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RAYLEIGH-TAYLOR INSTABILITY OF A COMPOSITE MIXTURE THROUGH POROUS MEDIUM

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The frictional effect of collisions of ionized with neutral atoms on the Rayleigh—Taylor instability of a composite mixture through porous medium is considered in the presence of a horizontal magnetic field. For the case of two uniform fluids separated by a horizontal boundary, the magnetic field completely stabilizes certain wave-number band. For the case of exponentially varying density, the collisions are found to have no effect as such on the stratification. However for the stable stratification, the growth rates increase with the increase in permeability of the medium whereas for the unstable stratification, the growth rates may be both increasing or decreasing.

1. Introduction

CHANDRASEKHAR [1] has given a detailed account of the stability of superposed fluids in the presence of magnetic field through non-porous medium. When a fluid permeates a porous material, the actual path of an individual particle of fluid cannot be followed analytically. The effect, as the fluid slowly percolates through the pores of the rock, is represented by a macroscopic law. This is the usual Darcy's law. As a result of this, the usual viscous term in the equations of fluid motion is replaced by the resistance term $(\mu/k_1)\mathbf{q}$, where μ is the viscosity of the fluid, k_1 the permeability of the medium and $\mathbf{q}$ the velocity of the fluid, calculated from Darcy's law. WOODING [2] has considered the Rayleigh instability of a thermal boundary layer in flow through a porous medium.

It is quite frequent that the medium is not fully ionized and may be permeated with neutral atoms. The medium has been idealized therefore, following HANS [3], as a composite mixture of a hydromagnetic (ionized) component and a neutral component, the two interacting through mutual collisional (frictional) effects. HANS [3] and BHATIA [4] have shown that the collisions have a stabilizing effect on the Rayleigh—Taylor instability. However, for the Kelvin—Helmholtz configuration, RAO and KALRA [5] and HANS [3] have found that the collisional effects are in fact destabilizing for a sufficiently large collision frequency.

In the present paper we study the collisional and porosity effects on the Rayleigh—Taylor instability of a composite mixture through porous medium in hydromagnetics.

2. Perturbation equations

Consider an incompressible composite layer consisting of an infinitely conducting hydromagnetic fluid of density ϱ, permeated with neutrals of density ϱ_d, arranged in horizontal strata; through porous medium and acted on by gravity force $\mathbf{g}(0, 0, -g)$ and horizontal magnetic field $\mathbf{H}(H, 0, 0)$. Assume that both the ionized fluid and neutral fluid behave like continuum fluids and that effects on the neutral component resulting from the fields of gravity and pressure are neglected. The magnetic field interacts with the hydromagnetic component only.

Let $\delta\varrho$, δp, $\mathbf{q}(u, v\ \ w)$ and $\mathbf{h}(h_x, h_y, h_z)$ denote respectively the perturbations in density, pressure, velocity and magnetic field $\mathbf{H}$; $\mathbf{q}_d$, ν_c, μ_e and μ denote the velocity of the neutral fluid, the mutual collisional (frictional) frequency between the two components of the composite medium, the magnetic permeability of the medium and the viscosity of the hydromagnetic fluid, respectively. Then the linearized perturbation equations governing the motion of the composite medium are

$$\varrho \frac{\partial \mathbf{q}}{\partial t} = -\nabla \delta p + \mathbf{g}\delta\varrho + \frac{\mu_e}{4\pi}(\nabla \times \mathbf{h}) \times \mathbf{H} + \varrho_d \nu_c(\mathbf{q}_d - \mathbf{q}) - \frac{\varrho \nu}{k_1}\mathbf{q}, \tag{1}$$

$$\frac{\partial \mathbf{q}_d}{\partial t} = -\nu_c(\mathbf{q}_d - \mathbf{q}), \tag{2}$$

$$\nabla \cdot \mathbf{q} = 0, \ \nabla \cdot \mathbf{h} = 0, \tag{3}$$

$$\frac{\partial}{\partial t}\delta\varrho = -w\frac{d\varrho}{dz}, \tag{4}$$

$$\frac{\partial \mathbf{h}}{\partial t} = \nabla \times (\mathbf{q} \times \mathbf{H}). \tag{5}$$

Analyzing the disturbances into normal modes, we seek solutions whose dependence on space coordinates x, y, z and time t is of the form

$$f(z) \exp(ik_x x + ik_y y + nt), \tag{6}$$

where $k_x, k_y \left(k = \sqrt{(k_x^2 + k_y^2)}\right)$ are the wave numbers along x and y directions respectively, $f(z)$ is some function of z and n is a complex constant.

Eliminating $\mathbf{q}_d$ between Eqs. (1) and (2) and using (6), Eqs. (1)—(5) give

$$\left(n' + \frac{\nu}{k_1}\right)\varrho u = -ik_x\,\delta p, \tag{7}$$

$$\left(n' + \frac{\nu}{k_1}\right)\varrho v = -ik_y\,\delta p + \frac{\mu_e H}{4\pi}(ik_x h_y - ik_y h_x), \tag{8}$$

$$\left(n' + \frac{\nu}{k_1}\right) \varrho w = -D\delta p - \frac{\mu_e H}{4\pi}(Dh_x - ik_x h_z) - g\delta\varrho, \tag{9}$$

$$ik_x u + ik_y v + Dw = 0, \tag{10}$$

$$ik_x h_x + ik_y h_y + Dh_z = 0, \tag{11}$$

$$n\delta\varrho = -wD\varrho, \tag{12}$$

$$nh_x = ik_x Hu,\ nh_y = ik_x Hv,\ nh_z = ik_x Hw, \tag{13}$$

where

$$n' = n\left(1 + \frac{\alpha_0 \nu_c}{n + \nu_c}\right),\ \nu = \frac{\mu}{\varrho},\ \alpha_0 = \frac{\varrho_d}{\varrho} \quad \text{and} \quad D = \frac{d}{dz}.$$

Eliminating δp between Eqs. (7)—(9) and using Eqs. (10)—(13), we get

$$n'[D(\varrho Dw) - k^2 \varrho w] + \frac{1}{k_1}[D(\varrho\nu Dw) - k^2 \varrho\nu w] + \frac{\mu_e k_x^2 H^2}{4\pi n} \times (D^2 - k^2) w + \frac{gk^2}{n}(D\varrho) w = 0. \tag{14}$$

3. Two uniform fluids separated by a horizontal boundary

Consider the case of two uniform fluids of densities ϱ_1 (lower fluid) and ϱ_2 (upper fluid) separated by a horizontal boundary at $z = 0$. Eq. (14) for both regions of fluid reduces to

$$(D^2 - k^2) w = 0. \tag{15}$$

The general solution of Eq. (15) is

$$w = Ae^{+kz} + Be^{-kz}, \tag{16}$$

where A and B are arbitrary constants.

The boundary conditions to be satisfied in the present problem are as follows.

(i) The velocity w should vanish when $z \to +\infty$ (for the upper fluid) and $z \to -\infty$ (for the lower fluid).
(ii) $w(z)$ is continuous at $z = 0$.
(iii) The pressure should be continuous across the interface.

The continuity of pressure means that

$$n' \Delta_0(\varrho Dw) + \frac{1}{k_1}\Delta_0(\varrho\nu Dw) + \frac{\mu_e k_x^2 H^2}{4\pi n}\Delta_0(Dw) + \frac{gk^2}{n}\Delta_0(\varrho) w_0 = 0. \tag{17}$$

Applying the boundary conditions (i) and (ii), we can write

$$w_1 = Ae^{+kz}(z < 0)\,, \tag{18}$$

and

$$w_2 = Ae^{-kz}(z > 0)\,, \tag{19}$$

the same constant A has been chosen to ensure the continuity of w at $z = 0$.

Applying the condition (17) to the solutions (18) and (19), we get

$$n^3 + \left[\nu_c(\alpha_0 + 1) + \frac{1}{k_1}(\alpha_1\nu_1 + \alpha_2\nu_2)\right]n^2 + \left[\frac{\nu_c}{k_1}(\alpha_1\nu_1 + \alpha_2\nu_2) + 2k_x^2 v_A^2 + \right.$$
$$\left. + gk(\alpha_1 - \alpha_2)\right]n + [2k_x^2 v_A^2 + gk(\alpha_1 - \alpha_2)]\,\nu_c = 0\,, \tag{20}$$

where $v_A^2 = \mu_e H^2/4\pi(\varrho_1 + \varrho_2)$.

(a) Stable case ($\varrho_1 > \varrho_2$).

In this case, Eq. (20) does not allow any positive root as there is no change of sign. This means that the system is stable.

(b) Unstable case ($\varrho_2 > \varrho_1$).

In this case if

$$2k_x^2 v_A^2 < gk(\alpha_2 - \alpha_1)\,, \tag{21}$$

the constant term in Eq. (20) is negative. Eq. (20) therefore allows one change of sign and so has one positive root. The occurrence of positive root implies that the system is unstable. If

$$2k_x^2 v_A^2 > gk(\alpha_2 - \alpha_1)\,, \tag{22}$$

Eq. (20) does not admit of any change of sign and so no positive root occurs. The system is therefore stable.

Thus for the unstable case ($\varrho_2 > \varrho_1$), the system is stable or unstable according as $\varrho_2 - \varrho_1$ is less than or greater than $\mu_e H^2 k_x^2/2\pi gk$. In the absence of magnetic field, the system is unstable for $\varrho_2 > \varrho_1$, as one of the values of n given by Eq. (20) is positive. But the presence of magnetic field has got stabilizing effect and completely stabilizes the wave-number band $k > k_*$ where

$$k_* = \frac{2\pi g(\varrho_2 - \varrho_1)}{\mu_e H^2}\sec^2\theta\,, \tag{23}$$

and θ is the inclination of the wave vector $\mathbf{k}$ to the direction of magnetic field $\mathbf{H}$.

4. The case of exponentially varying density

Let us consider the density stratification in a continuously stratified medium of depth d as

$$\varrho(z) = \varrho_0 e^{\beta z}, \tag{24}$$

where ϱ_0 and β are constants. Let us assume that $\beta d \ll 1$, i.e., the variation of density at two neighbouring points in the velocity field, which is much less than the average density, has a negligible effect on the inertia of the fluid.

Following CHANDRASEKHAR [1], the boundary conditions for the case of two free surfaces are

$$w = D^2 w = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad d\,. \tag{25}$$

The proper solution of Eq. (14) satisfying (25) is

$$w = A \sin \frac{m\pi z}{d}\,, \tag{26}$$

where A is a constant and m is any integer.

Substituting (26) in (14) and neglecting the effect of heterogeneity on the inertia, we get

$$\left[n' + \frac{\nu}{k_1} + \frac{k_x^2 v^2}{n}\right]\left\{\left(\frac{m\pi}{d}\right)^2 + k^2\right\} - \frac{g\beta k^2}{n} = 0, \tag{27}$$

which on simplification gives

$$\begin{aligned} n^3 + \left[\nu_c(\alpha_0 + 1) + \frac{\nu}{k_1}\right] n^2 + \left[\frac{\nu}{k_1}\nu_c + k_x^2 V^2 - \frac{g\beta k^2}{L}\right] n + \\ + \left[k_x^2 v^2 - \frac{g\beta k^2}{L}\right]\nu_c = 0\,, \end{aligned} \tag{28}$$

where

$$v^2 = \frac{\mu_e H^2}{4\pi\varrho_0} \quad \text{and} \quad L = \left(\frac{m\pi}{d}\right) + k^2.$$

For the stable stratification ($\beta < 0$), Eq. (28) does not have any positive root implying thereby that the system is stable. For the unstable stratification ($\beta > 0$) and for $k^2 > k_x^2 v^2 L/g\beta$, the constant term in Eq. (28) is negative. This means that Eq. (28) possesses one positive root implying thereby that the system is unstable. Let n_0 denote the positive root of Eq. (28). Then

$$\begin{aligned} n_0^3 + \left[\nu_c(\alpha_0 + 1) + \frac{\nu}{k_1}\right] n_0^2 + \left[\frac{\nu}{k_1}\nu_c + k_x^2 v^2 - \frac{g\beta k^2}{L}\right] n_0 + \\ + \left[k_x^2 v^2 - \frac{g\beta k^2}{L}\right]\nu_c = 0\,. \end{aligned} \tag{29}$$

To find the role of collisions concerning the growth rate of unstable modes, we examine the nature of $dn_0/d\nu_c$. Eq. (29) gives

$$\frac{dn_0}{d\nu_c} = -\frac{n_0^2(\alpha_0+1)+\frac{\nu}{k_1}n_0+\left(k_x^2 v^2-\frac{g\beta k^2}{L}\right)}{3n_0^2+2\left[\nu_c(\alpha_0+1)+\frac{\nu}{k_1}\right]n_0+\left[\frac{\nu}{k_1}\nu_c+k_x^2v^2-\frac{g\beta k^2}{L}\right]}. \tag{30}$$

Therefore if, in addition to $k^2 > k_x^2 v^2 L/g\beta$, which is a sufficient condition for instability, we have either of the conditions

$$k_x^2 v^2 - \frac{g\beta k^2}{L} \gtrless -\left[\frac{\nu}{k_1}\nu_c + 2n_0\left\{\nu_c(\alpha_0+1)+\frac{\nu}{k_1}\right\}+3n_0^2\right], \tag{31}$$

$dn_0/d\nu_c$ is always negative. Thus with the increase in collisional frequency, the growth rate decreases.

We conclude therefore that for $k^2 > k_x^2 v^2 L/g\beta$, the system is unstable and the growth rate, under either of the conditions (31), decreases with the increase of collisions. If $k^2 < k_x^2 v^2 L/g\beta$, the system is stable.

To find the effect of permeability of the medium on growth rates, we examine the nature of dn_0/dk_1. Eq. (29) gives

$$\frac{dn_0}{dk_1} = \frac{\frac{\nu}{k_1^2}(n_0+\nu_c)n_0}{3n_0^2+2n_0\left\{\nu_c(\alpha_0+1)+\frac{\nu}{k_1}\right\}+\left\{\frac{\nu}{k_1}\nu_c+k_x^2v^2-\frac{g\beta k^2}{L}\right\}}. \tag{32}$$

Eq. (32) implies that for stable stratification ($\beta < 0$), dn_0/dk_1 is positive; meaning thereby that with the increase in permeability of the medium, the growth rate increases for the stable stratification.

For unstable stratification ($\beta > 0$) and for

$$\frac{g\beta k^2}{L} \gtrless 3n_0^2 + 2n_0\left\{\nu_c(\alpha_0+1)+\frac{\nu}{k_1}\right\}+\left\{\frac{\nu}{k_1}\nu_c + k_x^2 v^2\right\}, \tag{33}$$

dn_0/dk_1 is negative or positive for the greater than or less than sign, respectively; meaning thereby that with the increase in permeability of the medium, the growth rates are both decreasing and increasing for the unstable stratification.

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