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PHYSICAL ASPECTS OF MATTER ACCRETION ON STARS

By

L. BARONI

ISTITUTO DI FISICA, UNIVERSITA DI BOLOGNA, BOLOGNA, ITALY

P. L. FORTINI

ISTITUTO DI ASTRONOMIA, BOLOGNA AND ISTITUTO DI FISICA, UNIVERSITA DI FERRARA, FERRARA, ITALY

C. GUALDI and G. CALLEGARI

ISTITUTO DI FISICA, UNIVERSITA DI FERRARA, FERRARA, ITALY

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The present status of the theory of matter accretion on stars is reviewed. The paper deals mainly with thermodynamics and hydrodynamics of accreting gas in the absence of a magnetic field. Particular emphasis is put on the main problems which are still unresolved.

Introduction

The physical problems in the theory of matter accretion on the stellar objects are reviewed. As it is well known, such theory has been recently used to explain X-ray emission by galactic binary systems.

In this paper we limit ourselves to the general physical features of accretion and therefore we do not take into account the particular problems connected with neutron stars and black-holes. We hope to give an account of this aspect of the theory in a forthcoming paper.

Section 1 deals with the early theories proposed by EDDINGTON, HOYLE and LYTTLETON in connection with star energy source and terrestrial climatic variations.

In Sections 2 and 3 we deal with the thermodynamics of falling matter, while the role of radiation pressure will be taken into account in Section 4. In Sections 5 and 6 we shall treat in detail the accretion on to objects moving through the ambient gas at a supersonic speed. Section 7 is concerned with BONDI's theory of subsonic accretion and Section 8 gives finally a review on the general hydrodynamical problem.

§ 1. Early accretion theories

The idea that matter withdrawn from interstellar gas by the gravitational force of a star can supply energy to the star itself by deposition of its kinetic energy onto the surface, can be traced back to EDDINGTON [7]. The hope was to explain the origin of stellar energy, as thermonuclear reactions were not known at that time. Neglecting the interaction of atoms among themselves and taking into account the constancy of energy and angular momentum with respect to the star centre, one finds that particles having a distance from the centre less than

$$R_0 = \left(\frac{2\,GMR}{v^2} \right)^{1/2} \tag{1}$$

must necessarily fall on the surface.

In the above formula G is the Newtonian gravitational constant, M and R are the mass and radius of the star, respectively, and v is the velocity of the star relative to the gas cloud. In fact if we call v' the velocity of a particle grazing the stellar surface, we have clearly:

$$R_0\, v = Rv' ,$$

$$v'^2 - v^2 = \frac{2GM}{R} ,$$

and therefore

$$\left(\frac{R_0}{R} \right)^2 = 1 + \frac{2GM}{Rv^2} .$$

As in cases of interest, $2GM/Rv^2 \gg 1$, one has at once formula (1). The rate of accreted mass is:

$$\frac{dM}{dt} = \pi R_0^2 \rho v = \frac{2\,\pi GMR\rho}{v} , \tag{2}$$

where ρ is the mass density of the cloud.

One easily sees that for a normal (i.e. not collapsed) star and a number density of the gas ~ 1 atom/cm^3, this process cannot supply the required energy by many orders of magnitude.

About after ten years, Hoyle and Lyttleton [9] proposed a theory of accretion in which the effects of collisions among particles in the cloud were taken into account.

The purpose of their paper was to explain the earth climatic changes during Geological Eras due to a variation in the solar luminosity when the sun enters a cosmic cloud in its motion around the galactic centre.

With reference to Fig. 1, the gas flows from left to right with collisions taking place in A, to the right of the sun S' as its gravitational attraction causes two opposite fluxes of particles to collide. The effect of such collisions is to destroy the particle angular momentum about the sun. If, after collision, the radial component of the velocity is less than the escape velocity at A, the particles fall on to the sun surface; therefore the accretion radius R_A can be calculated by requiring that in A, radial velocity is equal to parabolic velocity. This interpretation of facts, though quite rough, gives the right answer anyway: a more rigorous treatment has been given by BONDI and HOYLE [2] (see further on § 5). A gas element is subject to hyperbolic motion around the sun, i.e.:

$$\frac{p}{r} = 1 + e \cos \theta ,$$

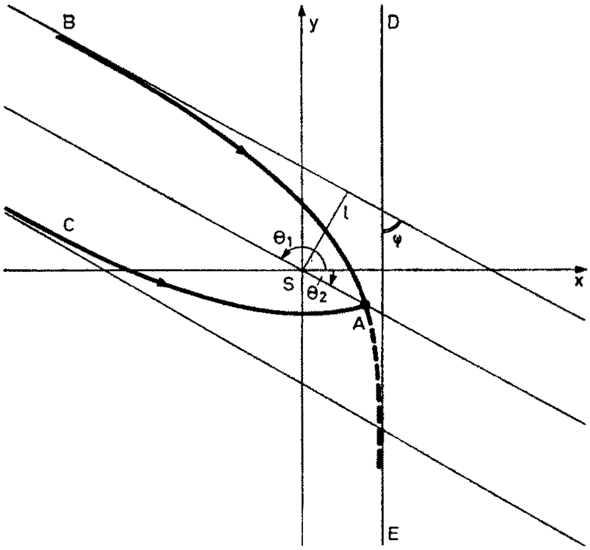

Fig. 1

where p and e are respectively the orbit parameter and eccentricity. The direction parallel to the initial assymptote is given by $r \to \infty$, that is:

$$e \cos \theta_1 + 1 = 0$$

and, as $\theta_1 - \theta_2 = \pi$, also:

$$e \cos \theta_2 + 1 = 0,$$

$$e \sin \theta_2 = (e^2 - 1)^{1/2},$$

from which follows:

$$SA = \frac{p}{2} .$$

Taking time derivative of the trajectory equation, one gets the radial component of the velocity:

$$\dot{r} = \frac{e}{p} r^2 \dot{\theta} \sin \theta = \frac{eh}{p} \sin \theta ,$$

since

$$r^2 \dot{\theta} = h = \sqrt{GMp} = vl .$$

Therefore

$$|r_B| = e \, |\sin \theta_1| \frac{h}{p} = e \sin \theta_2 \frac{h}{p} = |\dot{r}_A|. \tag{3}$$

This means that the radial velocity at A is equal to the cloud velocity at infinity. The square of the parabolic velocity at A is given by

$$\frac{2GM}{SA} = \frac{4GM}{p}$$

and therefore the particles fall on the sun if the following inequality is satisfied

$$v^2 < \frac{4GM}{p} = \frac{4G^2 M^2}{v^2 l^2} ,$$

thus the accretion radius is given by

$$R_A = \frac{2GM}{v^2} ,$$

In physically interesting cases, R_A is always greater by several orders of magnitude than R_0, i.e. the accretion radius for non interacting particles (see formula (1)). The rate of matter accretion is now

$$\frac{dM}{dt} = \pi R_A^2 \varrho v = \frac{4\pi G^2 M^2 \varrho}{v^3} . \tag{4}$$

(Hoyle and Lyttleton [10], Hoyle and Lyttleton [11]).

Since the velocity of escape at the surface of the sun is very large, i.e. $6.2 \cdot 10^7$ cm/sec, one can assume that all the particles reaching the surface of the sun arrive with the escape velocity which, as one can easily see, corresponds to a kinetic energy $\sim 9 \cdot 10^{-9}$ ergs per hydrogen atom. Now the ionization energy of the hydrogen atom is about $4 \cdot 10^{-11}$ ergs, hence it can be concluded that the particle cannot get rid of any appreciable portion of its energy by ionization processes before reaching the sun: therefore the kinetic energy of the falling particles has the net effect to increase the sun's radiation, as the extra energy gained in this way must be reemitted.

The energy brought to the sun per second is easily obtained by (4) and turns out to be

$$4 \cdot 10^{68} \frac{\varrho}{v^3} \text{ erg|sec} .$$

We see that the increase in sun's radiation depends on the density of the cloud and on the velocity of the sun relative to it, being directly proportional to the first factor and inversely proportional to the cube of the latter factor. Thus slight changes in these factors bring about considerable ranges of variation in the solar radiation, and, for plausible values of ϱ and v, it may be caused to change from 0.1 to 1000% according to the density and velocity of the cloud.

According to the authors, if the increase in the solar luminosity is moderate ($<10\%$), we shall have on the earth an increase in the precipitation of snow (because of the enhanced evaporation) in those regions normally within the snow line and therefore the onset of an Ice Epoch.

If on the other hand the solar luminosity is increased by a factor greater than 2, a hot and humid climate will ensue even in polar regions: in this way the peculiarities of the carboniferous Epoch can be explained.

One easily sees that, assuming a constant density $\sim 10^{-18}$ gr/cm^3 of the cosmic clouds, the above figures can be obtained for a relative velocity of ~ 20 km/sec and 2 km/sec, respectively.

§ 2. Critical temperature of the accreting gas

The physical problems involved in accretion onto normal stars have been investigated for the first time by HOYLE and LYTTLETON [12]. Let us consider a radial flux of matter with a temperature T' toward a star of Mass M. Because of the continuity equation for a stationary flux ($\dot{M}$ = constant through a spherical surface of radius r) one gets:

$$\varrho = \frac{\dot{M}}{4\,\pi r^2 v} .$$

For free falling gas, $v = \sqrt{2GM/r}$ and therefore

$$\varrho = \frac{\dot{M}}{4\pi\sqrt{2GM}}\, r^{-3/2}. \tag{5}$$

Our assumption of a radial stationary flux requires that the gravitational force (which acts inwards) on a volume element dV, i.e. $GM\varrho dV/r^2$, is greater than the pressure gradient (which acts outwards) due to the density gradient. The latter is given by $R\varrho T' dV/r$, where R is the perfect gas constant and T' the gas temperature. As the gravitational force goes as $r^{-7/2}$ and the pressure gradient as $r^{-5/2}$, the latter is equal to the former for r sufficiently great and this for any temperature T'. The accretion can take place only if this value of r (which will be called r_T or thermal radius) is greater than the accretion radius (formula (3)).

From the above consideration, one sees that a critical temperature must exist above which $r_T < R_A$. For the sun, with $v = 20$ km/sec, such a temperature turns out to be $\sim$16 000 °K.

As is well known, the gas temperature of a cloud in which no high surface temperature star is present, cannot be greater than 10^4 °K, because of recombination effects. Therefore the problem of critical temperature for stars of 1 $M_\odot$ has no particular difficulty.

On the contrary one encounters some difficulties for massive stars, like V—Puppis stars, with masses $\sim$20 $M_\odot$, surface temperatures $T \sim$20 000 °K and relative velocity with respect to the cloud of about 5 km/sec. For such stars the critical temperature is of several thousands degrees: it is thereforevital to see if the cosmic cloud can emit radiation by processes other than inverse photoelectric effect. One easily sees that free-free transitions are ineffective. In fact the cross-section for the emission of 1 e.v photon at a temperature $\sim 10^4$ °K is $\sim 10^{-25}$ cm^2, while the cross section for the capture of an electron by a proton is $\sim 5 \cdot 10^{-21}$ cm^2: it turns out therefore that an electron is captured before it can emit by free-free transitions an appreciable fraction of its energy.

A more effective process is however infrared emission by hydrogen molecules, which will be considered in the next section.

§ 3. Cooling mechanism for the falling gas

Let us consider a hydrogen plasma (protons, electrons and H_2, H_2^+ molecules) lighted by a source at a temperature T. Every ionization contributes kT to the thermal energy of the gas. Calling ϱ the cloud density, X the fraction by weight of ionized matter, m_H the proton mass, u the mean velocity of electrons, and σ the cross section for recombination of H_2^+ or H^+ with the electron at the cloud temperature T', the number of electrons per c.c. is $X\varrho/m_H$ and the number of recombinations per c.c. per second is:

$$X^2 \left(\frac{\varrho}{m_H}\right)^2 \mu\sigma .$$

At equilibrium the ionizations per second must be equal to the recombinations per second and therefore the thermal energy given by electrons to molecules per c.c. and per second is:

$$XY \left(\frac{\varrho}{m_H}\right)^2 \mu\sigma' \, kT' ,$$

where Y is the fraction by weight of molecules and $\sigma' \sim 2 \cdot 10^{-18}$ cm^2 is the cross section for equipartition of energy between electrons and molecules. On equating these two formulae and remembering that

$$\sigma \sim \frac{4 \cdot 10^{-27}}{T'}$$

(STUCKELBERG—MORSE [16]), we get:

$$T' = \sqrt{40T \frac{X}{Y}} .$$

From this one sees that, unless $\frac{X}{Y}$ is very large, T' is small compared with T.

Let us now investigate whether the energy acquired by a molecule in one excitation act can be radiated away before the molecule undergoes a second excitation. As the electric dipole moment of a hydrogen molecule in the ground state vanishes, one has to do with forbidden transitions whose probability is less than the probability of allowed transitions by a factor of 10^8. Assuming a mean life of 10^{-6} sec for an allowed transition, the mean life of a forbidden one is therefore 10^2 sec.

The time elapsed between two successive excitations is given by $1/\sigma n_e v$ where $\sigma \sim 10^{-18}$ cm^2 is the excitation. For a gas with an electron number density $n_e = 10^3$ cm^{-3} and $v \sim 5 \cdot 10^7$ cm sec^{-1}, one gets a time $\sim 10^7$ sec: this means that the molecule can give away its excitation energy before a second process can take place.

Another process which can give energy to the gas is the angular momentum destruction induced by molecular collisions of the falling matter. The number of collisions to which a molecule is subject over a distance equal to the accretion radius is given by

$$\frac{2GM}{v^2} \sigma n_H ,$$

where $\sigma \sim 10^{-16}$ cm^2 is the geometrical collision cross section and $n_H = 10^3$ hydrogen atoms per c.c. For a star mass $M = 5\ M_\odot$ and $v = 5$ km/sec, this number is $\sim 10^3$ and therefore sufficient to ensure equipartition among molecular states. The characteristic time of accretion is $\sim GM/v^3$, which is $\gg 10^2$ sec: therefore the energy gained in this way is radiated away. Let us now investigate whether an appreciable number of molecular hydrogen can exist at a distance comparable with accretion radius, particularly near massive stars which emit a substantial amount of ionizing radiation. First of all the material sufficiently near the star is completely ionized and in this domain its temperature is the star surface temperature. Assuming radial accretion, the number of recombinations per c.c. per sec at a distance x from the star is

$$v(T)\sigma(T)\frac{\varrho^2(x)}{m_H^2},$$

where $v(T)$ is the velocity of electrons and $\sigma(T)$ is the recombination cross section. Remembering that $\varrho(x) = \varrho(r)(r/x)^{3/2}$the total number of recombination within a sphere of radius r is:

$$\frac{v(T)\sigma(T)}{m_H^2}\int_R^r 4\pi x^2 e^2(x)\,dx = \frac{4\pi}{m_H^2} v(T)\ \sigma(T)\varrho^2(r)r^3\log\frac{r}{R},$$

where R is the radius of the star.

As ionizations and recombinations per second must be equal and as about one half of the ionizing photons can be given by recombinations themselves and the energy required for a single ionization process is $\sim 2.5\cdot 10^{-11}$ erg, the star must supply an amount of energy

$$2\pi v(T)\sigma(T)\frac{\varrho^2(r)}{m_H^2}r^3\log\frac{r}{R}\cdot 2.5\cdot 10^{-11}\ \text{erg}|\text{sec}$$

in order to produce a number of ionizations about equal to one half recombinations. Assuming that the star radiates as a black-body and calling ε the amount of radiation with an energy greater than $2.5\cdot 10^{-11}$ erg, one must have:

$$\varepsilon = 2\pi v(T)\sigma(T)\frac{\varrho^2(r)}{m_H^2}r^3\log\frac{r}{R}\cdot 2.5\cdot 10^{-11}.$$

This equation allows one to calculate the radius r of the sphere within which matter is completely ionized (Strömgren's radius). In the case of V-Puppis $E\sim 10^{35}$ erg/sec, $R\sim 5\cdot 10^{11}$ cm, and therefore $r = 5\cdot 10^{16}$ cm, which is greater than $R_A\sim 10^{16}$ cm (and this always in the envisaged case of $\sim 10^3$ atoms cm^{-3}). Outside the Strömgren's sphere, only a small fraction of the material can be ionized. In fact the number of ionizing quanta at a distance $x > r$ goes down with an exponential law of the kind $e^{(r-x)/\tau}$, where τ is the ionization mean free path.

So, in order that an accretion process can take place onto stars like V-Puppis, the gas density must be greater than 10^3 atoms cm^{-3}.

§ 4. Effects of radiation pressure

In order to evaluate the effect of radiation pressure it is first of all necessary to calculate the mean life of atomic levels for the absorption of a quantum of radiation. This will be achieved through the estimate of the number of atoms, within a distance r from the star, which are excited by a quantum of energy $\hbar\omega_0$, where ω_0 is the frequency of the line. The probability that a quantum of energy between $h\omega$ and $\hbar(\omega + d\omega)$ is absorbed before it reaches a distance r, is approximately given by

$$\frac{\pi}{2} w_{ab}\,\gamma \frac{Nrc^2}{\omega_0^2(\omega - \omega_0)^2}$$

(HEITLER [8] p. 186, eq. (16)), where N is the number density of hydrogen atoms and therefore Nr is the number of atoms contained in a cylinder of unitary cross section and height r, w_{ab} is the transition probability for spontaneous emission between the states a and b, $\gamma = 2/3\ r_0\ \omega_0^2/c$ is the natural width of the line and $r_0 = 2.8 - 10^{-13}$ cm the classical radius of the electron (HEITLER [8], p. 35, formula (6)).

As $\gamma \sim w_{ab}$ (HEITLER [8], p. 184, formula (12)), one has:

$$\frac{\pi}{2} \gamma w_{ab} \sim 6 \cdot 10^{18}$$

for $\omega_0 \sim 2 \cdot 10^{16}$ rad/sec corresponding to the energy required for a transition from the ground state to the first excited level in a hydrogen atom ($\sim$10 eV). The quantum can be considered completely absorbed for a value of ω given by

$$6 \cdot 10^{18} \frac{Nrc^2}{\omega_0^2(w - \omega_0)^3} = 1$$

i.e.:

$$|\omega - \omega_0| \sim \sqrt{10^7\,Nr}\,.$$

The radiation emitted in this frequency interval is completely absorbed before it can reach a distance r.

If $u(\omega)$ is the black body radiation energy density, for a temperature 20 000 °K, $u(\omega_0) \sim 5 \cdot 10^{-14}$ erg/cm^3 Hz. Thus the required emission of absorbed radiation is approximately given by

$$4\pi R^2 c\mu(\omega_0)\, 2\, |\omega - \omega_0| \sim 10^2 R^2 (Nr)^{1/2}\,,$$

where R is the radius of the sphere.

As one erg corresponds to about $5 \cdot 10^{10}$ quanta, the number of atoms excited per second is $5 \cdot 10^{12}\, R^2(Nr)^{1/2}$. As the total number of atoms within a sphere of radius r is $4/3\ \pi N r^3$, the fraction of atoms excited per second within such a sphere is of the order of $10^2\, R^2 N^{-1/2} r^{-5/2}$, and therefore the mean life of the first excited level of hydrogen atom at a distance $r = 5 \cdot 10^{16}$ cm and with a density $N = 10^4$ cm^{-3} and a stellar radius $R = 5 \cdot 10^{11}$ cm, turns out to be:

$$10^{-12} R^{-2} N^{1/2} r^{5/2} \sim 10^9 \text{ sec} .$$

The exciting quantum has approximatively $5 \cdot 10^{-22}$ momentum units and therefore every atom gets in the mean about $5 \cdot 10^{-31}$ momentum units per second.

The gravitational force exerted by a star of mass $M \sim 20\ M_\odot$ on a hydrogen atom at a distance $\sim 5 \cdot 10^{16}$ cm is $\sim 10^{-30}$ dyne, so that the absorption of this line contributes 1/6 the total pressure necessary to sustain the atom against gravity.

It follows that none of the transitions $1s \to np$ can sustain hydrogen atoms because of the probability decrease with increasing n (CONDON and SHORTLEY, [3]).

If matter density is sufficiently high, line excitations cannot sustain atoms against gravity at the capture radius and matter moves therefore towards the star surface, as required by accretion theory.

On the other hand, the effect of radiation pressure on ionized matter is quite negligible. In fact, in this case, it is entirely due to the interaction of photons with free electrons (Thomson scattering) with a cross section $\sigma_T = 6.65 \cdot 10^{-25}$ cm^2.

The pair electron-proton is subjected to two forces:

1) the radiation pressure force acting on the electron and pointing outwards, which is given by

$$\frac{L}{4\pi r^2 c}\, \sigma_T \,,$$

where L is the star luminosity, r the distance of the pair from the star.

2) The gravitational force acting on the proton and pointing inwards, i.e.

$$\frac{G M m_H}{r^2} \,.$$

At equilibrium one gets:

$$L_E = \frac{4\pi G M m_H c}{\sigma_T} = 1.2 \cdot 10^{38} \frac{M}{M_\odot} \text{ erg/sec} . \tag{6}$$

This value of the luminosity is called the Eddington limit: for $L > L_E$ matter is "blown away" by radiation pressure.

For a 20 $M_\odot$ star, $L_E = 2.4 \cdot 10^{39}$ erg·sec^{-1}. As the effective luminosity is $\sim 10^{37}$ erg/sec, radiation pressure is entirely negligible.

§ 5. Supersonic accretion in the case of interacting particles

We have seen above that accretion can be treated in two extreme approximations: one is to consider non interacting gas particles, in which case the accretion radius is given by (1), the other is to consider interacting particles: the latter case leads to the accretion radius given by (3) and this applies when the mean free path of a particle is much less than R_A. A more rigorous treatment than the one given in Section 1 has been given by Bondi and Hoyle [2] for the case of non interacting as well as of interacting particles.

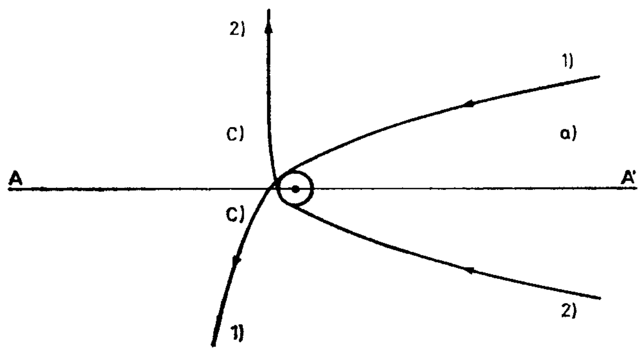

Fig. 2

Let us consider first the case of non interacting particles. With reference to Fig. 2, trajectories 1 and 2 represent the paths of particles grazing the star surface which is represented by the circle. These trajectories divide the space into three regions: a), b) and c). All particles moving in region a) hit the stellar surface. In region b) there is but one trajectory through a given point. In c) however there are two trajectories through a given point, apart from the points of the accretion axis, where there is an infinite number because of cylindrical symmetry. If we now consider the case of interacting particles with a low gas temperature (cf. § 3), there is no change in the properties of regions a) and b): in fact particles cannot collide as only one trajectory goes through each point. This is not true for region c), where particles collide even if their temperature is low: these collisions tend to prevent two fluxes of particles from passing through each point not on the axis. In fact it is clear that the two stream region cannot have dimensions much greater than the mean free path of the particles and therefore this region shall have dimensions much less than R_A (formula (3)). As a consequence the mechanism of accretion is determined by the four regions a), b), c1) and c2) shown in Fig. 3.

Region c1) is a two stream region with a thickness of the order of the mean free path and regions a), b) and c2) are all single stream. For a very high density of the cloud, region c1) becomes a surface of discontinuity. In c2) the pressure is very high as the density is great. This pressure causes a force on region c1) directed outwards which balance the momentum transverse component of matter coming in from b).

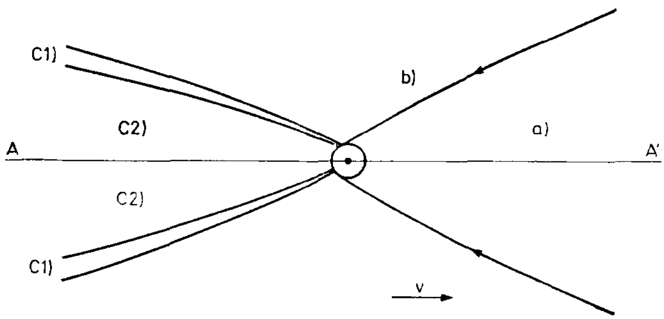

Fig. 3

Using a hydrodynamical terminology region c1) may be called a shock wave. The problem of accretion in its most general form is a hydrodynamical one and will be treated from this point of view in Section 8.

The behaviour of a particle crossing the shock can be qualitatively described in this way: the gas loses the component of the velocity perpendicular to the shock wave front, while the parallel component, i.e. the one directed radially, is left unchanged. With reference to Fig. 1, if the particle impact parameter is greater than R_A (cf. formula (3)), the radial velocity is greater than the parabolic velocity and the gas goes to infinity, while if the impact parameter is less than R_A, the gas falls on to the star after crossing the shock wave front.

This result, strictly rigorous in case of Fig. 1, where it was assumed that collisions took place only on the axis, is still true also in the more general case of a shock, provided the Mach cone is narrow enough, i.e. for relative velocities V much greater than the sound velocity a; the opening θ of such a cone is given by

$$\sin \theta = \frac{a}{V} .$$

From these considerations one already sees the importance of sound speed in accretion problems.

§ 6. Braking force

A very important dynamical effect is the braking force produced by the cloud on the star. This force is due to the particles changing their momenta, via the interaction with the gravitational field of the star. This process has been investigated for the first time by BONDI and HOYLE [2] and by DODD and McCREA [5—6].

Let us consider first the case of non interacting particles. With references to Fig. 1 we call ψ the angle between asymptotes of the hyperbolic trajectory of a particle coming from B whose impact parameter is l. The particle comes in from infinity and goes to infinity, in the direction of the asymptote DE, thereby changing only the velocity direction (not modulus !), through an angle ψ. From standard formulae of celestial mechanics (OGORODNIKOV, [14]) one has:

$$\operatorname{tg}\frac{\psi}{2}=\frac{GM}{lV^2},$$

where M is the mass of the star and V the relative velocity between star and particle.

The vectorial change in the velocity is given in modulus by $2V\sin\psi/2$ and makes an angle $\pi+\psi/2$ with the arrival direction. The component of $\Delta\vec{V}$ along axis SA is therefore $|\Delta\vec{V}|=2V\ \sin^2\psi/2$. On expressing $\sin^2\psi/2$ by $\operatorname{tg}\psi/2$ one gets:

$$|\Delta\vec{V}|=\frac{2V}{1+\left(\dfrac{lV^2}{GM}\right)^2}\,.$$

Consider now a ring of radius l, thickness dl centered on SA and area $2\pi l\,dl$. The total mass which crosses this area per unit time is $\varrho V\,2\pi l\,dl$, where ϱ is the cloud density. These particles act on the star with a force:

$$dF=\varrho\Delta VV2\pi ldl$$

from which we get the total force:

$$F=4\pi V^2 e\int_0^{l_1}\frac{ldl}{1+\dfrac{l^2V^4}{G^2M^2}}=2\pi\varrho\frac{G^2M^2}{V^2}\ln\left(1+\frac{l_1^2V^4}{G^2M^2}\right),$$

where l_1 is the extension of the cloud. This force is not negligible if

$$\frac{l_1^2V^4}{G^2M^2}\gg 1$$

i.e. when the linear dimensions of the cloud are much greater than the accretion radius $\sim GM/V^2$. This means that the braking effect is essentially due to the particles far away and not to those falling on the star.

One important consequence of this fact is that the same expression for the force is valid also in the case of interacting particles.

As accretion increases when relative velocity decreases (cf. formula (3)), Mc CREA [6] has investigated the problem if the braking of stars by interstellar gas in the galaxy is responsible for the exceedingly great luminosity of some of them and SALPETER [15] has examined whether the emission of quasars can be explained in terms of the braking of massive objects by diffuse matter.

§ 7. Hydrodynamics of accretion in the case of subsonic relative velocity of stars and clouds

Let us now consider the case where the relative velocity between the star and the cloud is equal or less than the sound velocity in the cloud. This problem has been investigated by BONDI [1] in the hydrodynamical approximation. See also ZEL'DOVICH and NOVIKOV [17], p. 435. A star of mass M is at rest in an infinitely extended gas cloud with a density ϱ_∞ and pressure p_∞. The motion of the gas is stationary and spherically symmetric. We shall neglect the increase of the star mass, so that the field of force is constant. The gas can be characterized by its adiabatic index γ, density ϱ, pressure p and sound velocity:

$$a = \left(\frac{\gamma p}{\varrho}\right)^{1/2}.$$

The phenomenon obeys the following equations:

1) Continuity equation:

$$4\pi r^2 \varrho v = A \text{ (constant)}, \tag{7}$$

where r is the radial coordinate, v the velocity of the gas directed toward the star and A is a constant which represents the accretion rate in gram per second.

2) Bernoulli equation:

$$\frac{v^2}{2} + \int_{p_\infty}^{p} \frac{dp}{\varrho} - \frac{GM}{r} = \text{const} = 0 .$$

The integration constant is zero because of boundary conditions at infinity.

3) Adiabatic equation:

$$\frac{p}{p_\infty} = \left(\frac{\varrho}{\varrho_\infty}\right)^\gamma . \tag{8}$$

By means of (8), the Bernoulli equation takes the form:

$$\frac{v^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p_\infty}{\varrho_\infty} \left[\left(\frac{p}{\varrho_\infty}\right)^{\gamma-1} - 1\right] = \frac{GM}{r} . \tag{9}$$

Eqs. (8) and (9) are valid in case no heat exchange takes place between two neighbouring fluid elements. We can however take account of heat exchange by suitably modifying γ, which in any case must always lie between 1 and 5/3.

The adiabatic index is in general defined as:

$$\gamma = \left(\frac{\partial \log p}{\partial \log \varrho}\right)_{S = \mathrm{const.}} ,$$

where S is the entropy per gram given by

$$S = -\frac{\mathfrak{R}}{\mu} \ln\left(\frac{\varrho}{m\mu}\right) + \frac{3}{2} \frac{\mathfrak{R}}{\mu} \ln kT + \text{constant.}$$

Here $\mathfrak{R} = 8.31 \cdot 10^7$ erg/°K · gr is the perfect gas constant, $K = 1.38 \cdot 10^-$ erg/°K is the Boltzmann constant with $\mathfrak{R} = N_0 K$ and $N_0 = 6.023 \cdot 10^{23}\ \mathrm{gr}^{-1}$ is Avogadro's number (inverse of proton mass); μ is the molecular weight defined, for a neutral gas, as the number of nucleons in a nucleus. For a neutral gas the number of particles per c.c. n is given by $n = \dfrac{\pi}{\mu m_H}$. The perfect gas equation is:

$$p = nkT = \frac{\varrho}{\mu m_H} kT = \frac{\varrho}{\mu}(N_0 k)T = \frac{\varrho \mathfrak{R} T}{\mu} .$$

If the gas is completely ionized the number of particles per c.c. n is given by the number of nuclei per c.c., i.e. $\dfrac{\varrho}{\mu m_H}$ plus the number of electrons $z\dfrac{\varrho}{\mu m_H}$, i.e.:

$$n = \frac{1 + z}{\mu} \frac{\varrho}{m_H} .$$

In this way to a completely ionized gas can be attributed a molecular weight

$$\mu' = \frac{\mu}{1 + z} ,$$

where μ is the molecular weight for the neutral gas.

So for neutral hydrogen and helium one has respectively: $\mu = 1$ and $\mu = 4$; for the same gases when completely ionized, $\mu' = 1/2$ and $\mu' = 4/3$, respectively. When the gas is either completely ionized or neutral, γ can be calculated by the above formulae getting:

$$\ln p = \text{const} + \frac{5}{3} \log \varrho ,$$

from which

$$\gamma = \frac{5}{3} .$$

In general however the ionization state of a gas depends on pressure and density, i.e. γ is a function of p and ϱ bounded between 1 and 5/3. (Cf. ZEL'DOVICH and NOVIKOV [17] p. 213).

As during accretion the physical conditions of the gas are variable, one can expect a variation of γ too.

These configurations are particularly important for accretion onto compact objects, while for normal stars, γ can be considered fairly well constant. Eqs. (7) and (9) can be put in a dimensional form by introducing the sound speed at infinity:

$$a_\infty^{2} = \gamma \frac{p_\infty}{\varrho_\infty}$$

and putting:

$$r = x \frac{GM}{a_\infty^2} , \; v = y a_\infty , \; \varrho = z \varrho_\infty$$

the continuity equation takes the form:

$$x^2 y z = \lambda , \tag{10}$$

where λ is given by:

$$A = \frac{4\pi\lambda (GM)^2 \varrho_\infty}{a_\infty^3} . \tag{11}$$

The Bernoulli equation has the form:

$$\frac{1}{2} y^2 + \frac{z^{\gamma-1} - 1}{\gamma - 1} = \frac{1}{x} . \tag{12}$$

To solve (10) and (11), let us put

$$u = y z^{-\frac{\gamma-1}{2}} , \tag{13}$$

where u is the ratio of the local velocity of the gas v to the local sound speed $\left(\frac{\gamma p}{\varrho}\right)^{1/2}$ In fact going over to original variables

$$u = \frac{v}{a_\infty}\left(\frac{\varrho}{p_\infty}\right)^{-\frac{\gamma-1}{2}}.$$

But

$$a = \left(\frac{\gamma p}{\varrho}\right)^{1/2} = \left(\frac{\gamma p_\infty}{\varrho_\infty}\right)^{1/2}\left(\frac{\varrho_H}{p_\infty}\cdot\frac{p}{\varrho}\right)^{1/2}$$

and therefore, because (8):

$$a = a_\infty\left[\frac{\varrho_\infty}{\varrho}\left(\frac{\varrho}{\varrho_\infty}\right)^{\gamma}\right]^{1/2} = a_\infty\left(\frac{\varrho}{\varrho_\infty}\right)^{\frac{\gamma-1}{2}}$$

so that

$$u = \frac{v}{a}.$$

From (13) and (10), we get:

$$y = u^{\frac{2}{\gamma+1}}\left(\frac{\lambda}{x^2}\right)^{\frac{\gamma-1}{\gamma+1}},\quad z = \left(\frac{\lambda}{x^2 u}\right)^{\frac{2}{\gamma+1}}$$

and (12) becomes:

$$u^{\frac{4}{\gamma+1}}\left(\frac{1}{2}+\frac{1}{\gamma-1}\,\frac{1}{u^2}\right) = \lambda^{-2\frac{\gamma-1}{\gamma+1}}\left[\frac{x^{\frac{\gamma-1}{\gamma+1}}}{\gamma-1} + x^{-\frac{5-3\gamma}{\gamma+1}}\right]. \tag{14}$$

The right and left hand side of this equation are separately the sum of a positive and a negative power of their variables, and therefore each of them has a minimum. The left hand side minimum occurs when $u = 1$ and is given by $\frac{1}{2}\frac{\gamma+1}{\gamma-1}$. The x dependent part of the right hand side has a minimum when $x = \frac{1}{4}(5-3\gamma)$, the value of which is

$$\frac{1}{4}\,\frac{\gamma+1}{\gamma-1}\left[\frac{1}{4}(5-3\gamma)\right]^{-\frac{5-3\gamma}{\gamma+1}}. \tag{15}$$

On substitution of these values in (14) one gets the results that λ cannot be greater than

$$\lambda_c = \left(\frac{1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\left(\frac{5-3\gamma}{4}\right)^{-\frac{5-3\gamma}{2(\gamma-1)}}. \tag{16}$$

Therefore the accretion rate cannot be greater than

$$A_c = \frac{4\pi\,\lambda c\,(GM)^2\,\varrho_\infty}{a_\infty^3}. \tag{17}$$

A_c takes on values between 1.12 (for $\gamma = 1$) and 0.75 (for $\gamma = 5/3$). This means that if $\lambda > \lambda_c$, the problem has no solution and accretion cannot take place. A simple graphical discussion (see ZEL'DOVICH and NOVIKOV [17], p. 436) shows that if $\lambda > \lambda_c$, the velocity of the gas is everywhere less than the speed of sound (subsonic accretion everywhere). If $\lambda > \lambda_c$, there exists a distance above which the velocity of the gas is less than the sound speed and beneath which it is greater (supersonic accretion).

In this case, as for $x = 1/4(5-3\gamma)$, $u = 1$, i.e. $v = a$, the radius at which transition to supersonic accretion takes place is

$$r_s = \frac{5-3\gamma}{4}\,\frac{GM}{a_\infty^2}\,.$$

At this radius $y = \left(\frac{2}{5-3y}\right)^{1/2}$ and so:

$$v_s = a_s = a_\infty \left(\frac{2}{5-3\gamma}\right)^{1/2}.$$

As $z = \left(\frac{2}{5-3\gamma}\right)^{\frac{1}{\gamma-2}}$, we get for the density:

$$\varrho_s = \varrho_\infty \left(\frac{2}{5-3\gamma}\right)^{\frac{1}{\gamma-1}},$$

Only supersonic accretion an can give energy to the star, while the subsonic case can be considered as a settling of the gas on the stellar atmosphere, the latter case is possible if pressure, near the star surface is sufficiently high.

To ensure supersonic accretion, the existence of r_s is not sufficient, but it is also necessary that phenomena taking place in stellar atmosphere do not perturb the gas conditions at a distance r_s.

From the barometric formula, one can evaluate the height of the atmosphere:

$$H = \frac{N_0 kTR^2}{GM\mu}\,.$$

The above condition is therefore

$$H + R < r_s\,.$$

If now $r_s \gg R$, supersonic accretion is certainly possible for $H \ll R$ and this entails:

$$T \ll 10^7 \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right).$$

Condition $r_s \gg R$ is definitely verified, for a realistic state of gas at infinity, i.e. $T \leq 10^4$ °K, even for stars with a radius substantially greater then R_0. A fortiori this condition holds for collapsed stars.

We have therefore a vast class of stars for which accretion is supersonic and therefore can be a source of energy.

In computations, λ_c can be considered of unity order, so that from (17) we get

$$A = \frac{dM}{dt} = \frac{4\pi G^2 M^2 \varrho_\infty}{a_\infty^3} \tag{18}$$

If a star moves with respect to intersteller gas at a speed less than the speed of sound, accretion is essentially dominated by sound velocity, while for supersonic relative velocity (cf. formula (4)) it is dominated by the star velocity.

The hydrodynamical problem in the case of arbitrary relative velocity has not yet been solved. We have only BONDI's conjecture (BONDI [1]) that accretion rate is given by

$$A = \frac{4\pi (GM)^2 \varrho_\infty}{(v^2 + a_\infty^2)^{3/2}}, \tag{19}$$

which admits, as limiting cases, formulae (4) and (18). BONDI's conjecture has not been confirmed, not invalidated: there is only a partial confirmation by DODD [4]. In any case, formula (19) agrees with one's intuition and certainly gives the correct order of magnitude.

In order to make up one's mind which of the theories exposed so far is to be applied to real cases, one must first of all check whether the body has a subsonic or supersonic velocity with respect to the gas.

If the velocity is subsonic, one can apply (approximately) the theory discussed in this section (body at rest), if on the contrary one has to do with supersonic velocity one must apply the theory of Section 1.

In the cases so far examined, i.e. those relative to stars, the geometrical radius of a body turns out to be always many orders of magnitude less than the various accretion radii considered; in the case of galaxies or of clusters of galaxies the geometrical radius and accretion radius are comparable.

§ 8. The general hydrodynamical problem

The general hydrodynamical problem, i.e. the solution of hydrodynamical time dependent equations at various Mach's number, has been studied for the first time by HUNT [13] though in an incomplete form. The procedure consists in integrating the time dependent equations of fluid dynamics, from a given initial time up to the time when a stationary solution is reached. The fundamental equations are:

$$\frac{\partial \varrho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 m)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(n\sin\theta)=0$$

(conservation of mass per unit volume),

$$\frac{\partial m}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(p+\frac{m^2}{\varrho}\right)\right]+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{mn}{\varrho}\sin\theta\right)=$$

$$=-\frac{\varrho}{r^2}+\frac{v_\theta n}{r}+\frac{2p}{r}$$

(conservation of radial momentum per unit volume),

$$\frac{\partial n}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2 mn}{\varrho}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\left(p+\frac{n^2}{\varrho}\right)\sin\theta\right]=-\frac{v_\theta m}{r}+\frac{p}{r}\cot\theta$$

(conservation of transverse momentum per unit volume),

$$\frac{\partial E}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(E+p)\frac{m}{\varrho}\right]+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[(E+p)\frac{n}{\varrho}\sin\theta\right]=-\frac{m}{r^2}$$

(conservation of total energy per unit volume),

$$p=(\gamma-1)\left(E-\frac{1}{2}\frac{m^2+n^2}{\varrho}\right)$$

(equation of state).

The solution found by HUNT by numerical integration of the above equations are not general, because:

a) the mass M of the star is taken as a constant.
b) The braking forces of Section 6 are neglected.
c) The cooling of the gas is not taken into account.
d) Only $\gamma = 5/3$ is considered.

While assumption from a) to c) are well justified in many cases of physical interest, assumption d) is a severe restriction to the generality of the solution (the author is fully aware of this limitation).

HUNT's results (for Mach's numbers 0.6; 1.4; 2.4) confirm the result obtained in preceding sections. In particular:

1) for subsonic relative velocities one has practically spherical symmetric accretion and therefore BONDI's theory for a body at rest can be applied.

2) In the case of supersonic relative velocities there appears a shock front, which, for increasing Mach's number, approaches the body and shrinks downstream to the axis of accretion (HOYLE—LYTTLETON—BONDI's theory).

§ 9. Conclusions

The physical theory of accretion, as it appears from the above considerations turns out to be in a fairly satisfactory state. In fact one can confidently use the laws discussed in the preceding sections to get reliable order of magnitude estimates. It appears also that the main features were already clear by the mid fifties and no substantial progress has been made since then. There remains however a set of important problems still to be solved or deepened.

1) A thorough investigation of the general hydrodynamical problem with varying γ and any star velocity is still lacking. This problem is crucial for the theory of accretion onto collapsed stars in binary systems where the velocities are highly supersonic. HUNT's solution, which takes into account Mach's numbers up to 2.4 is clearly inadequate. It would be very interesting also a proof of BONDI's conjecture (19).

2) Due to the great difficulties of the magneto-hydrodynamical equations, the influence of a magnetic field on accretion has not yet been satisfactorily investigated. What one can find in the literature is only a host of partial results which, though very important, are not yet systematically arranged in a general framework. This state of affairs is particularly relevant for accretion onto neutron stars and black-holes, in connection with the theory of galactic X-ray sources. We hope to give a survey of these partial results in a forthcoming paper.

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