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AN EXACT SOLUTION OF THE PROBLEM OF MHD UNSTEADY VISCOUS FLOW THROUGH A POROUS STRAIGHT CHANNEL

By

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The exact solution of the problem of unsteady incompressible viscous flow under a time-varying pressure gradient in a straight channel with two parallel porous walls with uniform suction and injection at the walls has been obtained by PRAKASH [1]. MATHUR [2] dealt with the unsteady flow of an electrically conducting, viscous and incompressible fluid between two parallel uniform porous walls in the presence of transverse magnetic field when there is a constant injection on the lower wall and an equal suction at the upper wall. The present note is concerned with the study of unsteady flow of an electrically conducting, viscous fluid through a straight channel with two parallel porous flat plates under a time varying pressure gradient when there is equal and uniform suction and injection on the walls. The exact solution of the problem has been obtained when pressure gradient is constant and then the case of steady flow under a constant pressure gradient has been deduced taking the time since the start of motion to be infinite. The flow takes place in the presence of a uniform vertical magnetic field.

Consider an unsteady electrically conducting two-dimensional incompressible flow through a straight channel with two parallel porous flat plates situated at a distance h apart. We take x and y values along and transverse to the parallel plates and assume a uniform magnetic field H_0 acting along y-axes. The fluid is being injected into the channel through the wall at $y = 0$ and is being sucked through the wall at $y = h$ with a uniform velocity V_0. Elastic field E is assumed to be zero. The induced magnetic field due to electrical current flow in the fluid is assumed to be very small and the electric conductivity σ of the fluid is sufficiently large.

At a sufficiently large distance from the origin the flow is fully developed and the physical quantities depend on y and t only. Then the governing equations of the problem are

$$\frac{\partial u}{\partial t} + V_0 \frac{\partial u}{\partial y} = - \frac{1}{\varrho} \frac{\partial p}{\partial x} + v \frac{\partial^2 u}{\partial y^2} - \frac{\sigma}{\varrho} B^2 u , \tag{1}$$

$$0=\frac{\partial p}{\partial y}, \tag{2}$$

with the initial and boundary conditions

$$\begin{array}{ll} 0\leq y\leq h: u=0, \quad v=0 & \text{for} \quad t\leq 0, \\ \left.\begin{array}{l} y=0: u=0, \quad v=V_0 \\ y=h: u=0, \quad v=V \end{array}\right] & \text{for} \quad t>0. \end{array} \tag{3}$$

Analysis

Introducing non-dimensional quantities as

$$u_1=\frac{u}{V_0}, \quad x_1=\frac{x}{h}, \quad y_1=\frac{p}{\varrho V_0^2}, \quad p_1=\frac{t}{h/V_0}, \tag{4}$$

into Eq. (1)—(2), which reduce to

$$\frac{\partial u_1}{\partial t_1}+\frac{\partial u_1}{\partial y_1}=-\frac{\partial p_1}{\partial x_1}+\frac{1}{R_s}\frac{\partial^2 u_1}{\partial y_1^2}-m_1 u_1, \tag{5}$$

$$0=\frac{\partial p_1}{\partial y_1}, \tag{6}$$

$$\begin{array}{l} 0\leq y_1\leq 1: u_1=0 \text{ for } t_1\leq 0, \\ y_1=0,1: u_1=0 \text{ for } t_1>0, \end{array} \tag{7}$$

where $R_s=V_0 h/\nu=$ suction and injection Reynolds number,

$$m_1^{1/2}=\left(\frac{h}{V_0}\cdot\frac{\sigma}{\varrho}\mu_e^2 H_0^2\right)^{1/2}=R_M=\text{magnetic parameter},$$

$$m_1=\frac{M_2}{R_s}, \quad M=\left[\frac{\sigma}{\mu}\mu_e^2 H_0^2 h^2\right]=\text{ Hartmann number.}$$

Now assuming $\partial p_1/\partial x_1=-f(t_1)$, thus (5) reduces to

$$\frac{\partial u_1}{\partial t_1}+\frac{\partial u_1}{\partial y_1}=f(t_1)+\frac{1}{R_s}\frac{\partial^2 u_1}{\partial y_2^2}-m_1 m_1. \tag{8}$$

To obtain the solution of (8), we will apply here Laplace transformation which is defined for velocity u, as

$$\bar{u}_1=\int_0^\infty u_1 e^{-\varphi t_1}dt_1. \tag{9}$$

Thus (8) and (7) transform to

$$\frac{d^2\bar{u}_1}{dy_1^2}-R_s\frac{d\bar{u}_1}{dy_1}-R_s(\lambda+m_1)=-R_s\bar{f}(\lambda) \tag{10}$$

$$\bar{u}_1=0 \text{ at } y_1=0,1, \tag{11}$$

where
$$\bar{f}(\lambda) = \int_0^\infty f(t_1)\, e^{-\lambda_1}\, dt_1 \,. \tag{12}$$

The solution of (10) subject to the boundary condition (11) is

$$\begin{aligned}\bar{u}_1 = \frac{\bar{f}(\lambda)}{(\lambda+m)\sinh\sqrt{B}}\Big[&-\exp\Big\{(y_1-1)\frac{R_s}{2}\Big\}\sinh\big(\sqrt{B}y_1\big) - \\ &-\exp\Big\{y_1\frac{R_s}{2}\Big\}\sinh\{(1-y_1)\sqrt{B}\}\Big] + \frac{\bar{f}(\lambda)}{\lambda+m}\,,\end{aligned} \tag{13}$$

where

$$B = \frac{R_s^2 + 4R_s(\lambda+m)}{4}\,,$$

thus

$$\begin{aligned}u_1 = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\bar{f}(\lambda)}{(\lambda+m_1)\sinh\sqrt{B}}\Big[&-\exp\Big\{(y_1-1)\frac{R_s}{2}\Big\}\sinh\big(\sqrt{B}y_1\big) - \\ &-\exp\Big\{y_1\frac{R_s}{2}\Big\}\sinh\{(1-y_1)\sqrt{B}\} + \frac{\bar{f}(\lambda)}{(\lambda+m)}\Big]e^{\lambda t_1}\, d\lambda.\end{aligned} \tag{14}$$

After assuming pressure gradient constant (i.e., $\partial p_1/\partial x_1 = -f(t_1) = P$, P is positive constant), (14) becomes

$$\begin{aligned}u_1 = \frac{1}{B\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Big[\frac{P}{[-\lambda(\lambda+m_1)\sinh\sqrt{B}} &- e^{(y_1-1)\frac{R_1}{2}}\sinh\big(\sqrt{B}y_1\big) - \\ &- e^{y\frac{R_1}{2}}\sinh\{(1-y_1)\sqrt{B}\}] + \frac{P}{\lambda(\lambda+m_1)}\Big]e^{\lambda t_1}\, d\lambda.\end{aligned} \tag{15}$$

Therefore solution for constant pressure gradient with the help of poles and residue method is given by

$$\begin{aligned}u_1 = \frac{P}{m}&\left[1 - \frac{e^{(y_1-1)\frac{R_s}{2}}\sinh\left\{\sqrt{\frac{R_s^2+4R_s m_1}{4}}\,y_1\right\} + e^{y_1\frac{R_s}{2}}\sinh\left\{\sqrt{\frac{R_s^2+4R_s m_1}{4}}\,(1-y_1\right\}}{\sinh\sqrt{\frac{R_s^2+4R_s m_1}{4}}}\right] + \\ &+ \frac{p}{m}e^{-m_1 t_1}\left[1 + \frac{e^{(y_1-1)\frac{R_s}{2}}\sinh\left\{\frac{R_s}{2}y_1\right\} - e^{y_1\frac{R_s}{2}}\sinh\left\{\frac{R_s}{2}(1-y_1)\right\}}{\sinh\frac{R_s}{2}}\right] + \\ &+ 32PR_s\pi\sum_{n=0}^{\infty}\left[\frac{nc^{\frac{R_s}{2}y_1}\sinh(n\pi y_1)\left\{e^{-\frac{R_s}{2}}(1)^n - 1\right\}e^{-\left(\frac{R_s^2+4\pi n^2}{z}+m_1\right)t_1}}{(R_s^2+4\pi^2n^2)^2+4R_s m_1(R_s^2+4\pi^2n^2)}\right] \\ &\qquad (n = 0, 1, 2, 3, \ldots)\,.\end{aligned} \tag{16}$$

Solution for steady state

Solution for steady state from (16) can be obtained by taking its limit $t \to W$,

$$u_1 = \frac{P}{m}\left[1 - \frac{e^{(y_1-1)\frac{R_s}{2}} \sinh\left\{\sqrt{\frac{R_s^2+4R_s m_1}{4}}\, y_1\right\} + e^{y_1 \frac{R_1}{2}} \sinh\left\{\sqrt{\frac{R_s^2+4R_s m_1}{4}}\,(1-y_1)\right\}}{\sinh\sqrt{\frac{R_\beta^2+4R_s m_1}{4}}}\right]. \tag{17}$$

Now we shall obtain the steady state solution directly from the equation of motion (8) which, after substituting P for $f(t_1)$, reduces to

$$\frac{d^2 u_1}{dy_1^2} - R_s \frac{du_1}{dy_1} - m_1 R_s u_1 = -pR_s \tag{18}$$

with boundary conditions

$$u_1 = 0, \quad y_1 = 0.1\,. \tag{19}$$

It may be easily seen that solution of (18) subject to the boundary conditions (19) is (17).

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