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ELASTIC SHEAR WAVES IN THE PRESENCE OF COUPLE STRESSES

By

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In this paper the effect of the presence of couple stresses in an elastic medium on the propagation of shear waves is discussed. Two different analytical representations for the solution are obtained. Diffusion is seen to dominate near the source. Far down the source two wave fronts appear. One is associated with damping, the other, corresponding to the classical one, is associated with dispersion and its variation for large time is obtained in terms of *Airy function.* By using the Fourier-Laplace transforms, the displacement is expressed in terms of integrands of the modified Bessel function. The integrals are numerically calculated for all time and the results are presented by suitable graphs.

1. Introduction

Continuum mechanics is the study of the response of a medium to deformation. The conservation laws for mass, momentum and energy have to be supplemented by a constitutive law to characterise the medium. The statement of a constitutive law is based on the *Hypothesis* of the existence of a *Stress Vector.* As the body deforms it is assumed that contiguous parts exert a mutual action across bounding surfaces. It is *further assumed* that these surface forces reduce, in the limit of vanishing area, to a *single force* inclined, in general, to the common normal. Symmetry of the stress tensor follows from this *further assumption.* It is important to realise that this symmetry is an assumption. There is nothing in the derivation of basic laws to prove it. Once it is recognised as an assumption, it is natural to inquire into the consequence of rejecting it. The mutual action has then to be assumed to reduce to a force and a couple, without loss of any generality.

Several attempts have been made to construct a new theory of elasticity based on this broader assumption. Recently the subject has attracted attention again. The different authors have, sometimes, varied attitudes to the development. TOUPIN [1] has given a general review of earlier work and the formulation of this general theory. MINDLIN and TIERSTEN [2] have rederived the general and linearised equations. They give solutions to a number of new problems. They conclude that the existence of couple stresses may be of microscopic character and may not show up in ordinary problems of engineering interest. KOITER [3] reviews earlier work by himself and others. His original enquiry was to seek an explanation of fatigue by use of this theory. He also gives solutions to a number of simple problems where the new theory can be tested. KRÖNER [4] gives a quite novel explanation. He traces the difficulty to the limiting procedure involved. In the reduction of mutual action to a resultant, one proceeds to the limit of vanishing areas. But there exists a lower bound dictated by the interatomic distances beyond which one cannot shrink areas. This lower bound below which the dimensions of an element cannot shrink, gives rise to couple stresses of various orders in macroscopic phenomena. HUNTINGTON [5] gives an interesting discussion of the physical circumstances under which couple stresses are possible. It is possible that the best indication of the existence of couple stresses is given by moving dislocations, since it is one of the bridges connecting the microscopic and the macroscopic states of a body.

Wave propagation is one of the important experimental methods of evaluating elastic constants. So in the following we attempt to study the propagation of shear waves. The dilational wave propagated is unaffected by the existence of couple stresses. The theory of singular surfaces gives a sharp wave front for this irrotational wave while it does not lead to any discontinuous variation of rotation. So it is only the shear wave that is affected by the new theory, therefore we study the simplest shear wave generated by a source. The existence of couple stresses drastically modifies these 'waves'. The governing equation is no more hyperbolic. We first give an exact integral representation of the solution. Using the interesting technique followed by STEKETEE [6] in the study of magnetohydrodynamic waves in the presence of viscosity and electrical conductivity, we obtain the solution as a superposition of solutions of 'elliptic' equations. We then give the Laplace transform solution. We obtain now 'two' 'wave-fronts'. One is exponentially damped while the other, corresponding to the classical wavefront, decays in amplitude as the cube root of the inverse distance for large distances, near to the wave front, while everywhere else it appears to fall off exponentially.

2. Statement and solution of the problem

Referred to a Cartesian System (x, y, z) let the displacement vector be $(0, v\,(x, t), 0)$. We then seek the solution of the problem [2]:

$$\left(1 - l^2 \frac{\partial^2}{\partial x^2}\right) \frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 v}{\partial t^2} + \frac{A}{c^2} \delta'(x)\delta(t). \tag{2.1}$$

Here the right-hand member consisting of delta functions, gives the 'source' term; A may-be taken as the strength of this source; $c = (\mu/\varrho)^{1/2}$ gives the shearwave velocity in absence of couple stresses; l is a new parameter, of the dimension of the length, and, is the square root of the ratio of the new elastic constant to shear modulus. We choose $ct \to t$ so that the new wave velocity is unity and the new time has the dimension of length. We further set $x \to lx$, $t \to lt$, $v \to vl$ and take the coefficient of the source term as unity. We then have

$$\left(1 - \frac{\partial^2}{\partial x^2}\right) \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2} + \delta'(x)\, \delta(t). \tag{2.1a}$$

In the absence of couple stresses $l = 0$ and we have the simple wave equation. Since the new parameter multiplies the highest derivatives it should exhibit boundary layer behaviour. But in addition it changes the basic character of the equation, which is no more hyperbolic. This change in the character of the equation is brought out very clearly by the following solution. We also feel that this procedure offers an additional novel way of treating transform problems.

The fourth order differential operator in (2.1a) does not formally separate into two second order ones. We are able to effect this 'factorisation' by the following technique.

Taking the Laplace transform of (2.1a), we obtain

$$(1 - D^2)D^2 \bar{v} = p^2 \bar{v} + \delta'(x)\,, \tag{2.2}$$

where a bar denotes Laplace transform, p the transform variable, and D denotes differentiation with respect to x. This can be rewritten as

$$(D^4 - D^2 + p^2)\, \bar{v} = -\, \delta'(x)\,. \tag{2.3}$$

Formal 'factorisation' leads to

$$(D^2 - \sqrt{2}\sqrt{p+1/2}\, D + p)\,(D^2 + \sqrt{2}\sqrt{p+1/2}\, D + p)\, v = -\, \delta'(x)\,. \tag{2.4}$$

If $v = \exp(-t/2)u$, then $v(x, p) = u(x, p + 1/2)$ [7]. Replacing p by $(p - 1/2)$ and v by u, we get,

$$(D^2 - \sqrt{2p}\, D + p - 1/2)\,(D^2 + \sqrt{2p}\, D + p - 1/2)\, \bar{u} = -\, \delta'(x)\,. \tag{2.5}$$

Further if

$$u(x, t) = \int_0^\infty \frac{T}{2\sqrt{\pi t^3}} \exp\left(-\frac{T^2}{4t}\right) w(x, T)\, dT\,, \tag{2.6a}$$

then we have [7],

$$\bar{u}(x, p) = \bar{w}(x, \sqrt{p})\,. \tag{2.6b}$$

So replacing $\sqrt{p}$ by p and $\bar{u}$ by $\bar{w}$ we obtain

$$(D^2 - \sqrt{2}pD + p^2 - 1/2)\,(D^2 + \sqrt{2}pD + p^2 - 1/2)\,\bar{w} = -\,\delta'(x)\,. \tag{2.7}$$

The equation has now formally 'factored' out. The above is equivalent to

$$\left(\frac{\partial^2}{\partial x^2} - \sqrt{2}\frac{\partial^2}{\partial x\partial t} + \frac{\partial^2}{\partial t^2} - \frac{1}{2}\right)\left(\frac{\partial^2}{\partial x^2} + \sqrt{2}\frac{\partial^2}{\partial x\,\partial t} + \frac{\partial^2}{\partial t^2}\frac{1}{2}\right) w = = -\delta'(x)\delta(t)\,. \tag{2.8}$$

To solve this we exploit the factorisation and introduce

$$L_1 w_1 = \frac{\partial^2 w_1}{\partial x^2} - \sqrt{2}\frac{\partial^2 w_1}{\partial x\,\partial t} + \frac{\partial^2 w_1}{\partial t^2} - \frac{1}{2}w_1 = -\,\delta(x)H(t)\,, \tag{2.9a}$$

$$L_2 w_2 = \frac{\partial^2 w_2}{\partial x^2} - \sqrt{2}\frac{\partial^2 w_2}{\partial x\,\partial t} + \frac{\partial^2 w_2}{\partial t^2} - \frac{1}{2}w_2 = -\,\delta(x)H(t)\,. \tag{2.9b}$$

Note that $L_1w_1 = L_2w_2$. Also L_1, L_2 are linear differential operators and so commute. Using this property we get

$$L_1L_2(w_1 - w_2) = L_2(L_1w_1) - L_1(L_2w_2) = (L_2 - L_1)\big(-\,\delta(x)H(t)\big)$$
$$= 2\sqrt{2}\,\frac{\partial^2}{\partial x\,\partial t}\,-\big(\delta(x)H(t)\big) = -2\sqrt{2}\,\delta'(x)\delta(t)\,. \tag{2.10}$$

Comparing (2.8) and (2.10) we obtain

$$w = \frac{1}{2\sqrt{2}}\,(w_1 - w_2)\,. \tag{2.11}$$

We further note that (2.6a) can be integrated by parts to give

$$u(x,t) = \frac{1}{\sqrt{\pi t}}\left\{\left[-\,\exp\left(-\frac{T^2}{4t}\right) w(x,T)\right]_{T=0}^{\infty} + \int_0^{\infty} \exp\left(-\frac{T^2}{4t}\right)\frac{\partial \omega}{\partial t}(x,T)\,dT\right\} =$$
$$= \frac{1}{\sqrt{\pi t}}\int_0^{\infty} \exp\left(-\frac{T^2}{4t}\right)\frac{\partial w}{\partial t}\,(x,t)\,dT. \tag{2.12}$$

Since we thus need $(\partial w/\partial t)$, we need only $z_1 = (\partial w_1/\partial t)$ and $z_2 = (\partial w_2/\partial t)$ satisfying the differential equations

$$\frac{\partial^2 z_1}{\partial t^2} - \sqrt{2}\frac{\partial^2 z_1}{\partial x \partial t} + \frac{\partial^2 z_1}{\partial t^2} - \frac{1}{2} z_1 = - \delta(x)\delta(t) , \tag{2.13a}$$

$$\frac{\partial^2 z_2}{\partial t^2} + \sqrt{2}\frac{\partial^2 z_2}{\partial x \partial t} + \frac{\partial^2 z_2}{\partial t^2} - \frac{1}{2} z_2 = - \delta(x)\delta(t) . \tag{2.13b}$$

To obtain z_1 we change it to the canonical form by the following transformation of variables. Let

$$\xi = x + \frac{1}{\sqrt{2}} t, \ \eta = \frac{1}{\sqrt{2}} t . \tag{2.14}$$

Using this we obtain the equation for z_1 as

$$\frac{\partial^2 z_1}{\partial \xi^2} + \frac{\partial^2 z_1}{\partial \eta^2} - z_1 = - 2\,\delta(\xi)\,\delta(\eta)\frac{\partial(\xi, \eta)}{\partial(x, t)} = - \sqrt{2}\;\delta(\xi)\delta(\eta) . \tag{2.15}$$

The last Jacobian transforms the delta functions to the new variables. But the Eq. (2.15) is an elliptic equation. Thus the solution of the basic problem of 'wave propagation' is obtained by superposing solutions of 'elliptic equation'. The new variables are of course different. In view of the transformation (2.12) corresponds to $\sqrt{t}$ and not to t. So the solution may be said to be a superposition of the solutions of a parabolic equation. However strict parabolicity would have been there only if the term in l^2 in (2.1) were positive. The above method is a novel approach to deal with problems and also provides an interesting revelation of the change in the character of the basic equation which does not belong to any standard type.

It is now straightforward to write solutions for z_1 and z_2.

We have,

$$z_1 = \sqrt{2}\, K_0\big(\sqrt{x^2 + t^2 + \sqrt{2}\, xt}\big) ,$$
$$z_2 = \sqrt{2}\, K_0\big(\sqrt{(x^2 + t^2 - \sqrt{2}\, xt}\big) . \tag{2.16}$$

Using these we finally obtain u as

$$u(x, t) = - \frac{1}{2\sqrt{\pi t}}\int_0^\infty \exp\left(- \frac{T^2}{4t}\right) \left[K_0\big(\sqrt{x^2 + t^2 + \sqrt{2}\, xt}\big) - \right.$$
$$\left. - K_0\big(\sqrt{x^2 + t^2 + \sqrt{2}\, xt}\big)\right] dT =$$
$$= - \frac{1}{\sqrt{\pi}}\int_0^\infty \exp(- u^2) \left[K_0\big(\sqrt{x^2 + 4tu^2 + 2\sqrt{2t}\, xu}\big) - \right. \tag{2.17}$$
$$\left. - K_0\big(\sqrt{x^2 + 4tu^2 - 2\sqrt{2t}\, xu}\big)\right] du .$$

The original variable v is easily obtained as $v = \exp(-1/2\, t)u$. The solution clearly exhibits $v(0, t) = v(x, 0) = 0$. The form of representation indicates $(x/\sqrt{t})$ as a similarity variable.

It was not possible to convert it into simpler representations. We could represent the modified second type of Bessel function by use of addition theorem and convert it into a series of integrals involving confluent hypergeometric functions. But that does not help to reveal the nature of propagation any more clearly than (2.17). For large x, K_0 can be replaced by exponentials by use of their asymptotic forms. The contribution to the integral is then seen to be provided by the vicinity of the origin. This indicates the dominant behaviour near $x \sim t$. To see these things clearly we obtain below the Laplace transform solution by a straightforward process.

3. Asymptotic forms

We take the Fourier and Laplace transform of (2.1a) and perform the Fourier inversion. We then obtain for, $x > 0$ the Laplace transform of v as

$$\bar{v} = \frac{1}{2\sqrt{p^2 - \frac{1}{4}}} \exp\left(-\sqrt{p + \frac{1}{2}}\frac{x}{\sqrt{2}}\right) \sin\left(\sqrt{p - \frac{1}{2}}\frac{x}{\sqrt{2}}\right) \tag{3.1}$$

$$= \frac{1}{\sqrt{1-4p^2}} \exp\left(-\sqrt{p + \frac{1}{2}}\frac{x}{\sqrt{2}}\right) \sinh\left(\sqrt{\frac{1}{2} - p}\frac{x}{\sqrt{2}}\right). \tag{3.1a}$$

If we replace the original dimensional variables x and t, we can in fact pass the limit $l \to 0$ of the two exponentials in (3.1a),

$$m_{1,2} = \exp\left[-\frac{x}{2}(\sqrt{1 + 2\,lp} \pm \sqrt{1 - 2lp})\right], \tag{3.1b}$$

$$m_1 \to \exp(-x/l), \; m_2 \to (-px).$$

This solution corresponding to m_1 thus tends to zero as $l = 0$, while that corresponding to m_2 leads to the solution

$$v_c = -\frac{1}{2}\delta(t - x). \tag{3.2}$$

This solution (3.2) is the solution of the original problem (2.1) in absence of couple stresses viz. $l = 0$. We then have a wave front travelling without change of form.

It is important to note that the Laplace transform is defined as a single valued function on the Browmwhich contour which is any line parallel to the imaginary axis in the complex p-plane, such that all the singularities lie to its left. Any line now such that $\mathrm{Re}\, p > 1/2$ satisfied the requirement. However, it must be noted that $p = 1/2$ is not a singularity of the integrand, while $p = -1/2$ is the only singularity, being a branch point.

For $u \ll 1$, and small x, we expand the sine function and perform the inversion to obtain

$$v = \frac{x}{2\sqrt{2}} \exp\left(\frac{1}{2}t\right) \sum_0 \frac{(-1)^n}{(2n+1)!} \left(\frac{x^2}{2}\right)^n \frac{d^n}{dx^n}\left[\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{x^2}{8t} - t\right)\right]. \quad (3.3)$$

The series (3.3), exhibits the diffusive nature of the solution near the boundary. Again since $p = 1/2$ is not a branch point the imaginary axis is an admissible contour. Taking this we get

$$\begin{aligned} v = \frac{1}{\pi}\int_0^\infty \frac{\cos\left(ut - x\sqrt{\sqrt{u^2+\frac{1}{4}} - \frac{1}{2}}\right)}{\sqrt{u^2+\frac{1}{4}}}\, du\, - \\ - \frac{1}{\pi}\int_0^\infty \frac{\cos(ut)\exp\left(-x\sqrt{\sqrt{u^2+\frac{1}{4}} - \frac{1}{2}}\right)}{u}\, du\,. \end{aligned} \quad (3.4)$$

The second integral is everywhere exponentially small for large x and so it may be disregarded. The first integral is, however, of the known form discussed in the literature [8]. We obtain

$$v \sim \left(\frac{2}{3x}\right)^{1/3} Ai\left[\left(\frac{2}{3x}\right)^{1/3}(t - x)\right], \quad (3.5)$$

where Ai denotes the Airy function.

The variation of the amplitude as inverse cube root of distance near the wave front $t \sim x$ is clear from above. The same conclusions can be seen from the method of *steepest descent* in greater detail. The exponentials in the inversion of (3.1a) can be written as

$$\exp[-x\{m(p) - \delta p\}]\,, \quad (3.6)$$

where $\delta(t/x) = 1/k$.

For each fixed δ or k, for large x, the major contribution to the integral comes from the neighbourhood of the stationary points, given by $m'(p) = \delta$ as

$$p_{1,2} = \mp \frac{1}{8}\left([8 - 4k^2 - k^4 + 8\sqrt{1-k^2}]^{1/2} \pm [8 - 4k^2 - k^4 - 8\sqrt{1-k^2}]^{1/2}\right). \quad (3.7)$$

Of the above two roots it can be verified $p_{1,2}$ belong to $m_{1,2}$ respectively. However, p_1 gives an exponentially small contribution. Limiting attention to the neighbourhood of the wave-front, $t \sim x$ with $k^2 = 1 + \varepsilon$, we obtain

$$p_2 = \frac{2}{\sqrt{3}} \sqrt{\varepsilon} \,. \tag{3.8}$$

Thus for $t = x$, origin is a saddle point. The path of steepest descent is such that

$$\mathrm{Im}[m(p) - p] = 0, \ \mathrm{Re}[m(p) - p] > 0 \,. \tag{3.9}$$

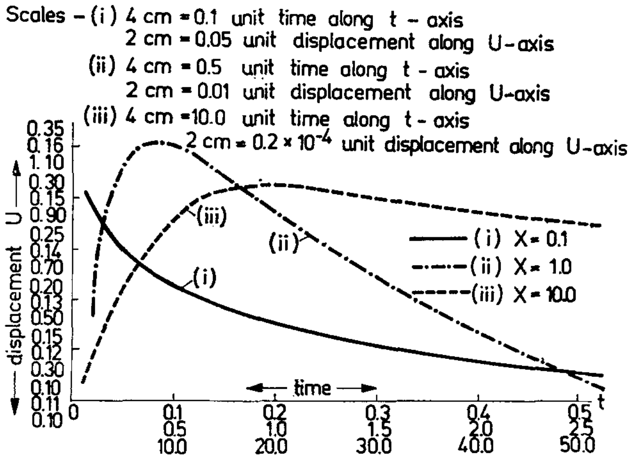

Fig. 1. The curve of displacement against time for $x = 0.1$, $x = 1.0$ and $x = 10.0$

It can be verified that the path starts from the origin, at angles $\pm(2\pi/3)$ and at infinity, is asymptotic to $\sqrt{2\varrho} \sin(\theta/2) = \pm 1$. The analytical expression for the path is not easy to obtain. However, $[m(p) - p]$ behaves, near the origin as p^3 and so, with an exponential error, we can take the path of definition of Airy integral [8]. Then we are led essentially to the same conclusions as in (3.8). For $\varepsilon > 0$, $t < x$, p_2 is real, giving an exponentially small contribution. This shows that there is a sharp fall in the amplitude just in front of $t = x$ i.e. ahead of the classical wave. However, just behind the front it has an oscillatory behaviour since p_2 gives two points on the imaginary axis.

The Figure shows the solution as calculated from (2.17). For different values of x the ranges of the values of t and u are taken as follows.

$x = 0.1$,	$t\,(0.01,\ 3.0)$,	$u\,(0.0,\ 11.10)$.
$x = 1.0$,	$t\,(0.10,\ 6.0)$,	$u\,(0.0,\ 11.10)$.
$x = 10.0$,	$t\,(1.0,\ 60.0)$,	$u\,(0.0,\ 11.10)$.

4. Conclusion

The existence of couple stresses changes the character of the propagation basically. A wave unchanged in form in their absence is modified drastically. Near the source it is highly oscillatory and diffusive, whereas far from the source, the amplitude decays as the inverse cube root of distance, falling off exponentially ahead and in an oscillatory manner behind the classical shear-wave front.

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