Acta Physica Academiae Scientiarum Hungaricae, Tomus 50 (3), pp. 233—236 (1981)

HOMOGENEOUS SELF-GRAVITATING PERFECT FLUID

By

SUBHENDU P. CHAKRAVARTI

PHYSICS DEPARTMENT, B.N. MAHAVIDYALAYA, ITACHUNA 712147 DT. HOOGHLY (W. BENGAL), INDIA

and

UTPAL K. DE

PHYSICS DEPARTMENT, JADAVPUR UNIVERSITY CALCUTTA, 700032, INDIA

(Received 24. XI. 1980)

Homogeneous solutions of self-gravitating perfect fluids in spherically symmetric and also cylindrically symmetric cases have been presented. In spherically symmetric case, the condition of comoving coordinates has been imposed whereas in the case of cylindrical symmetry it follows from the homogeneity condition.

I. Introduction

Recently a number of papers [1, 2, 3, 4, 5] have appeared on self-gravitating perfect fluid. While TABENSKY and TAUB [1] developed the theory of a fluid where the density of the distribution equals the pressure in relativistic units and also solved the problem in case of a plane symmetric distribution, LETELIER [2] attacked the problem in case of a cylindrically symmetric distribution. RAY [3] rectified some oversights committed in LETELIER's paper. However, in all previous works, the complete solution of the space-time geometry was not attained.

In the present paper, the authors present exact solutions of self-gravitating perfect fluids having spherical symmetry and cylindrical symmetry. However, in both cases, the spatial part of the metric was conformally flat, so only homogeneous solutions are considered. Besides, in spherical symmetric case, the restriction of a comoving coordinate system is imposed from the beginning, while this situation comes out automatically in cylindrically symmetric case from the homogeneity condition. Under these circumstances, our equations simplified to a great extent which helped us to find out complete solutions.

In Part II, the general field equations are given, while in Part III, the spherically symmetric solution is presented. The cylindrically symmetric solution is given in Part IV. The paper ends with a discussion in Part V.

II. Field equations

Einstein's field equations for a self-gravitating perfect fluid with pressure P equal to rest energy ϱ and four-velocity u_a are equivalent to the field equations [1]:

$$R_{ab} = -2\sigma_{,a}\sigma_{,b}\,, \tag{2.1}$$

$$\Box\sigma = (\sqrt{-g}\cdot\sigma_{,a}g^{ab})_{,b}/\sqrt{-g} = 0\,. \tag{2.2}$$

When irrotationality is imposed, i.e.

$$u_a = \sigma_{,a}/[\sigma_{,b}\,\sigma^{,b}]^{1/2}\,. \tag{2.3}$$

The units chosen are such that for the velocity of light $c = 1$ and Newton's constant of gravitation $G = 1/8\pi$. Partial derivative with respect to the index is shown as a comma.

The pressure P and the energy-momentum tensor T_{ab} are related to σ by

$$P = \varrho = \sigma_{,a}\,\sigma^{,a} \tag{2.4}$$

and

$$T_{ab} = 2\sigma_{,a}\,\sigma_{,b} - g_{ab}\,\sigma_{,c}\,\sigma^{,c}\,. \tag{2.5}$$

III. Spherically symmetric self-gravitating perfect fluid

The metric, taken in isotropic form is given by:

$$dS^2 = e^\nu\,dt^2 - e^\mu(dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2)\,, \tag{3.1}$$

where ν and μ are function of both t and r.

Since, we assume the coordinate system to be comoving, from Eq. (2.3), only $\sigma_{,0}$ will exist and so, the field equations (2.1) and (2.2) under the metric (3.1) can easily be written as:

$$\sigma^{,0}\,\sigma_{,0} = \frac{3}{4}\,e^{-\nu}\,\dot{\mu}^2 - e^{-\mu}\left(\mu'' + \frac{\mu'^2}{4} + \frac{2\,\mu'}{r}\right), \tag{3.2}$$

$$\mu'^2 - \nu'^2 + 2\mu'\,\nu' + \frac{2}{r}(\mu' + \nu') - 2(\mu'' + \nu'') = 0\,, \tag{3.3}$$

$$e^{\nu/2} = \dot{\mu}\,T(t) \tag{3.4}$$

and

$$\sigma_{,0} = B(r)e^{-1/2(a\mu-\nu)}\,, \tag{3.5}$$

where $T(t)$ and $B(r)$ are, respectively, arbitrary functions of t and r only. Here dashes and dots denote partial derivatives with respect to r and t, respectively.

Using (3.4) in (3.3) we get the equation for μ as

$$\mu'^2\,\dot{\mu} - 4\,\mu'\,\dot{\mu}' + \frac{2}{r}\,\mu'\,\dot{\mu} + \frac{4}{r}\,\dot{\mu}' - 2\mu''\,\dot{\mu} - 4\dot{\mu}'' = 0\,. \tag{3.6}$$

We have not been able to solve Eq. (3.6) in a general way but when we consider μ in the form

$$\mu = X(r) + Y(t)\,, \tag{3.7}$$

we easily get

$$e^{\mu} = \left(H - \frac{r^2}{4G}\right)^{-2} \cdot e^{Y(t)}. \tag{3.8}$$

H and G are just arbitrary constants.

A simple check for consistency of Eq. (3.2) for the above expression for μ leads us to

$$H = 0$$

and

$$e^{3Y(t)} = A[T(t)]^2, \tag{3.9}$$

where A is a constant except zero.

Thus, the metric (3.1) becomes

$$dS^2 = \left(\frac{4}{9A^2}\right) \cdot dt^2 - \frac{16\,G^2}{r^4}\left(A\,[T(t)]\right)^2 (dr^2 + r^2 d\theta + r^2 \sin^2\theta d\varphi^2)\,.$$

The above metric may also be written after transformation of time coordinate as

$$dS^2 = dt^2 - \frac{M^2}{r^4}\,[T(t)]^2\,(dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\varphi^2)\,, \tag{3.10}$$

where M is an arbitrary constant.

From Eq. (2.4), we easily get

$$P = \varrho = \frac{3}{4}\,[T(t)]^{-2}. \tag{3.11}$$

IV. Cylindrically symmetric self-gravitating perfect fluid

If we take the isotropic cylindrically symmetric metric

$$dS^2 = e^{2\mu}\,dt^2 - e^{2\psi}\,(dr^2 + dz^2 + r^2\,d\varphi^2)\,, \tag{4.1}$$

where μ and ψ are the functions of r and t, we get from Eq. (2.1)

$$-2\sigma_{,1}\,\sigma^{,1} = e^{-2\psi}\,(2\psi'\mu' - \psi'' - \mu'' - \mu'^2 + \psi'^2)\,, \tag{4.2}$$

$$\psi' = -\mu'\,, \tag{4.3}$$

$$-2\sigma_{,0}\,\sigma^{,0} = e^{-2\mu}\,(3\ddot{\psi} + 3\dot{\psi}^2 - 3\dot{\psi}\dot{\mu}) - e^{-2\psi}\left(\mu'' + \mu'^2 + \mu'\psi' + \frac{1}{r}\,\mu'\right). \tag{4.4}$$

It is easy to conclude from (4.3) that

$$\psi = -\mu + B(t), \tag{4.5}$$

where $B(t)$ is an arbitrary function of time only.

Thus from (4.2)

$$\sigma_{,1} = 0 \tag{4.6}$$

and

$$\psi' = -\mu' = 0 \tag{4.7}$$

so that

$$\mu = f(t), \tag{4.8}$$

$$\psi = [B(t) - C(t)] \tag{4.9}$$

and

$$\mu = C(t)\,. \tag{4.10}$$

Obviously, Eq. (4.6) gives us the comoving nature of the case. From Eqs. (2.2) and (4.8)

$$\sigma = \frac{E(r)}{r} \int \exp\left[4C(t) - 3B(t)\right] dt + F\,. \tag{4.11}$$

We must have $\dfrac{E(r)}{r}$ = constant = G (say) in view of (4.8), where $E(r)$ and F are integration constants, $E(r)$ being a function of r also.

So, the metric (4.1) takes the form

$$dS^2 = \exp\left[2C(t)\right[- \exp 2\left[B(t) - C(t)\right]\left[dr^2 + dz^2 + r^2 d\varphi^2\right]. \tag{4.12}$$

and from (2.4)

$$P = \varrho\, G^2 \exp 6\left[C(t) - B(t)\right]. \tag{4.13}$$

V. Discussion

In both solutions, the temporal behaviour of the matter density as well as the pressure depends solely upon the temporal development of spatial part as is expected in homogeneous solutions.

If the space is expanding, contracting or oscillating, the matter density or pressure will accordingly reduce, increase or oscillate.

It seems that the method of TABENSKY and TAUB [1] is much simpler if one is interested in studying perfect fluids only.

*

One of us (S.P.C.) gratefully acknowledges the Teacher Fellowship in Jadavpur University awarded by the U.G.C., New Delhi during which this work was completed.

REFERENCES

1. ROMUALDO TABENSKY and A. H. TAUB, Commun. Math. Phys., **29**, 61, 1973.
2. P. S. LETELIER, J. Math. Phys., **16**, 1488, 1975.
3. DIPANKAR RAY, J. Math. Phys., **17**, 1171, 1976.
4. T. SINGH and R. B. S. YADAV, Acta Phys. Hung., **45**, 107, 1978.
5. P. S. LETELIER and R. R. TABENSKY, Il Nuovo Cim., **B 28**, 408, 1975.