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Structure of Positive Radial Solutions of Matukuma's Equation

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We study the structure of solutions of a semilinear elliptic equation called Matukuma's equation. This equation is a mathematical model for describing the dynamics of a globular cluster of stars. It is known that, under some conditions, there exists a solution called a positive entire solution with finite total mass. It is conjectured that the finite total mass solution is unique. In this paper the structure of positive radial solutions is made clear and an affirmative answer is given to the conjecture.

Key words: semilinear elliptic equation, uniqueness

1. Introduction

We consider a semilinear elliptic equation

$$\Delta u(x) + \frac{1}{1+|x|^2} u(x)^p = 0, \quad x \in \mathbf{R}^n. \tag{1}$$

Throughout this paper, we assume $n \geqq 3$ and $1 < p < \frac{n+2}{n-2}$. Equation (1) with $n = 3$ was proposed by Matukuma [3] as a mathematical model of a globular cluster of stars, in which $u > 0$ represents the gravitational potential and

$$\int_{\mathbf{R}^3} \frac{1}{4\pi(1+|x|^2)} u(x)^p dx$$

represents the total mass. The main purpose of this paper is to prove the uniqueness of a positive radial entire solution with finite total mass.

Any radially symmetric solution $u = u(r)$, $r \equiv |x|$, of Eq. (1) satisfies the ordinary differential equation

$$\begin{aligned} &u_{rr}(r) + \frac{n-1}{r} u_r(r) + \frac{1}{1+r^2} u(r)^p = 0, \quad r > 0, \\ &u(0) = \alpha > 0, \quad u_r(0) = 0. \end{aligned} \tag{2}$$

It was shown by Li and Ni (see Theorem 2.41 of [1]) that solutions of this equation are classified into the following three types:

Type I: $u(r)$ has a finite zero;
Type II: $u(r)$ is positive on $[0, +\infty)$ and $u(r) \sim (\log r)^{1/(1-p)}$ at $+\infty$;
Type III: $u(r)$ is positive on $[0, +\infty)$ and $u(r) \sim r^{2-n}$ at $+\infty$.

Here the notation "$v(r) \sim w(r)$ at $+\infty$" means that there exist positive constants C_1 and C_2 such that $C_1 w(r) < v(r) < C_2 w(r)$ at $+\infty$.

Any solution $u(r)$ of Type II satisfies

$$\int_0^{+\infty} \frac{1}{1+r^2} u(r)^p r^{n-1} dr = +\infty,$$

so $u(r)$ of Type II is a positive entire solution with infinite total mass, while any solution $u(r)$ of Type III satisfies

$$\int_0^{+\infty} \frac{1}{1+r^2} u(r)^p r^{n-1} dr < +\infty,$$

so $u(r)$ of Type III is a positive entire solution with finite total mass. Hereafter we simply denote a solution of Type II by an infinite total mass solution and denote a solution of Type III by a finite total mass solution.

It was proved by Ni and Yotsutani [4] that $u(r)$ has a finite zero for every sufficiently large α and that $u(r)$ is an infinite total mass solution for every sufficiently small α. Recently it was proved independently by Li and Ni [2] and Noussair and Swanson [5] that there exists at least one α such that $u(r)$ is a finite total mass solution. It is conjectured by these authors that the finite total mass solution is unique. The purpose of this paper is to give an affirmative answer to this conjecture.

Our main result is the following theorem:

THEOREM. *Let $u(r)$ be a solution of Eq.(2). There exists a unique $\alpha^* > 0$ such that*

(i) *if $\alpha > \alpha^*$, then $u(r)$ has a finite zero,*
(ii) *if $\alpha = \alpha^*$, then $u(r)$ is a finite total mass solution, and*
(iii) *if $\alpha < \alpha^*$, then $u(r)$ is an infinite total mass solution.*

Recently it was proved by Li and Ni that any bounded positive entire solution with finite total mass is necessarily radially symmetric about the origin if $p > (n-1)/(n-2)$ (see Theorem 2.2 of [2]). As a direct consequence of their result and the above theorem, we have the following corollary:

COROLLARY. *If $(n-1)/(n-2) < p < (n+2)/(n-2)$, Eq.(1) has exactly one positive entire solution with finite total mass.*

2. Properties of Solutions of Matukuma's Equation

Since we are only concerned with positive solutions, we may consider, instead of (2),

$$(3) \qquad \begin{aligned} &u_{rr}(r) + \frac{n-1}{r} u_r(r) + \frac{1}{1+r^2} u^+(r)^p = 0, \quad r > 0, \\ &u(0) = \alpha > 0, \quad u_r(0) = 0, \end{aligned}$$

where $u^+(r) \equiv \max\{u(r), 0\}$.

Solutions of this equation have the following properties:

LEMMA 2.1.

(i) *$u_r(r) < 0$ for all $r > 0$.*

(ii) *If $u(r) > 0$ for all $r \geqq 0$, then $\lim_{r\to+\infty} ru_r(r) = 0$.*

(iii) *For all $r > 0$, the following Pohozaev identity holds:*

$$r^n u_r(r)^2 + (n-2)r^{n-1}u(r)u_r(r) + \frac{2r^n}{(p+1)(1+r^2)}u^+(r)^{p+1}$$
$$= \int_0^r G(s)u^+(s)^{p+1}ds,$$

where

$$G(r) \equiv \frac{(n-2)r^{n-1}}{(p+1)(1+r^2)^2}\left\{\frac{n+2}{n-2} - p - (p-1)r^2\right\}.$$

Proof of (i). This is proved in Proposition 4.1 (b) of [4].

Proof of (ii). Equation (3) can be written as

$$\{r^{n-1}u_r(r)\}_r = -\frac{r^{n-1}}{1+r^2}u(r)^p. \tag{4}$$

Hence we have

$$ru_r(r) = -\int_0^r \frac{s^{n-1}}{1+s^2}u(s)^p ds/r^{n-2}.$$

By the l'Hôspital's rule and Theorem 2.41 of [1], we obtain

$$\lim_{r\to+\infty} ru_r(r) = -\lim_{r\to+\infty}\frac{r^{n-1}}{1+r^2}u(r)^p/(n-2)r^{n-3} = 0.$$

Proof of (iii). This is a special case of Proposition 4.3 of [4]. Using (3), we have

$$\frac{d}{dr}\left\{r^n u_r(r)^2 + (n-2)r^{n-1}u(r)u_r(r) + \frac{2r^n}{(p+1)(1+r^2)}u^+(r)^{p+1}\right\}$$
$$= \left\{\frac{2}{p+1}\left(\frac{r^n}{1+r^2}\right)_r - \frac{(n-2)r^{n-1}}{1+r^2}\right\}u^+(r)^{p+1}$$
$$= G(r)u^+(r)^{p+1}.$$

Integrating this over $[0, r)$, we obtain the desired identity.

Note that if we put

$$R \equiv \left\{\frac{(n+2)/(n-2) - p}{p-1}\right\}^{1/2} > 0,$$

$G(r)$ satisfies

$$\begin{cases} G(r) > 0 & \text{if } 0 < r < R, \\ G(r) < 0 & \text{if } R < r < +\infty. \end{cases} \tag{5}$$

LEMMA 2.2. *Let $u = \phi(r)$ be a finite total mass solution of Eq.(2). Then*
(i) $\lim_{r\to+\infty} r^{n-2}\phi(r)$ *and* $\lim_{r\to+\infty} r^{n-1}\phi_r(r)$ *exist and are finite,*
(ii) $r\phi_r(r) + (n-2)\phi(r) > 0$ *for all* $r \geqq 0$,
(iii) $\int_0^r G(s)\phi(s)^{p+1}ds > 0$ *for all* $r > 0$.

Proof of (i). This is clear from Lemma 7.1 of [4].

Proof of (ii). From (3), we have

$$\{r\phi_r(r) + (n-2)\phi(r)\}_r = -\frac{r}{1+r^2}\phi(r)^p < 0.$$

Integrating this over $[r, +\infty)$ and using $r\phi_r(r) + (n-2)\phi(r) \to 0$ as $r \to +\infty$, we obtain

$$r\phi_r(r) + (n-2)\phi(r) = \int_r^{+\infty} \frac{s}{1+s^2}\phi(s)^p ds > 0.$$

Proof of (iii). If $0 < r \leqq R$, this is clear from (5). By (i), the left-hand side of the Pohozaev indentity for $\phi(r)$ tends to 0 as $r \to +\infty$. This means

$$\int_0^{+\infty} G(s)\phi(s)^{p+1}ds = 0.$$

Hence, if $R < r < +\infty$, it follows from (5) that

$$\int_0^r G(s)\phi(s)^{p+1}ds = -\int_r^{+\infty} G(s)\phi(s)^{p+1}ds > 0.$$

3. Main Lemmas

Let $\phi(r)$ be a finite total mass solution of Eq.(2) and let $u(r)$ be any solution of (3). In this section we shall prove that $u(r) - \phi(r)$ has one and only one zero if $u(0) \neq \phi(0)$.

LEMMA 3.1. *If $u(0) \neq \phi(0)$, then $u(r) - \phi(r)$ has at least one zero.*

Proof. From (3), we have

$$\{r^{n-1}u_r(r)\}_r + \frac{r^{n-1}}{1+r^2}u^+(r)^p = 0,$$

$$\{r^{n-1}\phi_r(r)\}_r + \frac{r^{n-1}}{1+r^2}\phi(r)^p = 0.$$

Multiplying these equalities by $\phi(r)$ and $u(r)$, respectively, and integrating by parts, we obtain

$$
\begin{aligned}
0 &= \int_0^r \left\{ \{r^{n-1}u_r\}_r(s) + \frac{s^{n-1}}{1+s^2}u^+(s)^p \right\} \phi(s)ds \\
&\quad - \int_0^r \left\{ \{r^{n-1}\phi_r\}_r(s) + \frac{s^{n-1}}{1+s^2}\phi(s)^p \right\} u(s)ds \\
&= \left[s^{n-1}u_r(s)\phi(s) - s^{n-1}\phi_r(s)u(s)\right]_0^r \\
&\quad + \int_0^r \left\{ \frac{s^{n-1}}{1+s^2}\{u^+(s)^p\phi(s) - \phi(s)^p u(s)\} \right\} ds \\
&= r^{n-1}u_r(r)\phi(r) - r^{n-1}\phi_r(r)u(r) \\
&\quad + \int_0^r \left\{ \frac{s^{n-1}}{1+s^2}\{u^+(s)^{p-1} - \phi(s)^{p-1}\} \right\} \phi(s)u(s)ds.
\end{aligned}
$$

Hence we obtain an identity

$$
\begin{aligned}
&r^{n-1}\{\phi_r(r)u(r) - u_r(r)\phi(r)\} \\
&\qquad = \int_0^r \left\{ \frac{s^{n-1}}{1+s^2}\{u^+(s)^{p-1} - \phi(s)^{p-1}\} \right\} \phi(s)u(s)ds.
\end{aligned}
\tag{6}
$$

First suppose $u(r) > \phi(r)$ for all $r \geqq 0$. Then $u(r)$ must be a positive entire solution so that $u(r) \sim (\log r)^{1/(1-p)}$ or $u(r) \sim r^{2-n}$ at $+\infty$. In either case, by Lemmas 2.1 (ii) and 2.2 (i), the left-hand side of (6) tends to 0 as $r \to +\infty$. However, since it is supposed that $u(r) > \phi(r)$ for all $r \geqq 0$, the right-hand side of (6) is positive as $r \to +\infty$. This is a contradiction.

Next suppose $u(r) < \phi(r)$ for all $r \geqq 0$. Then, either $u(r) \sim r^{2-n}$ at $+\infty$ or $u(r)$ has a finite zero. If $u(r) \sim r^{2-n}$ at $+\infty$, then the left-hand side of (6) tends to 0 as $r \to +\infty$ while the right-hand side is negative as $r \to +\infty$, which is a contradiction. If $u(r)$ has a finite zero, say r_0, then

$$
\begin{aligned}
&r_0^{n-1}\{\phi_r(r_0)u(r_0) - u_r(r_0)\phi(r_0)\} \\
&\qquad = -r_0^{n-1}u_r(r_0)\phi(r_0) > 0,
\end{aligned}
$$

which contradicts

$$
\int_0^{r_0} \left\{ \frac{s^{n-1}}{1+s^2}\{u^+(s)^{p-1} - \phi(s)^{p-1}\} \right\} \phi(s)u(s)ds < 0.
$$

Thus it is shown that $u(r) - \phi(r)$ has at least one zero.

LEMMA 3.2. *If $u(0) \neq \phi(0)$, then $u(r) - \phi(r)$ has at most one zero.*

Proof. Suppose that $u(0) > \phi(0)$ and $u(r) - \phi(r)$ has two or more zeros and let r_1 and r_2 denote the first zero and second zero, respectively. By the uniqueness of a solution of the initial value problem, we have $u_r(r_1) < \phi_r(r_1)$ and $u_r(r_2) > \phi_r(r_2)$. Hence

$$\begin{cases} u(r) > \phi(r) & \text{if } 0 \leqq r < r_1, \\ u(r) < \phi(r) & \text{if } r_1 < r < r_2. \end{cases}$$

Then, by (6),

$$\phi_r(r)u(r) - u_r(r)\phi(r) > 0 \quad \text{if } 0 < r \leqq r_1.$$

On the other hand, since $u(r_2) = \phi(r_2)$ and $u_r(r_2) > \phi_r(r_2)$,

$$\phi_r(r_2)u(r_2) - u_r(r_2)\phi(r_2) = \{\phi_r(r_2) - u_r(r_2)\}\phi(r_2) < 0.$$

Hence, by the intermediate value theorem, there exists an $a \in (r_1, r_2)$ such that

$$\begin{cases} \phi_r(r)u(r) - u_r(r)\phi(r) > 0 & \text{if } 0 < r < a, \\ \phi_r(r)u(r) - u_r(r)\phi(r) = 0 & \text{if } r = a, \end{cases}$$

or equivalently,

$$\begin{cases} \dfrac{d}{dr}\left\{\dfrac{\phi(r)}{u(r)}\right\} > 0 & \text{if } 0 < r < a, \\ \dfrac{d}{dr}\left\{\dfrac{\phi(r)}{u(r)}\right\} = 0 & \text{if } r = a. \end{cases}$$

Put $c \equiv \phi(a)/u(a) > 1$. Since $\phi(r)/u(r)$ is a strictly increasing function for $r \in (0, a)$, we obtain

$$\begin{cases} cu(r) > \phi(r) & \text{if } 0 \leqq r < a, \\ cu(r) = \phi(r) & \text{if } r = a. \end{cases}$$

Moreover, since $\phi_r(a)u(a) - u_r(a)\phi(a) = 0$ and $cu(a) = \phi(a)$, we have $cu_r(a) = \phi_r(a)$. From the Pohozaev identities for $\phi(r)$ and $u(r)$, it follows that:

$$\begin{aligned} &c^{p+1}\left\{a^n u_r(a)^2 + (n-2)a^{n-1}u(a)u_r(a) + \frac{2a^n}{(p+1)(1+a^2)}u(a)^{p+1}\right\} \\ &\quad - \left\{a^n\phi_r(a)^2 + (n-2)a^{n-1}\phi(a)\phi_r(a) + \frac{2a^n}{(p+1)(1+a^2)}\phi(a)^{p+1}\right\} \\ &\quad = \int_0^a G(s)\{c^{p+1}u(s)^{p+1} - \phi(s)^{p+1}\}ds. \end{aligned}$$

Using $cu(a) = \phi(a)$ and $cu_r(a) = \phi_r(a)$, we obtain

$$\begin{aligned} &(c^{p-1} - 1)\{a^n\phi_r(a)^2 + (n-2)a^{n-1}\phi(a)\phi_r(a)\} \\ &\qquad = \int_0^a G(s)\{c^{p+1}u(s)^{p+1} - \phi(s)^{p+1}\}ds. \end{aligned} \tag{7}$$

If $R \geqq a$, since $cu(r) > \phi(r)$ and $G(r) > 0$ for $r \in (0, a)$, the right-hand side of (7) is positive. If $R < a$, put $d \equiv \phi(R)/u(R) > 0$. Since $\phi(r)/u(r)$ is a strictly increasing function of r in $(0, a)$, it follows that $d < c$ and

$$\begin{cases} du(r) > \phi(r) & \text{if } 0 \leqq r < R, \\ du(r) < \phi(r) & \text{if } R < r \leqq a. \end{cases}$$

Hence, by Lemma 2.2 (iii),

$$\begin{aligned}\int_0^a & G(s)\{c^{p+1}u(s)^{p+1} - \phi(s)^{p+1}\}ds \\ &> \int_0^a G(s)\{c^{p+1}u(s)^{p+1} - (c/d)^{p+1}\phi(s)^{p+1}\}ds \\ &= (c/d)^{p+1}\int_0^a G(s)\{d^{p+1}u(s)^{p+1} - \phi(s)^{p+1}\}ds.\end{aligned}$$

Since $G(r)$ and $d^{p+1}u(r)^{p+1} - \phi(r)^{p+1}$ are positive if $0 < r < R$ and are negative if $R < r < a$, the right-hand side of this inequality must be positive. Hence the right-hand side of (7) is positive. Since $c > 1$ and $\phi_r(a) < 0$, it follows from (7) that $a\phi_r(a) + (n-2)\phi(a) < 0$, which contradicts Lemma 2.2 (ii). Thus it is shown that $u(r) - \phi(r)$ has at most one zero in case $u(0) > \phi(0)$.

The proof in case $u(0) < \phi(0)$ can be obtained in the same manner. So we omit it.

4. Proof of Theorem

Let us complete the proof of Theorem. First we prove the uniqueness of the finite total mass solution. Suppose that Eq. (2) has two distinct finite total mass solutions $\phi(r)$ and $\psi(r)$. We assume $\psi(0) > \phi(0)$ without losing generality.

By virtue of Lemmas 3.1 and 3.2, $\psi(r) - \phi(r)$ has one and only one zero, i.e. there exists an $r_1 > 0$ such that

$$\begin{cases} \psi(r) > \phi(r) & \text{if } 0 \leqq r < r_1, \\ \psi(r) < \phi(r) & \text{if } r_1 < r < +\infty. \end{cases}$$

If $0 \leqq r \leqq r_1$, it follows from (6) that

$$\begin{aligned} r^{n-1}&\{\phi_r(r)\psi(r) - \psi_r(r)\phi(r)\} \\ &= \int_0^r \left\{\frac{s^{n-1}}{1+s^2}\{\psi(s)^{p-1} - \phi(s)^{p-1}\}\right\}\phi(s)\psi(s)ds > 0.\end{aligned}$$

If $r_1 < r < +\infty$, it follows from (6) and the fact that $r^{n-1}\{\phi_r(r)\psi(r) - \psi_r(r)\phi(r)\} \to 0$ as $r \to +\infty$,

$$\begin{aligned} r^{n-1}&\{\phi_r(r)\psi(r) - \psi_r(r)\phi(r)\} \\ &= -\int_r^{+\infty} \left\{\frac{s^{n-1}}{1+s^2}\{\psi(s)^{p-1} - \phi(s)^{p-1}\}\right\}\phi(s)\psi(s)ds > 0.\end{aligned}$$

These mean that $\phi_r(r)\psi(r) - \psi_r(r)\phi(r) > 0$ for all $r > 0$, or equivalently, $\phi(r)/\psi(r)$ is a strictly increasing function of $r > 0$. Hence, if we put $d \equiv \phi(R)/\psi(R)$, we have

$$\begin{cases} d\psi(r) > \phi(r) & \text{if } 0 \leqq r < R, \\ d\psi(r) < \phi(r) & \text{if } R < r < +\infty. \end{cases}$$

From the Pohozaev identities for $\phi(r)$ and $\psi(r)$, it follows that:

$$d^{p+1}\left\{r^n\psi_r(r)^2+(n-2)r^{n-1}\psi(r)\psi_r(r)+\frac{2r^n}{(p+1)(1+r^2)}\psi(r)^{p+1}\right\}$$
$$-\left\{r^n\phi_r(r)^2+(n-2)r^{n-1}\phi(r)\phi_r(r)+\frac{2r^n}{(p+1)(1+r^2)}\phi(r)^{p+1}\right\}$$
$$=\int_0^r G(s)\{d^{p+1}\psi(s)^{p+1}-\phi(s)^{p+1}\}ds.$$

The left-hand side tends to 0 as $r+\infty$. However, since $G(r)$ and $d^{p+1}\psi(r)^{p+1}-\phi(r)^{p+1}$ are positive if $0<r<R$ and are negative if $R<r<+\infty$, the right-hand side is positive as $r\to+\infty$. This is a contradiction. Thus the uniqueness of the finite total mass solution is proved.

Next let $\phi(r)$ be the unique finite total mass solution and let $u(r)$ be any solution of (3) satisfying $u(0)>\phi(0)$. If $u(r)>0$ for all $r\geqq 0$, from the uniqueness of the finite total mass solution, $u(r)$ satisfies $u(r)-(\log r)^{1/(1-p)}$ at $+\infty$. Hence $u(r)>\phi(r)$ for all sufficiently large r, which contradicts Lemmas 3.1 and 3.2. Hence $u(r)$ must have a finite zero.

Finally let $u(r)$ be a solution of (3) satisfying $u(0)<\phi(0)$. Since $u(r)-\phi(r)$ has one and only one zero, $u(r)$ does not have a finite zero. From the uniqueness of the finite total mass solution, $u(r)$ must be an infinite total mass solution. Thus the proof of Theorem is completed.

5. Concluding Remarks

In this paper we have established the uniqueness of the finite total mass solution of Matukuma's equation. We note that the method used in this paper is applicable to a more general equation

$$u_{rr}(r)+\frac{n-1}{r}u_r(r)+g(r)u(r)^p=0,$$

if the following conditions are satisfied:

(C1) $g(r)\geqq 0$ for all $r\geqq 0$ and $g(r)\sim r^{-2}$ at $+\infty$.

(C2) There exists an $R>0$ such that

$$G(r)\equiv\frac{2}{p+1}\{r^n g(r)\}_r-(n-2)r^{n-1}g(r)$$

satisfies $G(r)\geqq 0$ for $r\in[0,R]$ and $G(r)\leqq 0$ for $r\in[R,+\infty)$.

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