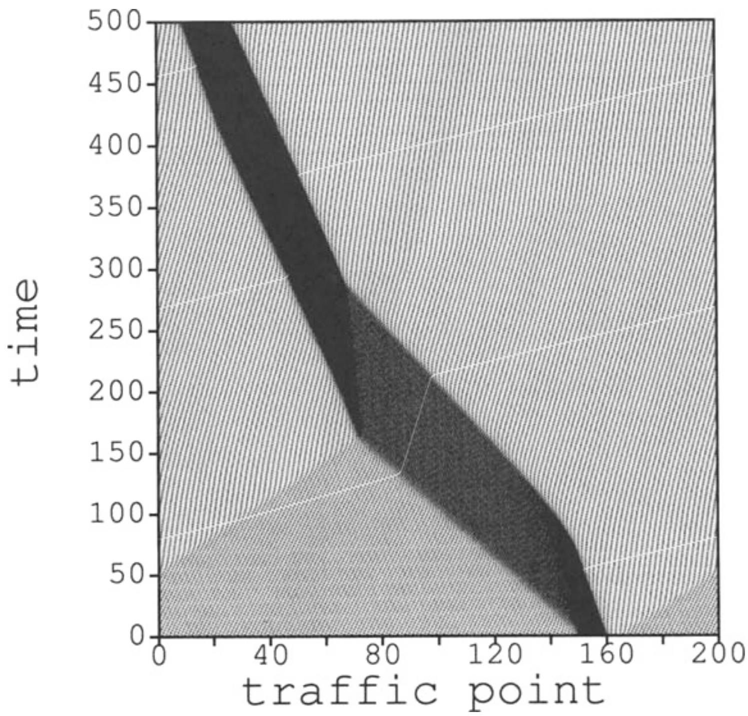

Fig. 5. Plot of the positions of all vehicles on the circuit with time development (x_n, t) for the case of $N = 100$ and $L = 200$. We take the initial clustering pattern gathered to a single cluster with the total size, $N_C = 30$. The simulation indicates that the total cluster in the final stage becomes $N_C = 50$.

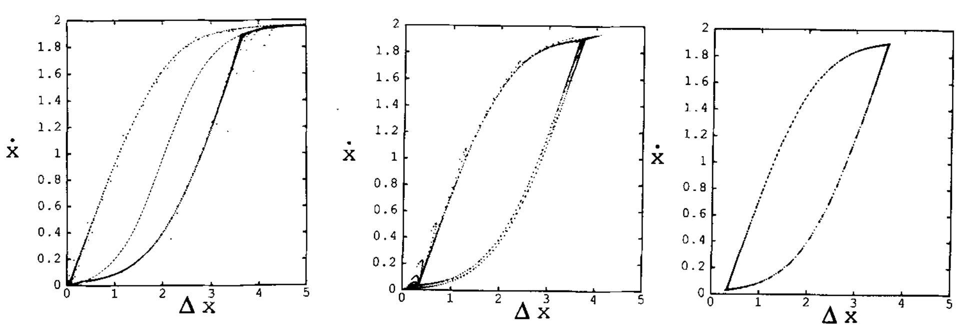

Fig. 6. The patterns of all the vehicle points in the phase space corresponding to Figure 4 during the organization of congestions.
Fig. 6a: All the vehicle points of first 100 steps,
Fig. 6b: those of 100th to 200th steps,
Fig. 6c: those after 400th step.

Fig. 7a Fig. 7b Fig. 7c

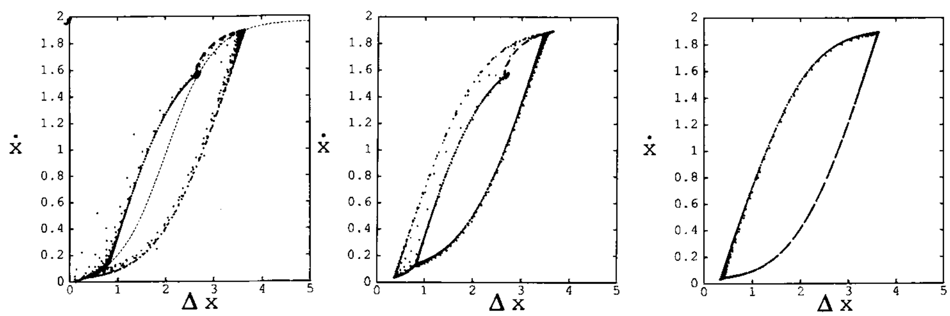

Fig. 7. The patterns of all the vehicle points in the phase space corresponding to Figure 5, during the organization of congestions.
Fig. 7a: All the vehicle points of first 100 steps,
Fig. 7b: those of 100th to 200th steps,
Fig. 7c: those after 400th step.

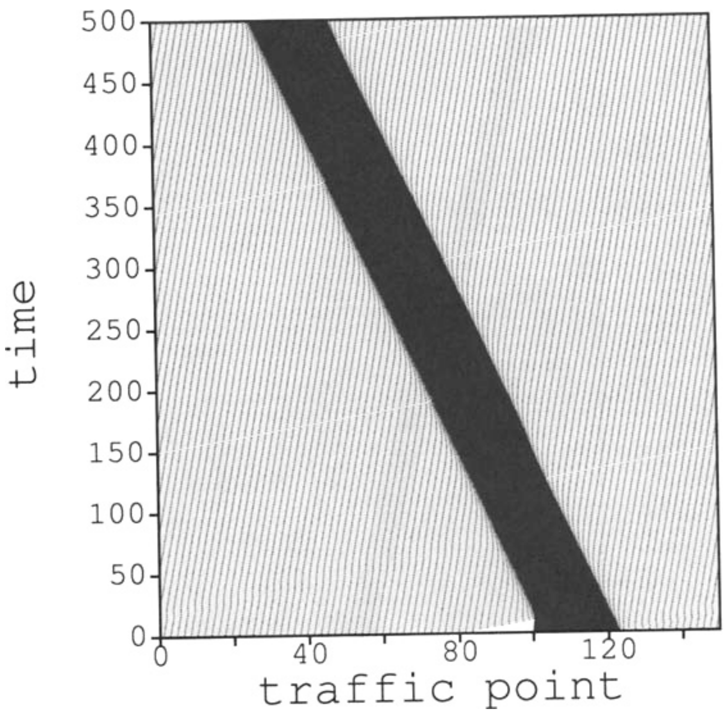

Fig. 8. Simulation with setting $N = 100$, $L = 150$. The simulation indicates that the total cluster size (initially set $N_C = 70$) in the final stage becomes $N_C = 65$.

We can thus confirm the stability of the total cluster size of congestion; the total cluster size N_C of the final stage is 50 independently of initial clustering. The above two typical cases have shown that all the vehicle points approach the unique hysterisis loop with the top and bottom cusp points $(\Delta x_n, \dot{x}_n) = (3.677, 1.896)$ and $(0.323, 0.032)$, respectively. This result is independent of whether the starting points are inside or outside of the hysterisis loop. Later we will see that the trajectory of one vehicle in this space $(\Delta x_n, \dot{x}_n)$ looks like that of a limit cycle in phase space.

We have seen in the several cases with fixed $N = 100$ and $L = 200$ that the total cluster size of congestion goes to 50 in any initial clustering conditions. This result indicates that the total cluster size is constant for fixed N and L independent of the initial conditions. That is, this size is determined merely by the number of vehicles N and the length of circuit L. Then, to investigate how the total cluster size is determined, we make simulations by setting various N and L, in the following two cases; $N = 100$, $L = 150$ and $N = 100$, $L = 250$. The numerical simulation indicates $N_C = 65$ in the first case (Figure 8) and $N_C = 35$ in the second case (Figure 9).

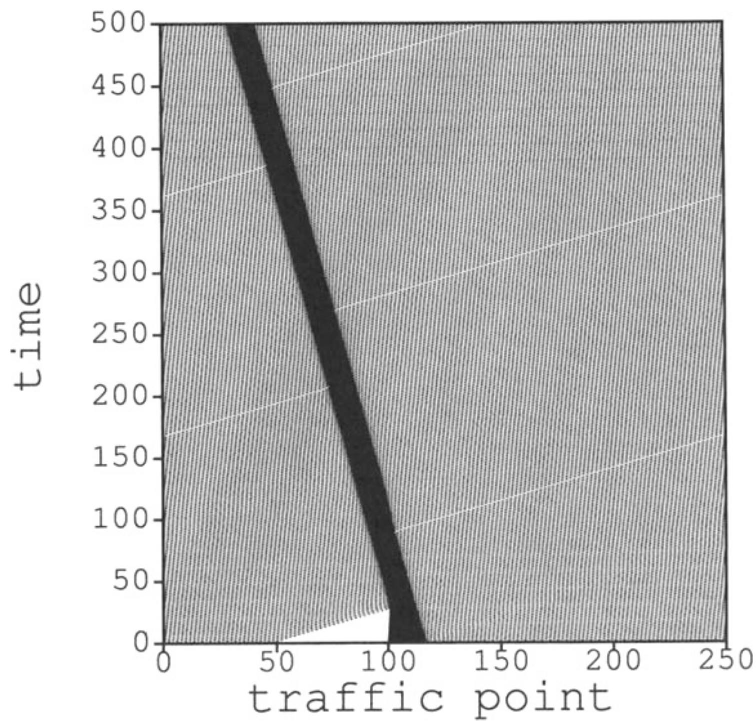

Fig. 9. Simulation with setting $N = 100$, $L = 250$. The simulation indicates that the total cluster size (initially set $N_C = 50$) in the final stage becomes $N_C = 35$.

We have found a remarkable result for the hysterisis loop of congestion, the shape of which is universal whatever values of N and L are taken and, of course, does not depend on initial conditions of clustering, which can be read off clearly from plots of the vehicles flow in the phase space corresponding to Figures 8 and 9. The vehicle points approach the same hysterisis loop as in previous typical cases and move along this loop.

This specific loop is determined only by the freedom of the dynamical equation; sensitivity a and legal velocity function $V(\Delta x)$. In practice this loop expresses the local balance of the intermediate region between high and low concentrations. This universality of hysterisis is independent of any other conditions. Consequently, $\Delta x_{\min} = 0.323$ (headway of the lower cusp point of the hysterisis loop) and $\Delta x_{\max} = 3.677$ (that of the upper cusp point) is determined universally.

With the help of this result, the total cluster size of congestion N_C is easily estimated from the following rough relation:

$$\begin{cases} \Delta x_{\min} N_C + \Delta x_{\max} N_F \simeq L, \\ N_C + N_F \simeq N, \end{cases} \tag{11}$$

where $\Delta x_{\max}$ and N_F are the headway and the sum of vehicles in the free flow region. $\Delta x_{\min}$ and N_C are those in the congestion region. We note that $\Delta x_{\max}$ and $\Delta x_{\min}$ are universal. Almost all the vehicles on the circuit are positioned at the top and the bottom cusp points of the hysterisis loop. In the above rough estimation, most of the contributions come from these two points. Exact relation of above variables are determined by taking into account the contribution of the residual vehicles on this loop. The equation (11) implies that the ratio of N_C and N_F is determined only by $\Delta x_{\min}$, $\Delta x_{\max}$ and $b = L/N$, the inverse of average density of vehicles.

$$N_C/N_F \simeq -\frac{b - \Delta x_{\max}}{b - \Delta x_{\min}}. \tag{12}$$

The above structure of congestion does not depend on the length of the circuit. Thus, we can confirm the justification for adopting the periodic boundary conditions of the circuit of traffic and the shape of the hysterisis loop of the final traffic flow is invariant, even if we take the infinity limit of the circuit length with the ratio L/N preserved constant. The universality of hysterisis loop guarantees the stability of total cluster size of congestion.

Let us see whether these equations (11) or (12) are available in the two cases in Figures 8 and 9. The calculated values are compared with those using N_C and N_F from simulations (Table 1).

On the other hand, the number of clusters is highly dependent on initial conditions. This has already been understood from the results shown in Figures 1, 3 and 13. In each case the congestion is stable, but there are different numbers of clusters. In the previous paper, where all vehicles move with the same constant initial velocity and identical spacing, 4 clusters of congestion have been induced as already seen in Figure 1. But it is not yet clear whether or not there exists any relation between the number of clusters and this initial condition.

Table 1.

	N_C/N_F (simulation)	$-\frac{b - \Delta x_{\max}}{b - \Delta x_{\min}}$ (estimation)
i) $N = 100, L = 150$	$\frac{65}{35} = 1.857$	$-\frac{1.5 - 3.677}{1.5 - 0.323} = 1.850$
ii) $N = 100, L = 250$	$\frac{35}{65} = 0.538$	$-\frac{2.5 - 3.677}{2.5 - 0.323} = 0.541$

5. The Hysterisis Loop as a Limit Cycle

The hysterisis loop in phase space $(\Delta x_n, \dot{x}_n)$ indicates the existence of an interesting phenomena in dynamical systems. After sufficient time, we observe that trajectories of vehicles converge to a unique closed loop. In some cases, values of $(\Delta x_n, \dot{x}_n)$ are initially set outside the hysterisis loop, while in other cases, initial points are set inside this loop. The results in the figures show us that whenever initial positions are set inside or outside the hysterisis loop, all vehicles are attracted to this hysterisis loop, forming a spiral loop around it. So, the trajectories of vehicles are separated by this closed loop into two domains. This behavior seems to be similar to that of the limit cycle commonly observed in nonlinear dynamical systems. In order to study the properties of this kind, it is necessary to investigate the orbit of one vehicle in this space. Obviously, the orbit of one vehicle coincides with the hysterisis loop finally after the congestion has formed completely. For the purpose of investigating the motion in the whole stage during the organization of congestion, we pick up data about the movement of one vehicle.

Since the organization process is most prominently seen in the case of Figure 1, this is the case we use here. Figures 10a $\sim$ 10d show the process of organization of congestion for the orbit of one vehicle at four stages. Roughly speaking, one loop of the trajectory indicates that this vehicle passes through one cluster of congestion. In the first 200 steps (Figure 10a) and the second 200 steps (Figure 10b) the trajectory has about 10 and 5 loops, with the elliptical curves. These loops correspond to rather obscure clusters of congestion, whose densities of vehicles do not show a sharp contrast to those of non-congestion regions. On the contrary, a loop with upper and lower cusp points indicates the existence of a clear congestion region. Drawing several loops as the vehicle passes clusters of congestion, the orbit gradually becomes larger and larger (Figure 10b) to reach the hysterisis loop finally (Figures 10c and d).

In the case of Figures 10, we have set the initial condition in the inner domain of the hysterisis loop. This initial point is unstable and the flow of the vehicle point moves repulsively from this point becoming larger and larger to reach the attractive hysterisis loop. It is easy to check that if we set the initial point outside the hysterisis loop, the flow of a vehicle moves attractively to the hysterisis loop from the outer domain of this loop. Anyway, the flow of vehicles moves attractively to this hysterisis loop, whether starting positions are inside or outside this loop. Finally, a flow begins to move along the hysterisis loop and maintain this behavior. From the above analysis, we can regard this hysterisis loop as a kind of a "limit cycle" observed in the usual dynamical systems.

Before closing this section, we show the distributions of vehicles of one cluster, concentrating on to a specific one among 4 clusters. We choose here the most noticeable congestion cluster of Figure 1. Picking up the vehicles located in this

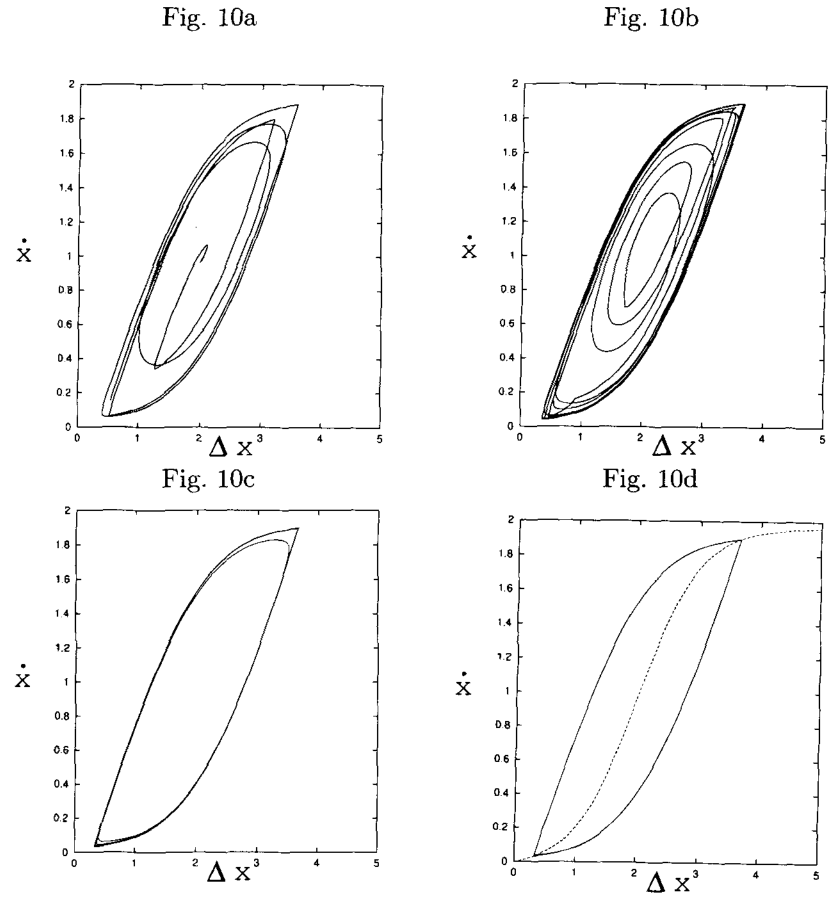

Fig. 10. The orbit of one vehicle is shown dividing into four stages. The sample vehicle is picked up from the vehicles in Figure 1.
Fig. 10a: The trajectory of the first 200 steps,
Fig. 10b: that of 200th to 400th steps,
Fig. 10c: that of 400th to 600th steps,
Fig. 10d: that after 800th step.

congestion region (Figure 11), we plot all the vehicle points in the phase space, dividing the organization into 6 stages. Each stage of Figure 12 is indicated by an arrow in Figure 11. This demonstrates clearly how all the points expand uniformly to the final attractive hysterisis loop.

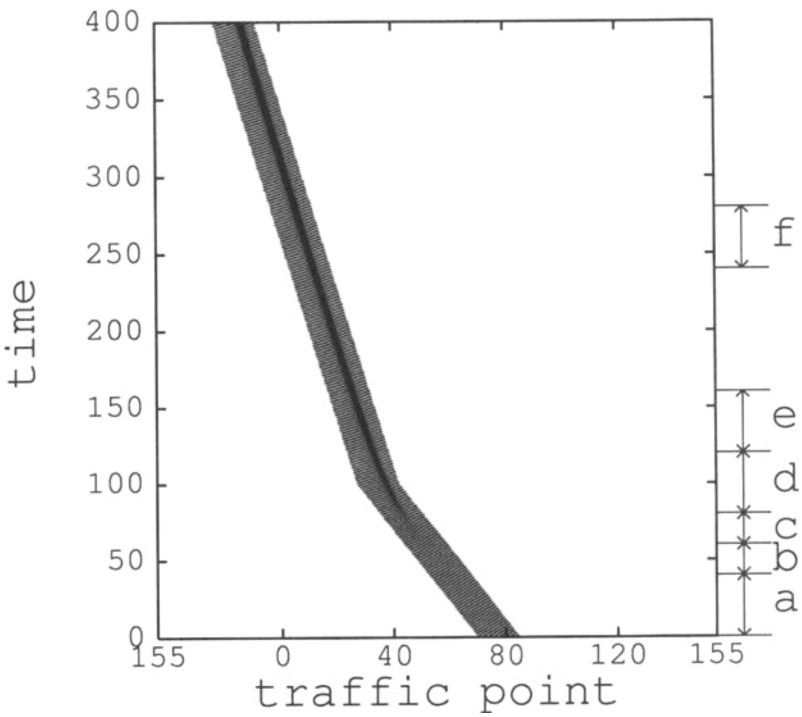

Fig. 11. A specific congestion cluster is chosen as a sample among 4 stable clusters in Figure 1.

6. The Dynamical Behavior of Clusters of Congestion

In this section, we want to investigate the instability of a cluster of vehicles and discuss some properties of dynamical behavior of clusters, such as combination, absorption or disappearance of clusters observed in the intermediate stage of forming the final stable structure of congestion. For this purpose we initially set several sizes of clusters with several distances on the circuit. Figure 13 is the result of a demonstrative simulation. Clusters are designed to have 2, 3, 4 and other numbers of vehicles with distances of 3 or 7 successive vehicles. The headway in each cluster of congestion and in the smoothly moving area are set on the maximum and minimum values of the hysterisis loop. We can observe several typical behaviors of clusters from the result of simulation.

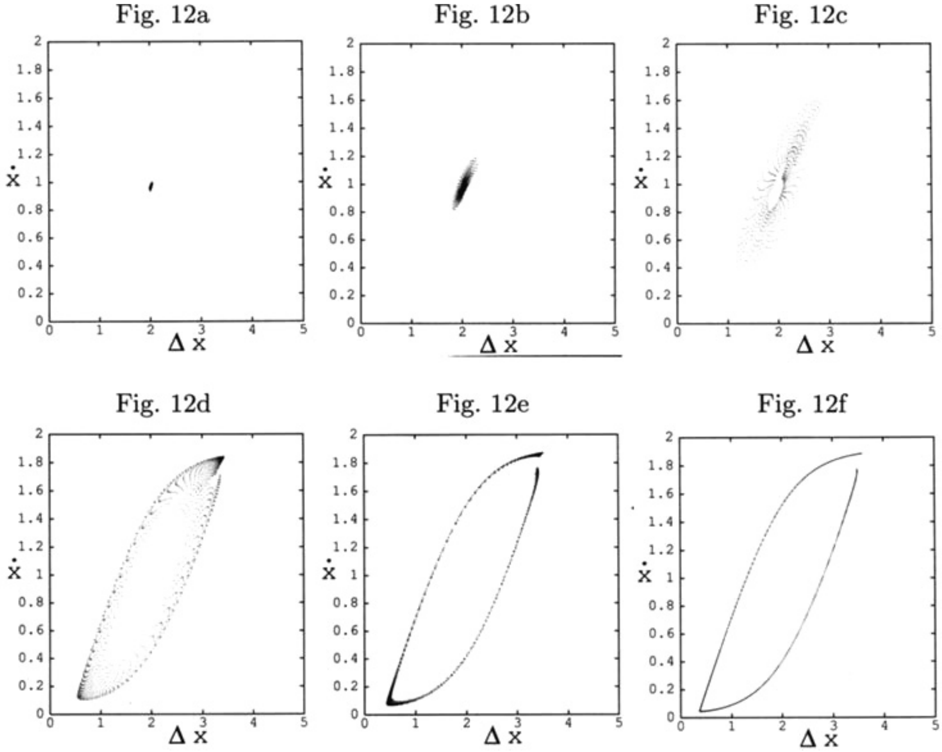

Fig. 12. Plots of all the vehicle points in the phase space, dividing the organization into 6 stages. Each stage of Figure 12 is indicated by an arrow in Figure 11.

First, we can recognize the instability of a cluster. The isolated cluster with the number of vehicles less than 3 (the first and second ones in Figure 13) is unstable, and that with more than 4 vehicles is stable. A cluster of less than 3 vehicles is not big enough for a vehicle moving into this cluster to reach the headway $\Delta x_{\min}$ of stable congestion before the preceding vehicle accelerates and exits the cluster. Therefore such small clusters cannot maintain the headway of stable congestion which is determined from the hysterisis loop. As a result, the congestion becomes broad and disappears into the smoothly moving area.

Next, in Figure 13, we can observe be observed the phenomena where two neighbor clusters (4th and 5th ones) are combined as if there exists an attractive interaction. The length of the effective range of this interaction is the distance of 3 successive vehicles with the headway of $\Delta x_{\max}$. This effective range does not depend on the size of the cluster. It is found that two clusters are stable and move independently outside this range. This 'attractive force' acts on the preceding cluster and pulls it backward. Therefore a cluster is always absorbed by the following cluster, independently of whether the cluster is larger or smaller than the following one. Actually, the 5th cluster having 23 vehicles is absorbed by the 4th cluster of 7 vehicles. This phenomena is explained as follows. In the case where the distance

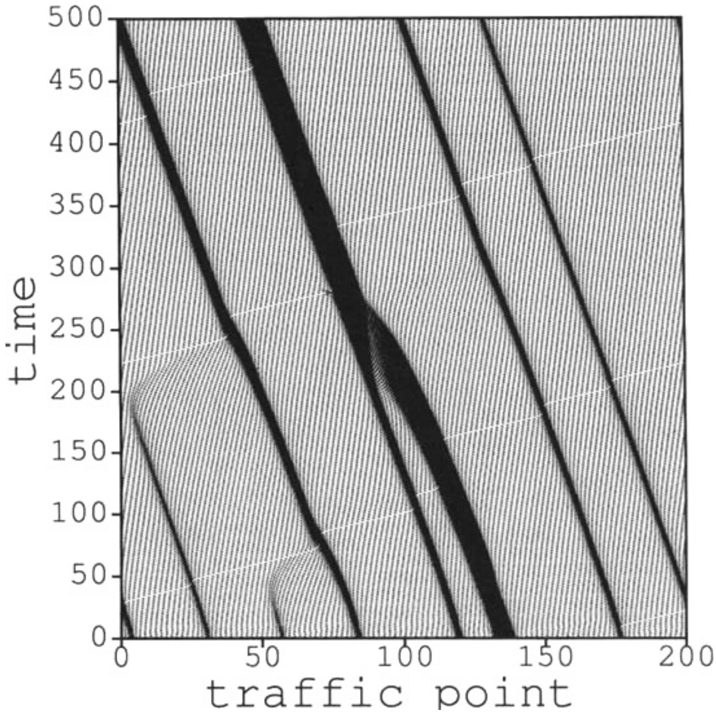

Fig. 13. The organization of congestion by taking an initial clustering pattern scattered into several clusters. Clusters are designed to have 3, 2, 6, 7, 23, 5 and 4 numbers of vehicles with distances of 7, 7, 7, 9, 3, 10 and 7 successive vehicles.

between two clusters is too small, a vehicle moving out of the second cluster can not accelerate up to the velocity $V(\Delta x_{\max})$ before reaching the first cluster. Therefore the hysterisis loop drawn by the motion of this vehicle becomes smaller than that of stable congestion, that is, the vehicle makes a congestion with a larger headway. Then, the density of the first cluster reduces and as the result, the cluster is stretched backwards. The distance becomes shorter and shorter, and finally, the first cluster is absorbed into the second cluster.

On the contrary, an unstable cluster disappears and is absorbed into a cluster ahead of it. We can distinguish the disappearance of cluster from the absorption of a cluster by the attractive interaction. These are two processes of the combination of two clusters in the intermediate stage of forming the stable clusters of congestion.

7. Summary and Discussions

We have presented an extremely simple but realistic model of traffic flow which induces traffic congestion naturally. We are now convinced that the dynamics of traffic flow is a collective motion problem. The evolution of traffic congestion is an appearance of this substantial property. This phenomena is a kind of phase transition induced by the nonlinear effect of dynamical equations of motion. We have investigated the characteristic properties of the traffic congestion in our model, especially the organization process and the stability of the structure of congestion. It turns out that these phenomena are well described by plotting motions of vehicles in the phase space of velocity and headway. The most remarkable feature is the universality of "the hysterisis loop" in this phase space, which is observed in the final stage of the congestion organization. This loop is understood as a limit cycle of the dynamical system. This universality guarantees the stability of total cluster size.

In earlier works on traffic dynamics, the attention of many investigators was focused on the time lag of a driver's response to the stimulus from other vehicles. Their models of traffic flow are essentially the first order differential equations with respect to time. On the contrary, our model introduces the effect of time lag through the second order differential equations based on the equation of motion in physics.

We discuss the effect of boundary conditions in our simulation results. As mentioned in Section 4, the structure of congestion does not depend on the length of the circuit. The ratio of the total number of vehicles in congestion and that of others are determined by the inverse average density of vehicles $b = L/N$ and $\Delta x_{\min}$ and $\Delta x_{\max}$ from hysterisis loop, which is highly universal and never depends on the initial and boundary conditions. Thus, we can insist that the periodic boundary condition of the circuit of traffic is not essential.

Here, we must comment on the transportation of vehicles in our model. Although the quantity should be checked against observational data, we only discuss here a comparison of the transport of the flow including congestion with that of the steady state flow with constant velocity. The capacity of transportation may be defined as the number of vehicles passing through a point per time. In our model, the capacity of transportation can be defined as follows.

After the congestion formation has been completed, the flow is steady with high and low density regions on the circuit, and all vehicles behave in the same way; a vehicle moves with the constant velocity $V(\Delta x_{\min})$ in a high density region, and $V(\Delta x_{\max})$ in a low density region. Every vehicle moves along the circuit with period T. The capacity of transportation is roughly estimated as N/T, where N is the total number of vehicles on the circuit. T is derived as follows. We must remark that there are two different periods: the period T where a vehicle moves around the circuit and the period T_0 where a vehicle passes all clusters of congestion. We have seen that the clusters move backward with the velocity V_{back} *). Using this velocity, T_0 is expressed as

$$T_0 \simeq \frac{\Delta x_{\max} \cdot N_F}{V(\Delta x_{\max}) + V_{\text{back}}} + \frac{\Delta x_{\min} \cdot N_C}{V(\Delta x_{\min}) + V_{\text{back}}}. \tag{13}$$

The relation between T and T_0 is given as

*) V_{back} is estimated as follows, $V_{\text{back}} \simeq \frac{\Delta x_{\min}}{\Delta x_{\max}} V(\Delta x_{\max}) - V(\Delta x_{\min})$.

$$T = \left(1 + \frac{V_{\text{back}}T}{L}\right) T_0, \tag{14}$$

where the difference between T and T_0 is obtained by estimating the ratio of the circuit length to the path which clusters move in the period T with the velocity V_{back}. Then, we obtain the capacity of transportation N/T. In the typical case of $L = 200$

$$N/T \simeq N\frac{-0.263N + 200.99}{389.2N - 1478.8}, \tag{15}$$

with $\Delta x_{\max}$ and $\Delta x_{\min}$ are read off from the data of hysterisis loop. Notice that the above equation is applicable for $59 \leq N \leq 606$ in case of circuit length $L = 200$. The minimum value bound means the critical value for at least one cluster of congestion to exist, which consists of more than 4 vehicles. $N = 606$ corresponds to the case where the whole circuit is occupied by clusters of congestion.

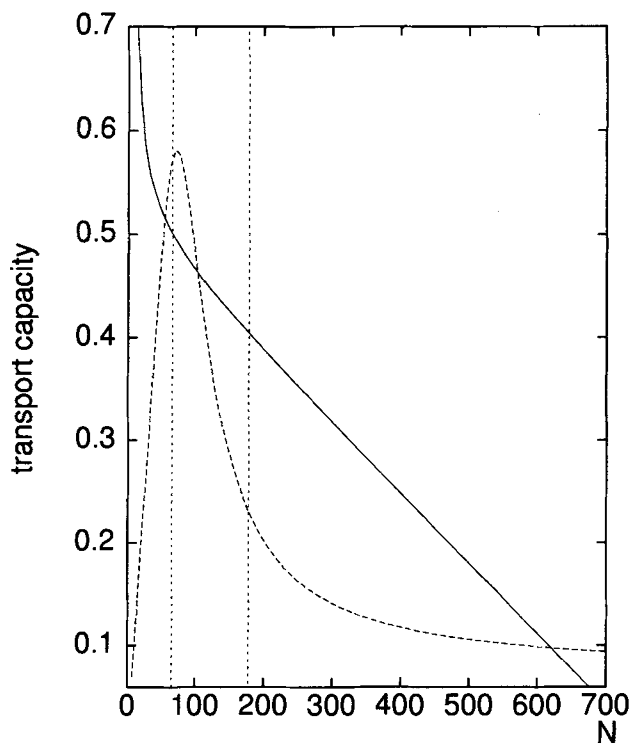

Fig. 14. The comparison of the capacity of transportation as variable N for the traffic flow with and without congestion. The solid and dashed lines denote the transport of flow with and without congestion and correspond to equations (15) and (16), respectively. The range between vertical dotted lines is unstable for the flow without congestion.

On the other hand, in the flow with no congestion, which denotes the steady flow of uniform distribution of vehicles on circuit moving with the constant velocity $V(b)$, where b is the identical headway $b = L/N$, the capacity of transportation is N/T', where $T' = L/V(b)$. In the case of $L = 200$,

$$N/T' \simeq \frac{N}{200}\left\{\tanh\left(\frac{200}{N} - 2\right) + \tanh 2\right\}. \tag{16}$$

The comparison of the capacity of transportation as variable N for two situations, equations (15) and (16) are shown in Figure 14. The solid and dashed lines denote the transport of flow with and without congestion, respectively. The uniform solution is unstable in the range between vertical dotted lines, in which the transition from free to congested flow occurs. In this case, the existence of congestion reduces the capacity of transportation in the region $N \leq 103$, and increases the capacity in the region $N \geq 104$. We would like to analyze the details of this behavior in a future paper. It must be remarked that these two curves are derived as solutions of two different phases from our model.

This feature is in sharp contrast to the other models studied before. We shall make further analysis of this point and check against observational data.

Several modifications of our simple realistic model may be interesting for further studies. We have assumed that sensitivity is independent of such variables as velocity, headway or the relative velocity of the vehicle ahead.

In earlier works, these kinds of improvements have been made extensively to make the model more realistic, which can also be done also for our model. Sensitivity may also depend on each drivers' character. This variation might be more interesting and may provide us with as yet unknown pattern formations of congestion.

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