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EINSTEIN—MAXWELL FIELDS WITH NULL KILLING VECTOR

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The field equations for Einstein—Maxwell fields admitting a normal null Killing vector are reduced to a 2-covariant system of equations, which can be derived from a variational principle. Using the invariance of the associated Lagrangian one can generate a class of Einstein—Maxwell fields from the corresponding vacuum solutions.

1. Introduction

For stationary Einstein—Maxwell fields it is possible to construct from the field tensor F_{ab} and the time-like Killing vector ζ^a *scalar potentials*, and the field equations follow from a 3-dimensional variational principle [1, 2]. The Lagrangian contains these potentials and their first partial derivatives. The SU (2, 1) symmetry [3] of the Lagrangian leads to the possibility to generate new solutions [1]. Similar results hold for a space-like Killing vector. The trajectories of a *non-null* Killing vector determine a 3-dimensional space V_3 [4], and the Einstein—Maxwell equations can be written as 3-covariant equations over V_3. This relevant property breaks down in the case of a null Killing vector. Therefore, this case has been excluded from considerations on generating new solutions. However, a twistfree null Killing vector k^a

$$k_{(a;b)} = 0, \qquad k_a k^a = 0, \qquad k_a = W u_{,a} \tag{1}$$

admits finite 2-dimensional surfaces V_2 orthogonal to k^a [5]. The reduction of the field equations on equations over V_2 is possible. Moreover, we can introduce scalar potentials and find a simple Lagrangian for Einstein—Maxwell fields under the conditions (1) (with $W = 1$).

DEBNEY [6] investigated expansionfree Einstein—Maxwell fields which are of Kerr—Schild type and for which the preferred null direction is simultaneously an eigendirection of the electromagnetic field tensor F_{ab}. In place of these restrictions we impose the conditions (1) on the null vector field.

2. Coordinate system

In particular, the conditions (1) imply that k^a is a geodesic, shearfree, expansionfree, and twistfree congruence. We introduce coordinates $x^i = (x, y, v, u)$ adapted to this null congruence [5],

$$k^i = \delta^i_3 , \qquad k_i = W\delta^4_i = g_{3i} . \tag{2}$$

The space-like coordinates $x^A = (x, y)$ are chosen orthogonal to k^a. It is always possible to take a conformally flat metric in the 2-spaces V_2 $(u, v = \text{const})$.

$$g_{ij} = \begin{pmatrix} p^2 & 0 & 0 & m_1 \\ 0 & p^2 & 0 & m_2 \\ 0 & 0 & 0 & W \\ m_1 & m_2 & W & -2H \end{pmatrix} g_{ij,3} = 0, \ \sqrt{\det(-g_{ij})} = Wp^2 . \tag{3}$$

In general, a coordinate transformation making $W = 1$ would destroy the v-independence of g_{ij}. The following transformations preserve the form of the metric (3):

$$\begin{aligned} &\text{(a)} \quad z' = F(z, u), \qquad z = x + \mathrm{i}y, \\ &\text{(b)} \quad u' = h(u), \\ &\text{(c)} \quad v' = v + g(x, y, u) \end{aligned} \tag{4}$$

By means of the last transformation we can achieve $m_2 = 0$.

3. Electromagnetic null field. Scalar potential

For a geodesic null congruence k^a one obtains from the identity

$$2k_{a;[b;c]} = k_d R^d_{abc} \tag{5}$$

an equation for the derivative of the complex expansion Z with respect to the affine parameter v [8],

$$\frac{dZ}{dv} + Z^2 + \sigma\bar{\sigma} = -\frac{1}{2} R_{ab}k^a k^b . \tag{6}$$

Thus, the conditions (1) have the immediate consequence

$$R_{ab}\, k^a k^b = 0 \ . \tag{7}$$

From the conditions (1) we get the relation

$$k_{a;b} = W_{,[b}u_{,a]}, \qquad W_{,a}\ u'^{a} = 0\ . \tag{8}$$

Calculating the contraction of the Ricci identity (5) we obtain

$$k^{b;a}{}_{n} = R^{ab}k_{b} = \lambda k^{a}\ . \tag{9}$$

i.e., k^a is an eigenvector of the energy-momentum tensor T_{ab}. In Section 5 we shall show that the electromagnetic field is necessarily a null field,

$$\begin{aligned} F_{ab} &= 2p_{[a}k_{b]}, \qquad & p_a k^a &= 0\ , \\ T_{ab} &= nk_a k_b, & n &= p_a p^a\ . \end{aligned} \tag{10}$$

The eigenvalue λ in (9) must vanish,

$$\lambda = 0 = W_{,A,A} \qquad A = 1, 2, \tag{11}$$

so that the function W satisfies a potential equation in V_2. In the case of a time-like Killing vector ζ^a the complex electromagnetic potential Φ has been defined by

$$\zeta^a F^{*}_{ab} = \Phi_{,b}, \qquad F^{*}_{ab} \equiv F_{ab} + \frac{i}{2}\,\varepsilon_{abcd}F^{cd} \tag{12}$$

[1]. It does not make sense to substitute ζ^a by k^a in this equation. The investigation of the relation

$$2A_{[b,a]} = F_{ab} = 2p_{[a}k_{b]} \tag{13}$$

in the metric (3) with (2) shows that with the aid of a gauge transformation

$$\tilde{A}_a = A_a + \chi_{,a} \tag{14}$$

the vector potential A_a can always be transformed to the form

$$A_a = \psi\, u_{,a}, \qquad \psi = \psi(x^A, u). \tag{15}$$

The gauge function χ is linear in v. Eq. (15) defines a real scalar potential ψ. The vector potential (15) satisfies the Lorentz gauge condition

$$A^{a}{}_{;a} = 0 \leftrightarrow \psi_{,a}k^a = 0\ . \tag{16}$$

Thus, the Lie derivative of the field tensor

$$F_{ab} = 2\psi_{,[a}u_{,b]} \tag{17}$$

with respect to k^a vanishes. F_{ab} determines the vector p^a in (13) up to a term proportional to k^a. This freedom can be used such that a gradient $\psi_{,a}$ appears in the representation (17). The Maxwell equations

$$F_{ab}{}^{;b} = -\psi_{,b}{}^{;b} u_{,a} = 0 = \psi_{,A,A} \tag{18}$$

demand that the potential ψ is the real part of a function $f(z, u)$ analytic in z. For the complex self-dual field tensor F^*_{ab} we get

$$F^*_{ab} = 2f_{,[a} u_{,b]}, \quad f = f(z, u) \equiv \psi + i\varphi. \tag{19}$$

The real and imaginary parts of f are related by the Cauchy—Riemann equations

$$\varphi_{,A} = -\varepsilon_{AB}\psi_{,A}\ , \tag{20}$$

so that the full system of the Maxwell equations

$$F^{*;b}_{ab} = 0 \tag{21}$$

is fulfilled because of (18).

4. Einstein equations

We have to solve the Einstein equations for the null field (17),

$$R_{ab} = \varkappa \psi_{,c}\psi^{,c} u_{,a} u_{,b}\ . \tag{22}$$

The solutions are contained in the general class investigated by KUNDT [5, 9]. We use the coordinate system (3) and apply the transformations (4) to simplify the metric.

Starting with the potential equation (11) we have to distinguish two cases:

$$\begin{aligned} \text{I.} \quad & W = 1, \\ \text{II.} \quad & W = x. \end{aligned} \tag{23}$$

The first case is characterized by the existence of a covariantly constant null vector,

$$W = 1 : k_{a;b} = 0\,. \tag{24}$$

In the second case the coordinate transformation (4a) has been used. Without the special choice $W = x$ in case II we get from the equations $R_{AB} = 0$:

$$p^2 = W^{-1/2} W_{,A} W_{,A}\ . \tag{25}$$

The Eqs. (22) lead to the statements listed in Table I where the transformations (4a—c) used are indicated.

Table I

	I.		II.	
$R_{3A} = 0$:	$W = 1$		$W = x$	(4a)
$R_{AB} = 0$:	$p^2 = 1$	(4a)	$p^2 = x^{-1/2}$	(4b)
$R_{4A} = 0$:	$m_1 = 0$	(4a, c)	$m_1 = N(u)yx^{-3/2}$	(4c)

$N(u)$ is an arbitrary function of u. The last field equation of (22) is a differential equation for the remaining function H. In the case II we introduce a new function M,

$$M \equiv x^{-1} H + \frac{2}{3} x^{-3/2} \left(\frac{dN}{du} y - \frac{1}{3} N^2 \right) . \tag{26}$$

The second term in (26) takes into account the nonvanishing function m_1. In the case I the functions H and M coincide. Then, the total system of the Einstein—Maxwell equations reduces to very simple equations over the 2-spaces V_2 ($u, v =$ const) or, equivalently, over the Euclidean plane:

$$\psi_{,A,A} = 0 , \tag{27}$$
$$(WM_{,A})_{,A} = \varkappa \psi_{,A} \psi_{,A} , \quad W = 1; \; W = x .$$

Derivatives with respect to u do not occur.

We consider the two cases separately.

Case I. ($W = 1$):

In terms of the complex coordinate z we have the equations

$$\psi = \frac{1}{2} (f + \bar{f}), \quad \frac{\partial^2 H}{\partial z \partial \bar{z}} = \varkappa \frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial \bar{z}}$$

leading to the final form of the metric

$$ds^2 = dz \, d\bar{z} + 2du \, dv - 2H \, du^2, \tag{28}$$
$$H = \varkappa f \bar{f} + g + \bar{g}, \quad f = f(z, u), \quad g = g(z, u),$$

where f and g are arbitrary analytic functions of z depending arbitrarily on the retarded time coordinate u.

If the gravitational field is entirely caused by the electromagnetic null field, the solution of the homogeneous equation for H can be put equal to zero, and H is just the squared modulus of an analytic function. The solutions (28) are in general of Petrov type N. For the special function $f = \alpha(u)\, z$ they are even conformally flat [6]. In the case under consideration we can derive the field equations (27) from a *variational principle* with the Lagrangian

$$\boxed{\begin{aligned} L &= \Gamma_{,A}\bar{\Gamma}_{,A} \\ \Gamma &\equiv H - \frac{\varkappa}{2}\psi^2 + i\psi^2. \end{aligned}} \tag{29}$$

The complex scalar potential Γ contains the gravitational potential H as well as the electromagnetic potential ψ. The invariance transformation

$$\Gamma' = e^{ia}\,\Gamma \tag{30}$$

generates solutions of the Einstein—Maxwell equations from vacuum *pp*-waves ($\psi = 0$). The parameter a in (30) can depend on u.

Case II. ($W = x$):

In this case the field equations (22) lead to one single inhomogeneous differential equation for the real function M,

$$2(z + \bar{z})\frac{\partial^2 M}{\partial z \partial \bar{z}} + \frac{\partial M}{\partial z} + \frac{\partial M}{\partial \bar{z}} = \varkappa \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}}\,. \tag{31}$$

The solutions are of Petrov type II or D (5). The metric

$$ds^2 = \frac{1}{\sqrt{x}}(dx^2 + dy^2) + 2x d\, udv - 2\varkappa C^2 x^2\, du^2, \tag{32}$$

$$\psi = Cx, \qquad C = \text{const}$$

provides the simplest example of an Einstein—Maxwell field of this kind. It can be interpreted as a stationary cylindrically symmetric field with rotating charges and curvature singularities on the axis of symmetry. If the electromagnetic field is switched off, the solution is not flat: For $C = 0$, the solution (32) is the static Levi—Civita metric which is of Petrov type D and admits two null Killing vectors.

5. Electromagnetic non-null field

Finally, we have to investigate the case of an electromagnetic non-null field with the eigendirection k^a,

$$F_{ab} = 2F(n_{[a}k_{b]} + \bar{r}_{[a}r_{b]}), \qquad T_{ab} = F\bar{F}(n_{(a}k_{b)} + \bar{r}_{(a}r_{b)}) . \tag{33}$$

The complex null tetrad $(r_a, \bar{r}_a, n_a, k_a)$ is adapted to the eigendirections of the electromagnetic field tensor. The eigenvalue λ in Eq. (9) must not vanish in this case

$$-\lambda = (2p^2W)^{-1}\, W_{,A,A} = \frac{\varkappa}{2} F\bar{F} \neq 0 . \tag{34}$$

We consider the Einstein equations

$$R_{AB} = \varkappa T_{AB} = -\lambda p^2 \delta_{AB} \begin{cases} \text{(a)} & R_{11} - R_{22} = 0 = R_{12} , \\ \text{(b)} & R_{11} + R_{22} = -2\lambda p^2. \end{cases} \tag{35}$$

From Eq. (35, a) we obtain

$$\sqrt{W}\lambda = \frac{-\partial A(W, u)}{\partial W}, \qquad q^{-2} = A(W, u), \qquad q^2 \equiv \sqrt{W} p^2 (W_{,A} W_{,A})^{-1}, \tag{36}$$

where $A(W, u)$ is an arbitrary function of its arguments. From the relations (34), (36) it follows that there exists a function $Y = Y(W)$ satisfying the potential equation $Y_{,A,A} = 0$, so that we can put $Y = x$. The remaining Eq. (36,b) requires $\lambda = 0$, which is contradictory to the premise (34). Therefore, under the conditions (1) solutions of the Einstein—Maxwell equations with electromagnetic non-null field do not exist.

6. Summary

If the existence of a twistfree null Killing vector k^a is presumed, the Einstein—Maxwell equations can be reduced to the system (27). These equations are derivable from a variational principle with the Lagrangian (29), provided that $k_a = u_{,a}$ (covariantly constant null vector). Only electromagnetic *null* fields are compatible with the conditions (1).

In this paper we have shown that there exists an internal invariance group which can be exploited to generate *pp*-wave solutions in the Einstein—Maxwell theory from the corresponding vacuum solutions. Of course, the resulting metrics are well-known. The main result is the new generation theorem for solutions admitting a *null* Killing vector.

To find Einstein—Maxwell fields with *twisting* null Killing vectors, it might be useful to apply similar methods: introduction of scalar potentials, reduction to equations containing only derivatives with respect to two spatial coordinates.

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