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SIMILARITY SOLUTIONS FOR PLANE RELATIVISTIC FLOW IN A HOMOGENEOUS MEDIUM

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Similarity solutions in closed forms for propagation of plane shock waves in a relativistic gas of uniform number density are obtained. The shock moves with constant speed.

Introduction

Similarity solutions for one-dimensional flow of a relativistic fluid headed by a shock front in a cold gas are studied by ELTGROTH [2], by assuming the velocity of the fluid as the similarity variable.

In the present paper similarity solutions in closed forms are obtained when the plane shock front moves through a homogeneous medium of uniform nucleon number density.

The origin of the (x, t) inertial co-ordinate frame is taken at a plane where an initial disturbance is given.

We find that such a flow for the similarity parameter $\xi = x^a t^b$, a and b being suitable constants, exists only when the shock moves with a constant velocity. The solutions given in this paper are applicable only to a medium of uniform pressure or a cold gas.

Equations of motion and boundary conditions

$$\frac{\partial}{\partial x}\left\{\frac{p+\beta^2 E}{1-\beta^2}\right\}+\frac{\partial}{\partial\, ct}\left\{\frac{\beta(p+E)}{1-\beta^2}\right\}=0, \tag{1}$$

$$\frac{\partial}{\partial x}\left\{\frac{\beta(p+E)}{1-\beta^2}\right\}+\frac{\partial}{\partial ct}\left\{\frac{E+\beta^2 p}{1-\beta^2}\right\}=0, \tag{2}$$

$$\frac{\partial}{\partial x}\left\{\frac{n\beta}{\sqrt{1-\beta^2}}\right\}+\frac{\partial}{\partial ct}\left\{\frac{n}{\sqrt{1-\beta^2}}\right\}=0, \tag{3}$$

where p is the pressure and E the proper energy density, both measured in the rest frame of the fluid; $c\beta$ is the fluid velocity in the (x, t) inertial frame, c being the speed of light and n is the nucleon number density.

The shock conditions are

$$\left[\frac{p+\beta^2 E}{1-\beta^2}\right]=\frac{dX}{dct}\left[\frac{\beta(p+E)}{1-\beta^2}\right], \tag{4}$$

$$\left[\frac{\beta(p+E)}{1-\beta^2}\right]=\frac{dX}{dct}\left[\frac{E+\beta^2 p}{1-\beta^2}\right], \tag{5}$$

$$\left[\frac{n\beta}{(1-\beta^2)^{1/2}}\right]=\frac{dX}{dct}\left[\frac{n}{(1-\beta^2)^{1/2}}\right]. \tag{6}$$

Here [ ] signifies the discontinuity sign and dX/dt is the velocity of the shock, X being the distance of the shock plane from the origin.

The nucleon density of the pre-shock stage is given by

$$n_0 = \text{constant}. \tag{7}$$

Solutions of the equations

We next introduce the following similarity transformations as done by COURANT and FRIEDRICHS [1] for non-relativistic motions of gases:

$$c\beta = \frac{x}{t}\,U(\xi),$$

$$p = x^{k+2}\,t^{\lambda-2}\,P(\xi),$$

$$E = x^{k+2}\,t^{\lambda-2}\,Z(\xi),$$

$$n = x^{k}\,t^{\lambda}\,\Omega(\xi),$$

where

$$\xi = x^a t^b.$$

Here λ, k, a and b are constants to be determined from the problem.

By their direct substitutions in Eqs. (1)—(3) and boundary conditions (4)—(6), we find that these forms are compatible only when we choose $a/b=-1$ and $\lambda+k=0$. Without any loss we take $a=1$ and $b=-1$, $\lambda=0$ and $k=0$.

For our subsequent work we choose the similarity parameter in dimensionless form as

$$\eta = \frac{x}{ct}\,. \tag{8}$$

At the shock front $\eta = \eta_0$ and is taken as constant. So,

$$\eta_0 = \frac{X}{ct} = \frac{V}{c}, \tag{9}$$

V being the constant velocity of the shock front.

The boundary conditions at the shock front may be re-written as

$$\frac{p_1}{p_0} = \frac{\beta_1 \eta_0 N + 1}{1 - \beta_1 \eta_0}, \tag{10}$$

$$\frac{E_1}{E_0} = \frac{\eta_0 N + \beta_1}{N(\eta_0 - \beta_1)}, \tag{11}$$

$$\frac{n_1}{n_0} = \eta_0 \frac{(1 - \beta_1^2)^{1/2}}{\eta_0 - \beta_1}, \tag{12}$$

where $N = E_0/p_0$ and the subscripts 1 and 0 stand respectively for quantities just behind the shock and just in front of it.

We find that unlike its non-relativistic analogue, there are two characteristic parameters, instead of one, namely, $\eta_0 = V/c$ and N, depending on the equation of state of matter in its pre-shock condition.

In the region behind the shock plane we take the equation of state,

$$p = \frac{1}{3} E. \tag{13}$$

Eqs. (1)—(3) are now transformed as

$$\frac{d}{d\eta}\left[\frac{E(1/3 + \beta^2)}{1 - \beta^2}\right] = \eta \frac{d}{d\eta}\left[\frac{4/3\, \beta E}{1 - \beta^2}\right], \tag{14}$$

$$\frac{d}{d\eta}\left[\frac{4/3\, \beta E}{1 - \beta^2}\right] = \eta \frac{d}{d\eta}\left[\frac{E(1 + \beta^2/3)}{1 - \beta^2}\right], \tag{15}$$

$$\frac{d}{d\eta}\left[\frac{n\beta}{\sqrt{1 - \beta^2}}\right] = \eta \frac{d}{d\eta}\left[\frac{n}{\sqrt{1 - \beta^2}}\right]. \tag{16}$$

Combining (1) and (2) we also find as ELTGROTH [2]

$$\frac{1}{E}\frac{dE}{d\eta} = \pm \frac{4}{\sqrt{3}} \cdot \frac{1}{(1 - \beta^2)} \frac{d\beta}{d\eta}. \tag{17}$$

Eqs. (1) and (2) are next re-arranged as

$$\frac{1}{E}\frac{dE}{d\eta} = \frac{d\beta}{d\eta} \cdot \frac{4}{(1 - \beta^2)} \cdot \frac{(\beta^2 \eta + \eta - 2\beta)}{(1 + 3\beta^2 - 4\beta\eta)} \tag{18}$$

and

$$\frac{1}{E}\frac{dE}{d\eta}=\frac{d\beta}{d\eta}\cdot\frac{4}{(1-\beta^2)}\cdot\frac{(2\beta\eta-1-\beta^2)}{(4\beta-3\eta-\eta\beta^2)}, \tag{19}$$

Comparing (18) and (19) with (17) when $d\beta/d\eta \neq 0$, we get,

$$\frac{\beta^2\eta+\eta-2\beta}{1+3\beta^2-4\beta\eta}=\frac{2\beta\eta-1-\beta^2}{4\beta-3\eta-\eta\beta^2}=l, \tag{20}$$

where

$$l=\frac{1}{\sqrt{3}} \text{ or } -\frac{1}{\sqrt{3}}.$$

The trial $l=\frac{1}{\sqrt{3}}$ yields as a solution

$$\beta=\frac{\sqrt{3}\,\eta-1}{\sqrt{3}-\eta} \tag{21}$$

and $l=-\frac{1}{\sqrt{3}}$, represents the solution

$$\beta=\frac{\sqrt{3}\,\eta+1}{\sqrt{3}+\eta}. \tag{22}$$

Case I

We next investigate the existence of solutions for the case $N=3$, which is appropriate for an ultra-relativistic initial state.

For the choice $n_0=$ const. (Eq. (7)), we easily find from the shock conditions that in this case both E_0 and p_0 are constants.

Eqs. (10) and (11) now give

$$\beta_1=\frac{3\eta_0^2-1}{2\eta_0},\ 0<\eta_0<1 \text{ and } |\beta_1|\leq 1. \tag{23}$$

Besides, the solution (21) is consistent with (23) only when $\eta_0=\frac{1}{\sqrt{3}}$ and $\eta_0=1$. Both these values of η_0 are not tenable, as $\eta_0=\frac{1}{\sqrt{3}}$ in this case implies the shock speed as equivalent to the speed of sound and $\eta_0=1$, the shock speed attaining the photonic speed. The solution (22) is also inconsistent with the requirement (23).

So, for the medium considered here, we should take the other alternative in Eqs. (18) and (19), namely, $d\beta/d\eta=0$

Hence,

$$\beta = \beta_1 \tag{24}$$

$$E = E_1 \tag{25}$$

and

$$n = n_1, \tag{26}$$

which are all constants.

Case II

Next we seek solutions for the other possibility $p_0 = 0$, appropriate for a cold gas. The modified shock conditions yield

$$\beta_1 = \frac{2\eta_0 \pm \sqrt{4\eta_0^2 - 3}}{3}\,. \tag{27}$$

Solution (21) is consistent with (27), only when $\eta_0 = \frac{\sqrt{3}}{2}$ and so $\beta_1 = \frac{1}{\sqrt{3}}$. Besides, it is found that solution (22) is unfavourable and extraneous.

For the value of β as given by (21), and from (18) and (16), respectively, we find that

$$\log \frac{E}{E_1} = \frac{2}{\sqrt{3}}\left[\log \frac{1+\eta}{1-\eta} + 2 \log (2 - \sqrt{3})\right], \tag{28}$$

$$\log \frac{n}{n_1} = \frac{\sqrt{3}}{2}\left[\log \frac{1+\eta}{1-\eta} + 2 \log (2 - \sqrt{3})\right]. \tag{29}$$

In this case if the gas is pushed instantaneously and thereby set into motion, there is a possible backflow as clearly indicated by our solutions. As $\eta \to -1$, both the energy density and number density tend to zero, thereby showing that a portion of matter moves backward and the edge of vacuum is at $\eta = -1$. At this boundary we find that the relativistic material moves with the speed of light into the vacuous region.

REFERENCES

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2. P. G. ELTGROTH, Phys. Fluids, **14**, 2631, 1971.