THE FUNDAMENTAL THEOREM OF CONTINUOUS TRANSFORMATIONS IN THE QUANTUM THEORY

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The unitary operator, the generator of symmetry transformations in the Hilbert-space, will be formed on the basis of the field equations together with the commutation law. Our method is the reversal of SCHWINGER's method used in the covariant formulation of the quantum theory and eliminates some insufficiencies of the previous treatments.

§ 1. The present paper deals with a problem of methodological interest, across which the author came in the course of his university lectures.

The transformations leaving the field equations (Lagrangian) as well as the commutation laws invariant play a significant role in the quantum theory. Such transformations of the coordinates x_i and of the field quantities ψ_μ

$$x_i \to x_i', \qquad \psi_\mu(x) \to \psi_\mu'(x') \tag{1}$$

are the so-called symmetry transformations. In the Hilbert-space the generator of a transformation which does not affect the commutation rules is a unitary operator :

$$\psi'(x) = U\,\psi(x)\,U^{-1}, \tag{2}$$

In the course of quantum theoretical applications the explicit form of the generator U becomes important. As is well known [1] in the classical theory always a conservation law corresponds to every symmetry transformation. There exists a close relation between this conservation law and the form of the unitary generator U. This operator gives the transition between the two state vectors with which observers in different "systems of reference" describe the same physical state :

$$|>' = U|>. \tag{3}$$

It is very remarkable that *in case the Lagrangian is known a method can be given for the unambiguous formation of the generator of a continuous symmetry transformation*. This method can be obtained by the aid of the *fundamental theorem of continuous transformations.*

In the SCHWINGER's covariant formulation of the quantum theory this fundamental theorem is regarded as an axiom of the theory and the form of the commutation rule was deduced from it [2]; in addition it is possible to deduce also the field equations. However, the reversed process can also be considered: the fundamental theorem can be deduced from the field equations and the commutation law. These latter can be introduced more easily in a correspondence-like way, therefore this kind of treatment seems to be methodically more advantageous (although SCHWINGER's method is mathematically more elegant). The deduction of the fundamental theorem from the commutation law has already been treated in the special case when the symmetry transformation does not affect the space coordinates (e.g. gauge transformation, the transformations of the isotopic space) and formerly the general case was handled, too, but in a rather indirect and cumbersome way [3]. The author, however, has not read any plain direct deduction of the fundamental theorem in a covariant manner which would be valid for all kinds of continuous symmetry transformations. This paper aims at presenting such a deduction which is of general validity. Only then will the equivalence (resp. the extent of the equivalence) of the Hamilton principle and the commutation law with the fundamental theorem become quite clear.

§ 2. As starting point for the covariant formulation of the theory serves the Lagrangian L. For the sake of simplicity let us assume that L is built up from the field quantities and their first derivatives. The physical field is described (in the classical theory as well as in the Heisenberg picture of the quantum theory) by such field quantities for which the integral

$$\int_W L(\psi(y), \partial\psi(y))\, dy \tag{3}$$

is stacionary (i.e. its first variation is zero) for values $\psi_\mu(x)$ fixed on the boundary of the four-dimensional domain W. This Hamilton principle will be regarded as the first axiom of the theory; from this the Lagrangian form of the field equation can be obtained:

$$\frac{\partial L}{\partial \psi_\mu} - \partial_i \frac{\partial L}{\partial \partial_i \psi_\mu} = 0. \tag{4}$$

When substituting the expression for $\psi_\mu(y)$ determined from (4) for the inner region of W into (3), we get the action integral

$$S = \frac{1}{ic} \int_W L(\psi(y), \partial\psi(y))\, dy, \tag{5}$$

which depends only on the extent of W and on the specific value of ψ_μ prescribed on the boundary of W.

Varying the four-dimensional domain W a surface point with coordinate x_i becomes a surface point with coordinate $\bar{x}_i = x_i + \delta x_i$. The value of the field quantity in this boundary surface point be also changed from $\psi_\mu(x)$ to $\bar{\psi}_\mu(\bar{x}) = \psi_\mu(x) + \delta\psi_\mu(x)$. (By using the changed functions $\bar{\psi}(\bar{y})$ corresponding to the modified boundary conditions the action integral S becomes of course stacionary in the interior of the domain.) The variation of the boundary conditions modifies also the value of the action integral. The variation of S (according to the boundary formula of the variation calculus) can be transformed into a surface integral:

$$\delta S = \frac{1}{ic} \oint_H \left\{ \frac{\partial L}{\partial \partial_k \psi_\mu} \delta\psi_\mu + \Theta_{ik} \delta x_i \right\} dF_k = \oint_H (\pi_\mu \delta\psi_\mu + p_i \delta x_i)\, dF, \tag{6}$$

(where the domain of integration H is the boundary surface of W). Let us use the following notations:

$$\Theta_{ik}(x) = L\delta_{ik} - \partial_i \psi_\mu \frac{\partial L}{\partial \partial_k \psi_\mu} \tag{7}$$

is the canonical energy momentum tensor, N_k is the normal unit vector of the surface element dF_k (i.e. $dF_k = N_k dF$, $N_k N_k = +1$). According to Pierre Weiss the quantity

$$\pi_\mu(x) = \frac{1}{ic} \frac{\partial L}{\partial \partial_k \psi_\mu} N_k \tag{8}$$

will be regarded as the canonical conjugate of the field quantity $\psi_\mu(x)$ and the function

$$p_i(x) = \frac{1}{ic} \Theta_{ik} N_k \tag{9}$$

as the canonical conjugate of the independent variable x_i [4]. (It should be noted that $\pi_\mu(x)$ and $p_i(x)$ are not pure local functions but they depend on the direction of the surface element at the point x.) They can be designated canonical conjugates as (in the classical theory as well as in the quantum theory) canonical equations can be deduced for them.

Let $f_i(x)$ be four arbitrarily given coordinate functions and let us form the integral

$$B = \int_F p_i(y) f_i(y)\, dF = \frac{1}{ic} \int_F \Theta_{ik}(y) f_i(y)\, dF_k = \int_F \left(\frac{1}{ic} L N_i - \partial_i \psi_\mu \cdot \pi_\mu \right) f_i\, dF \tag{10}$$

for an arbitrary three-dimensional not-bounded space-like hyperplane F. The integrand is a given expression of $\psi_\mu(y)$, $\partial_i \psi_\mu(y)$ and $\pi_\mu(y)$. Using (8) the normal derivate of ψ_μ can be expressed by ψ_μ and π_μ and thus B can be regarded as a functional of the values of ψ_μ and π_μ taken on the plane F.

By varying the values of $\psi_\mu(y)$ and $\pi_\mu(y)$ prescribed on the hyperplane we get for the variation of B (after identical transformations):

$$\delta B[\psi,\pi] = \frac{1}{ic}\int\left(\frac{\partial L}{\partial\psi_\mu} - \partial_\iota \frac{\partial L}{\partial\partial_\iota \psi_\mu}\right)\delta\psi_\mu\, dF + \\ + \int_F \partial_r\left[\frac{1}{ic}\frac{\partial L}{\partial\partial_r \psi_\mu} N_\iota f_i\, \delta\psi_\mu - \pi_\mu \delta\psi_\mu f_r\right] dF + \\ + \int_F \left\{\left[\partial_i (f_\iota \pi_\mu) - \frac{1}{ic}\frac{\partial L}{\partial\partial_r \psi_\mu} N_\iota \partial_r f_i\right]\delta\psi_\mu - [\partial_\iota \psi_\mu f_\iota]\,\delta\pi_\mu\right\} dF. \tag{11}$$

The first integral becomes zero due to (4), the second one is also zero (in spite of the fact that it is a four-sum) due to the vanishing of ψ_μ in the space-like infinity. From the remaining expression we can see that the functional derivatives are the following:

$$\frac{\delta B}{\delta\psi_\mu(x)} = \partial_\iota (f_i \pi_\mu) - \frac{1}{ic}\frac{\partial L}{\partial\partial_r \psi_\mu} N_\iota \partial_r f_\iota, \tag{12}$$

$$\frac{\delta B}{\delta\pi_\mu(x)} = -f_\iota \partial_\iota \psi_\mu . \tag{13}$$

In the special case when $f_i = \delta_{ik}$ (12) and (13) lead to the following canonical equation

$$\frac{\delta P_k}{\delta\psi_\mu(x)} = \partial_k \pi_\mu(x), \qquad \frac{\delta P_k}{\delta\pi_\mu(x)} = -\partial_k \psi_\mu(x), \tag{14}$$

where

$$P_k = \frac{1}{ic}\int_F \Theta_{\iota k}\, dF_k = \int_F p_\iota\, dF \tag{15}$$

is the four-momentum of the system.

§ 3. A symmetry transformation has to be regarded as continuous if 1) it is a differentiable function of the parameter a, 2) it turns into identical transformation when $a = 0$ and 3) the parameter can be chosen so that the successive transformations with parameter a_1 and a_2 correspond to a transformation with parameter $a_1 + a_2$.

The transformation changes the field quantity components ψ_μ in the given geometrical point P: they combine with each other

$$\psi'_\mu(P) = \sigma_{\mu\nu}\,\psi_\nu(P). \tag{16}$$

$\sigma_{\mu\nu}$ and thus also ψ'_μ are functions of the parameter a.

$$\sigma_{\mu\nu}(0) = \delta_{\mu\nu}\,, \qquad \sigma_{\mu\varrho}(a_2)\,\sigma_{\varrho\nu}(a_1) = \sigma_{\mu\nu}(a_2 + a_1)\,. \tag{17}$$

Let us form the following expressions:

$$\Delta\sigma_{\mu\nu} = \left[\frac{d\sigma_{\mu\nu}}{da}\right]_{a=0} = I_{\mu\nu}\,, \tag{18}$$

$$\Delta\psi_\mu(x) = \left[\frac{d\psi'_\mu(x')}{da}\right]_{a=0} = I_{\mu\nu}\,\psi_\nu(x). \tag{19}$$

$I_{\mu\nu}$ is the matrix of the infinitesimal transformation.

The symmetry transformation can change the coordinates of the point P too: $x \rightarrow x'(a) = sx$. Let us deal now with the following expression:

$$\Delta x_i = \left[\frac{dx'_i}{da}\right]_{a=0}. \tag{20}$$

If we regard the arguments of the field quantities in (16),

$$\psi'_\mu(x') = \sigma_{\mu\nu}\,\psi_\nu(x)\,, \tag{21}$$

i.e.

$$\psi'_\mu(x') = \sigma_{\mu\nu}\,\psi_\nu(s^{-1}x')$$

and change the notation of the independent variable $x'_i \rightarrow x_i$, we get:

$$\psi'_\mu(x) = \sigma_{\mu\nu}\,\psi_\nu(s^{-1}x)\,. \tag{22}$$

Thus, comparing the *functional forms* of the field quantities (i.e. their dependence on the coordinates as independent variables) instead of comparing their values taken at a given geometrical point according to (16) we can see that the expression $\psi_\mu\,(x)$ will be altered by the transformation due to two reasons: the components of the field quantities combine in a given point (σ) and the coordinates of this point will be changed in the argument (s). Thus the derivative of $\psi_\mu\,(x)$ with respect to the transformation parameter consists of two parts:

$$\Delta^*\psi_\mu(x) = \left[\frac{d\psi'_\mu(x)}{da}\right]_{a=0} = \left[\frac{d\sigma_{\mu\nu}}{da}\right]_{a=0}\psi_\nu(x) + \partial_i\,\psi_\mu(x)\left[\frac{d(s^{-1}x)_i}{da}\right]_{a=0} = \tag{23}$$
$$= \Delta\,\psi_\mu(x) - \partial_i\,\psi_\mu(x)\cdot\Delta x_i\,.$$

Let us consider now, following E. NOETHER [1], the action integral after the transformation as a function of the transformation parameter a:

$$S(a) = \frac{1}{ic} \int\limits_{W} L\left(\psi'(y'), \partial' \psi'(y')\right) dy' .$$

As the transformation is a symmetry transformation the action integral has to be equal to the non-transformed expression $S(0)$, i.e.

$$\Delta S = \left[\frac{dS(a)}{da}\right]_{a=0} = \int\limits_{W} \partial_k \left[\frac{1}{ic}\left(\frac{\partial L}{\partial \partial_k \psi_\mu} \Delta\psi_\mu + \Theta_{ik} \Delta x_i\right)\right] dx = 0 . \quad (24)$$

This relation, however, is valid for an arbitrary domain W only when

$$\partial_k j_k = 0 ,$$

where

$$j_k(x) = \frac{i}{c}\left(\frac{\partial L}{\partial \partial_k \psi_\mu} \Delta\psi_\mu + \Theta_{ik} \Delta x_i\right). \quad (25)$$

To every symmetry transformation corresponds a conservation law. The differential form of a conservation law is given in (25). This is *Noether's theorem*. The current density $j_k(x)$ is a pure local function independent of the transformation parameter and the surface direction.

Let us choose the four-dimensional domain W in (28) as the four-volume between the two hyperplanes F_0 and F and assume the normals of the hyperplanes as directed into the "future". Then using the Gauss theorem and the fact that the field quantities vanish in the space-like infinity, we get for (24)

$$\Delta S = Q(F_0) - Q(F) = 0,$$

i. e.

$$Q = \int\limits_{F} j_k \, dF_k = - \int\limits_{F} (\pi_\mu \Delta\psi_\mu + p_i \Delta x_i)\, dF \quad (26)$$

is independent of the hyperplane, i. e. it is a constant of motion and it can be transformed like a contravariant quantity with respect to the transformation parameter a. The form of Q can be written down directly when L is known. This is the integral form of NOETHER's theorem.

§ 4. In quantum theory besides the field equations the commutation laws must also be known. Let us consider a hyperplane F. The following commutation laws are valid for the field quantities taken at two points x, y of the hyperplane:

$$\{\psi_\mu(x), \pi_\nu(y)\} = i\hbar \delta_{\mu\nu}\, \delta(x-y) , \quad \{\pi_\mu(x), \pi_\nu(y)\} = 0, \quad \{\psi_\mu(x), \psi_\nu(y)\} = 0 . \quad (27)$$

Here

$$\{A, B\} = AB \pm BA,$$

where the sign depends on the statistics of the field investigated. The definition of the surface function $\delta(x)$ is the relation

$$\int_F f(y)\,\delta(y-x)\,dF(y) = f(x).$$

The "simultaneous" commutation rule (27) refers to the hyperplane F (due to the quantities π_μ, $\delta(x)$ and to the assumption $x, y \in F$). Comparing (27) and the field equations (4) the commutator $\{\psi_\mu(x), \psi_\nu(z)\}$ at two points of arbitrary location can be obtained and its value is already independent of F.

The results of the classical theory obtained in the above two paragraphs are valid also in the quantum theory as operator relations if we take care of the sequence of operators in the course of differentiations. The most obvious way is to regard all the products as ordered products.

§ 5. As the transformations dealt with are symmetry transformations the operators $\psi_\mu'(x)$ satisfy the same commutation rules as the operators $\psi_\mu(x)$ (the transformation is canonical). Thus the two operators can be related to each other by a unitary perator U, namely :

$$\psi_\mu'(x) = U\psi_\mu(x)\,U^{-1}. \tag{28}$$

Our main task is to determine the explicit form of the generator U. For the transformations dealt with here we can write

$$U(a_2)\,U(a_1) = U(a_2 + a_1), \qquad U(0) = 1.$$

Be

$$\Delta U = \left[\frac{dU}{da}\right]_{a=0}. \tag{29}$$

Differentiating (28) with respect to a and taking the derivative for $a = 0$ we get by taking into account (23) and (29)

$$\Delta^* \psi_\mu(x) = [\Delta U, \psi_\mu(x)]. \tag{30}$$

([...] means a minus commutator.) Evidently, if the operator ΔU satisfying (30), is determined, the operator U can easily be obtained. In the case of a very small transformation parameter one can write :

$$U(a) \approx U(0) + a\left(\frac{dU}{da}\right)_{a=0} = 1 + a\,\Delta U.$$

The relation becomes exact if $a \to 0$. The generator of a transformation with a finite parameter a can be obtained from the generator of a transformation having as parameter a/n in the following way:

$$U(a) = U\left(\frac{a}{n}\right)^n,$$

and if n is large enough

$$U\left(\frac{a}{n}\right) \approx 1 + \frac{a}{n}\,\Delta U\,, \qquad \text{thus} \qquad U(a) \approx \left(1 + \frac{a}{n}\,\Delta U\right)^n.$$

The equality sign applies if $\frac{a}{n} \to 0$, i.e. if $n \to \infty$. Thus we can write symbolically

$$U(a) = \lim_{n\to\infty} \left(1 + \frac{a}{n}\,\Delta U\right)^n = e^{a\Delta U}. \tag{31}$$

As a preparation for the determination of ΔU the commutators of Q and $\psi_\mu(x)$ have to be formed:

$$[Q, \psi_\mu(x)] = -\int [\pi_\nu(y)\,\Delta\psi_\nu(y), \psi_\mu(x)]\,dF(y) - [B, \psi_\mu(x)]\,. \tag{32}$$

Here

$$B = \int_F p_\iota(y)\,\Delta y_\iota\,dF. \tag{33}$$

As Q is independent of the position of F we have chosen a hyperplane laid through the fixed point x. For the evaluation of the first term on the right hand side of (32) the algebraic identity

$$[AB, C] = A\{B, C\} - \{C, A\}B \tag{34}$$

can be used and for the second term the general mathematical relation

$$[B, \psi_\mu(x)] = \frac{\hbar}{i}\,\frac{\delta B}{\delta\pi_\mu(x)} \tag{35}$$

which is valid as a result of (27) for all expressions B of interest. (It must be taken into account that only even number fermion operators can occur in B.) Thus we get

$$[Q, \psi_\mu(x)] = \int \Big(\{\psi_\mu(x), \pi_\nu(y)\}\,\Delta\psi_\nu(y) - \\ - \pi_\nu(y)\,I_{\nu\varrho}\{\psi_\varrho(y), \psi_\mu(x)\}\Big)\,dF(y) + i\hbar\,\frac{\delta B}{\delta\pi_\mu(x)}\,.$$

From the commutation laws (27) and from the general canonical equations (13) (by choosing $f_i(y) = \Delta y_i$) we get

$$[Q, \psi_\mu(x)] = i\hbar \left(\Delta\psi_\mu(x) - \partial_\iota \psi_\mu(x) \Delta x_\iota\right) = i\hbar \Delta^* \psi_\mu(x). \tag{36}$$

Comparing (30) and (36) we get for the operator ΔU:

$$\Delta U = -\frac{i}{\hbar} Q = \frac{i}{\hbar} \int (\pi_\mu \Delta\psi_\mu + p_\iota \Delta x_\iota) dF \tag{37}$$

and the generator of the finite symmetry transformation becomes

$$U(a) = e^{-a\frac{i}{\hbar}Q} = \exp \frac{ia}{\hbar} \int_F (\pi_\mu \Delta\psi_\mu + p_\iota \Delta x_\iota) dF = \exp \frac{a}{\hbar c} \int_F \left(\frac{\partial L}{\partial \partial_k \psi_\mu} \Delta\psi_\mu + \right. \\ \left. + L\Delta x_k - \frac{\partial L}{\partial\partial_k \psi_\mu} \partial_i \psi_\mu \Delta x_i\right) dF_k. \tag{38}$$

Thus we have succeeded in obtaining the generator of an arbitrary continuous symmetry transformation by the aid of the field equations and the commutation law. The generator is, according to § 3, a Lorentz invariant constant of motion, which is independent of F.

§ 6. As we mentioned already in the introduction, the fundamental theorem expressed by (28) and (38) was regarded by J. Schwinger as an axiom of the quantum theory from which the commutation law can be deduced.

Indeed: Let us consider such a symmetry transformation, for which $\Delta x_i = 0$. In this case it follows from the expression (37) (which is accepted as an axiom) that

$$\Delta\psi_\mu = \Delta^* \psi_\mu = [\Delta U, \psi_\mu(x)] = \frac{i}{\hbar} \int_F [\pi_\nu(y) \psi_\varrho(y), \psi_\mu(x)] \, I_{\nu\varrho} \, dF(y) = \\ = \frac{i}{\hbar} \int \left(\pi_\nu(y) \left\{\psi_\varrho(y), \psi_\mu(x)\right\} - \left\{\psi_\mu(x), \pi_\nu(y)\right\} \psi_\varrho(y)\right) I_{\nu\varrho} \, dF(y).$$

This requirement can be fulfilled for many possible expressions $I_{\nu\varrho}$ and point x and hyperplane F by choosing

$$\left\{\psi_\varrho(y), \psi_\mu(x)\right\} = 0, \qquad \left\{\psi_\mu(x), \pi_\nu(y)\right\} = i\hbar\delta_{\mu\nu} \delta(x-y).$$

Of course the fundamental theorem does not affect the type of the statistics. Conclusions can be drawn from the fundamental theorem also regarding the field equation. The theory is invariant against the displacement of the origo of the coordinate system, i.e.

$$x'_i = x_i - a_i\,, \qquad \psi'_\mu(x') = \psi_\mu(x) \tag{39}$$

are symmetry transformations. From this follows according to (26) the conservation of the field momentum P_i, the form of which can be obtained directly from (15) if the Lagrangian is known.

Applying (30) to the transformation (39) we get

$$i\hbar\,[P_i, \psi_\mu(x)] = \partial_i\,\psi_\mu(x)\,. \tag{40}$$

(40) determines the space and time variations of the field quantities and thus leads to the field equations.

REFERENCES

1. E. Noether, Gött. Nachr., 235, 1918.
2. J. Schwinger, Phys. Rev., **82**, 914, 1951.
3. See e.g. L. Rosenfeld, Ann. Phys., **5**, 113, 1930.
   R. Utiyama, Progr. Theor. Phys., **5**, 437, 1950.
   M. Jauch-R. Rohrlich, Quantum Theory of Photons and Electrons, Addison-Wesley Publ. Co. Cambridge, Mass. 1955.
   P. T. Matthews, The Relativistic Quantum Theory of Elementary Particles, Rochester Lectures, 1957. Preprint.
   J. Schwinger, Annals of Physics, **2**, 407, 1957.
4. P. Weiss, Proc. Roy. Soc. *A*, **169**, 102, 1938.

О ФУНДАМЕНТАЛЬНОЙ ТЕОРЕМЕ НЕПРЕРЫВНЫХ ПРЕОБРАЗОВАНИЙ В КВАНТОВОЙ ТЕОРИИ

Г. МАРКС

Резюме

Унитарный оператор, производящий преобразования симметрии в гилбертовом пространстве, строится на базе уравнений поля и перестановочных соотношений. Наш метод является обращением метода Швингера и он устраняет некоторые недостатки предыдущих трактовок.