DETERMINATION OF ELECTROSTATIC POTENTIALS BY SERIES

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By using the solutions of eigenvalue problems often occurring in the various fields of theoretical physics the method of series, already used in certain cases for solving the Poisson equation, has been reformulated. In case the charge distribution can be expressed by the Dirac δ-function, the potential distribution can be given in the form of a series, the convergence of which is sufficiently rapid. For simple cases the formulae can be analytically reduced to the known solutions of corresponding problems (e. g. in the case III/1). In other cases (III/2) the calculated potential distribution coincides with the potential distribution obtained by electrolytic tank measurements. In the case of the cylinder lens of electron optics (III/3) the method yields the potential distribution and the corresponding electrode shape for arbitrary slit width.

I. Introduction

The well-kown task of electrostatics is the determination of the electric field of charge distributions, when the electrode arrangements are given in advance. For the solution of the problem several methods have been worked out [1]. A common feature of most of these methods is that an electrode arrangement of particular symmetry is assumed and that for the individual arrangements the solutions are given separately in an explicit form. Thus these methods are in general too specialized for application to the actual electrode arrangement (required by some practical problems). The last possibility remaining in such cases is the experimental determination by electrolytic tanks which, however, may in cases not having symmetry properties prove to be a very complicated undertaking.

The solution of the fundamental problem of electrostatics in explicit form could be immediately given (although the possibility of its practical evaluation in slightly more complicated cases seems somewhat doubtful), if the distribution of the surface charge density on the metal electrodes were known. For a quite general electrode arrangement such an assumption would be indeed very audacious. In several cases of practical importance, however, the distribution of the surface charge density is exactly or at least approximately known. In the latter case when the distribution is approximately known one can proceed also in such a manner as to base the final execution of the shape of the electrode system on the equipotential surfaces of the field determined previously.

The basic idea of the method is the following. Let us consider an electrode system of which the distribution of the surface charge density is given and a closed surface on which the values of the electrostatic potential is prescribed. Knowing the latter the Green function of the fundamental equation of electrostatics can be determined and hereby the solution produced in the form of a series. In many instances a considerable reduction of analytical formulae can be achieved by producing the series in finite form.

The method developed is specialized first for the case of axially symmetric fields, then we show on a very simple example, the case of a cylinder capacitor, how the method may be applied. Afterwards the fields of three-electrode arrangements are determined which are used in the Penning's vacuum gauge, resp. which are differing from it in the placing of the anode and the cathode. These results are compared with the distributions measured by an electrolytic tank. Finally the potential field of an electrode arrangement important in electron optics (the two-cylinder lens of finite slit width) is determined in the form of a series.

It may be finally mentioned that the mathematical method applied here has recently been widely used for numerous problems [2]. Thus for instance also for the solution of the fundamental electrostatic problem in the case of simpler electrode arrangement [3].

II. General part. Survey of the method.

The fundamental problem of electrostatics is the determination of the solution of the Poisson equation,

$$\Delta \Phi = -4\pi\varrho, \tag{1}$$

i. e. determination of the potential distribution Φ for given boundary conditions, if the space charge density ϱ, respectively in other cases the surface charge density σ, the line densities γ or the dipole momenta of the surface double layers are known. In principle Φ can be determined from equation (1) when the above quantities are known, practically, however, it depends on the charge distribution and the boundary conditions whether the solution can be given. In the present paper the method is described for the case of all those potential distributions for which the charge distribution can be written down by the Dirac δ-function. (Hence for instance for any point charge distribution, for a surface and line charge distribution, provided they have suitable symmetry, etc.) In the following after the description of the general method we shall apply it to some actual instances.

The charge density of a single point charge occuring in the Poisson equation can be taken in the following form

$$\varrho = \varrho_0 \, \delta(\mathfrak{r} - \mathfrak{r}_0), \tag{2}$$

where ϱ_0 is the charge at the point $\mathfrak{r}_0$. Hence equation (1) takes the following form

$$\Delta\Phi = -4\pi\varrho_0\,\delta(\mathfrak{r}-\mathfrak{r}_0)\,, \tag{1a}$$

The boundary conditions are that the potential shall on a closed or open surface take up a value determined in advance.

The solutions Φ_i of the eigenvalue equation

$$\Delta\Phi = E\Phi \tag{2}$$

by which the above boundary conditions are satisfied, belong to the eigenvalues E_i. The Laplace operator is hermitian, and its eigenfunctions Φ_i form an orthonormal complete set of functions. As is well known the Dirac δ-function can be expanded in a complete set of orthonormalized functions, hence

$$\delta(\mathfrak{r}-\mathfrak{r}_0) = \sum_i \Phi_i^*(\mathfrak{r}_0)\,\Phi_i(\mathfrak{r})\,. \tag{4}$$

The functions Φ_i satisfy the same boundary conditions as the potential Φ which is to be determined, the latter can be expanded in the Φ_i-s

$$\Phi(\mathfrak{r}) = \sum_i c_i\,\Phi_i(\mathfrak{r})\,. \tag{5}$$

Substituting(5) and (4) into (1a) owing to the linearity of the Laplace operator

$$\sum_i c_i\,\Delta\Phi_i = -4\pi\varrho_0 \sum_i \Phi_i^*(\mathfrak{r}_0)\,\Phi_i(\mathfrak{r})\,. \tag{1b}$$

Using equation (3) from the comparison of the coefficients

$$c_i = -4\pi\varrho_0\,\frac{\Phi_i^*(\mathfrak{r}_0)}{E_i} \tag{6}$$

is obtained. Thus the potential is

$$\Phi(\mathfrak{r}) = -4\pi\varrho_0 \sum_i \frac{\Phi_i^*(\mathfrak{r}_0)\,\Phi_i(\mathfrak{r})}{E_i}\,.$$

(More generally see for instance [2].)

The method can be generalized without difficulties for the case of many point charges. Similarly all problems can be dealt with for which the charge density can be written as the superposition of terms of the form (2). In this case the determination of the potential can be reduced to the search of solutions of (3) satisfying suitable boundary conditions.

III. Special problems

In the following some axial symmetrical problems will be dealt with, on the one hand because of their physical importance (calculation of electron-optical cylinder lenses etc.) on the other because also from other sides the necessity of the solution of similar problems emerged. As a matter of course the method is also suitable for electrode arrangements having other adequate symmetries.

For axially symmetrical arrangements equation (3) can be written in the following form

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} = E\Phi \,, \tag{3a}$$

r and z are the well-known cylinder coordinates. In the following the solution of some special problem is searched.

1. As a first application such an electrode arrangement is dealt with for which calculations can easily be carried out also otherwise. An infinitely long metal cylinder of radius R_1 the axis of which coincides with the z axis of the system of coordinates is first considered. The surface charge density on the cylinder be σ_0. It is surrounded by an earthed metal cylinder of radius R (cylinder capacitor). In the case of this arrangement the charge density is

$$\sigma = \sigma_0 \,\delta\,(r - R_1) \;.$$

The potential depends owing to the symmetry properties only on r, and as is known

$$\left.\begin{aligned} \Phi &= 4\pi\sigma_0 \log \frac{R}{r}\,, \quad \text{if} \quad R_1 \leq r \leq R\,, \\ \Phi &= 4\pi\sigma_0 \log \frac{R}{R_1}\,, \quad \text{if} \quad 0 \leq r \leq R_1\,. \end{aligned}\right\} \tag{7}$$

Considering that the potential does not depend on z, the normalized solution of equation (3a) which vanishes at $r = R$ is

$$\Phi_i\,(r) = \frac{\sqrt{2}}{R J_0'(k_i)}\, J_0\left(\frac{k_i}{R}\, r\right)$$

for the eigenvalues

$$E_i = -\frac{k_i^2}{R^2},$$

where J_0 is the Bessel function of first kind of order zero and the k_i-s are the roots of this Bessel function. The dash means derivation into the argument. The potential can be produced according to the method given in the general part in the form

$$\Phi = 8\pi\sigma_0 \sum_i \frac{J_0\left(\frac{k_i}{R}R_1\right) J_0\left(\frac{k_i}{R}r\right)}{k_i^2 J_0'^2(k_i)}. \tag{8}$$

As can be easily seen by integration (8) is the expansion of (7) in a series of Bessel functions of different argument and order zero.

The fact that the series is convergent can be easily rendered plausible. According to the well-known asymptotical formula [4] referring to the Bessel functions of pth order

$$J_p(x) \approx \frac{\cos\varphi}{\sqrt{1/2\,\pi x}} \quad \text{and} \quad \varphi = x - (p - 0{\cdot}5)\frac{\pi}{2}, \tag{x}$$

if $x \to \infty$. Thus

$$J_0\left(\frac{k_i}{R}R_1\right) J_0\left(\frac{k_i}{R}r\right) \approx \frac{R}{\pi\sqrt{R_1 r k_i}}\left\{\cos\left[\frac{k_i}{R}(R_1 - r)\right] + \sin\left[\frac{k_i}{R}(R_1 + r)\right]\right\}.$$

On the other hand it can be demonstrated also by (x) that for sufficiently great i

$$J_1^2(k_i)\,k_i \approx \frac{2}{\pi}.$$

Thus the series is

$$\Phi(r) \approx \frac{4R\sigma_0\pi}{\sqrt{R_1 r}} \sum_i \frac{\cos\left[\frac{k_i}{R}(R_1 - r)\right] + \sin\left[\frac{k_i}{R}(R_1 + r)\right]}{k_i^2}.$$

Substituting the highest possible value of cos resp. sin, i. e. 1, and considering the asymptotical behaviour of the roots of the Bessel functions

$$k_i \approx \left(i - \frac{1}{4}\right)\pi, \quad \text{if} \quad i \to \infty ,$$

it can be seen that the series can be majorized by the absolutely convergent series $\sum_i \frac{1}{i^a}$ $(a > 1)$, which is a sufficient condition for convergence [5].

2. Let us calculate as another special problem the potential field of the following electrode arrangement: In a closed earthed cylinder of radius R and height l are placed in planes parallel with the base circle-shaped electrodes of radius R_1 provided with a charge density γ as illustrated in Figure 1 [electrode arrangement of the Penning's vacuum gauge]. Rotating the Figure around the z axis the electrode arrangement above described is obtained. In the case *a)* two rings of radius R_1 placed in heights z_1 and $l—z_1$ have been applied with the charge density $\gamma_A = \gamma_B$. Case *b)* — where in the middle plane of the cylinder one ring is present with a charge density γ_A — is, as can be seen, a special case of *a)*. Case *c)* is a combination of cases *a)* and *b)* where a ring having a charge density γ_C is placed in the middle plane while in the planes in heights z_1 and $l—z_1$ there are rings bearing charge densities $\gamma_A = \gamma_B$.

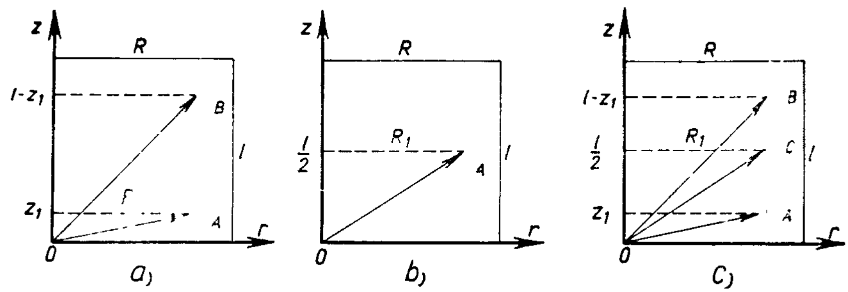

Fig. 1. Scheme of the electrode arrangement of the Penning type vacuum gauge

It can be immediately seen that in all the three cases the charge densities occuring in the Poisson equation can be expressed by one, two resp. three δ-functions and the charge density γ. The part depending on r of the solution of equation (3a) satisfying the boundary conditions will be also now the system of Bessel functions of order zero, $J_0\left(\frac{k_i}{R} r\right)$, whereas the part depending on z is produced by the set of function $\sin\left(n \frac{\pi}{l} z\right)$.

Taking into consideration the symmetry of the arrangement the following expressions are obtained for the potential distribution:

$$\left.\begin{aligned}
&a)\quad \Phi = \sum_{i,n} \frac{32\,\pi\gamma_A}{lR^2\,J_0'^2(k_i)}\,\frac{\sin\left(n\dfrac{\pi}{l}z_1\right)\sin\left(n\dfrac{\pi}{l}z\right)J_0\left(\dfrac{k_i}{R}R_1\right)J_0\left(\dfrac{k_i}{R}r\right)}{\dfrac{k_i^2}{R^2}+n^2\dfrac{\pi^2}{l^2}}\,,\\
&b)\quad \Phi = \sum_{i,n} \frac{16\,\pi\gamma_A}{lR^2\,J_0'^2(k_i)}\,\frac{\sin\left(n\dfrac{\pi}{2}\right)\sin\left(n\dfrac{\pi}{l}z\right)J_0\left(\dfrac{k_i}{R}R_1\right)J_0\left(\dfrac{k_i}{R}r\right)}{\dfrac{k_i^2}{R}+n^2\dfrac{\pi^2}{l^2}}\,,\\
&c)\quad \Phi = \sum_{i,n} \frac{16\,\pi\gamma_A}{lR^2\,J_0'^2(k_i)}\,\frac{\sin\left(n\dfrac{\pi}{l}z\right)J_0\left(\dfrac{k_i}{R}R_1\right)J_0\left(\dfrac{k_i}{R}r\right)}{\dfrac{k_i}{R^2}+n^2\dfrac{\pi^2}{l^2}}\left[\frac{\gamma_c}{\gamma_A}\sin\left(n\frac{\pi}{2}\right)+\right.\\
&\qquad\qquad\left.+\,2\sin\left(n\frac{\pi}{l}z_1\right)\right].
\end{aligned}\right\}\quad (9)$$

(n is odd in all three cases).

It should be mentioned, although this is not essential for the method, but is of considerable importance from the point of view of the numerical calculations, that in the double sum the summation over n can be easily carried out. The formulae (9) can be thus brought to the following form :

$$\left.\begin{aligned}
&a)\quad \Phi = \frac{8\pi\gamma_A}{R}\sum_i \frac{J_0\left(\dfrac{k_i}{R}R_1\right)J_0\left(\dfrac{k_i}{R}r\right)}{k_i J_0'^2(k_i)\,\mathrm{ch}\left(\dfrac{k_i l}{2R}\right)}\begin{cases}\mathrm{sh}\left(\dfrac{k_i}{R}z\right)\mathrm{ch}\left[\dfrac{k_i}{R}\left(z_1-\dfrac{l}{2}\right)\right], & \text{if } z_1>z>0,\\[2ex] \mathrm{sh}\left(\dfrac{k_i}{R}z_1\right)\mathrm{ch}\left[\dfrac{k_i}{R}\left(z-\dfrac{l}{2}\right)\right], & \text{if } l\text{-}z_1\geq z\geq z_1,\end{cases}\\
&b)\quad \Phi = \frac{4\,\pi\gamma_A}{R}\sum_i \frac{J_0\left(\dfrac{k_i}{R}R_1\right)J_0\left(\dfrac{k_i}{R}r\right)}{k_i J_0'^2(k_i)\,\mathrm{ch}\left(\dfrac{k_i l}{2R}\right)}\begin{cases}\mathrm{sh}\left(\dfrac{k_i}{R}z\right), & \text{if } \dfrac{l}{2}>z,\\[2ex] \mathrm{sh}\left[\dfrac{k_i}{R}(l-z)\right], & \text{if } \dfrac{l}{2}<z,\end{cases}\\
&c)\quad \Phi = \frac{4\,\pi\gamma_A}{R}\sum_i \frac{J_0\left(\dfrac{k_i}{R}R_1\right)J_0\left(\dfrac{k_i}{R}r\right)}{k_i J_0'^2(k_i)\,\mathrm{ch}\left(\dfrac{k_i l}{2R}\right)}\times\\
&\qquad\begin{cases}\dfrac{\gamma_c}{\gamma_A}\,\mathrm{sh}\left(\dfrac{k_i}{R}z\right)+2\,\mathrm{sh}\left(\dfrac{k_i}{R}z\right)\mathrm{ch}\left[\dfrac{k_i}{R}\left(z_1-\dfrac{l}{2}\right)\right], & \text{if } z_1>z>0,\\[2ex] \dfrac{\gamma_c}{\gamma_A}\,\mathrm{sh}\left[\dfrac{k_i}{R}(l-z)+2\,\mathrm{sh}\left(\dfrac{k_i}{R}z_1\right)\mathrm{ch}\left[\dfrac{k_i}{R}\left(z-\dfrac{l}{2}\right)\right]\right], & \text{if } l\text{-}z_1\geq z\geq\dfrac{l}{2}.\end{cases}
\end{aligned}\right\}\quad (9a)$$

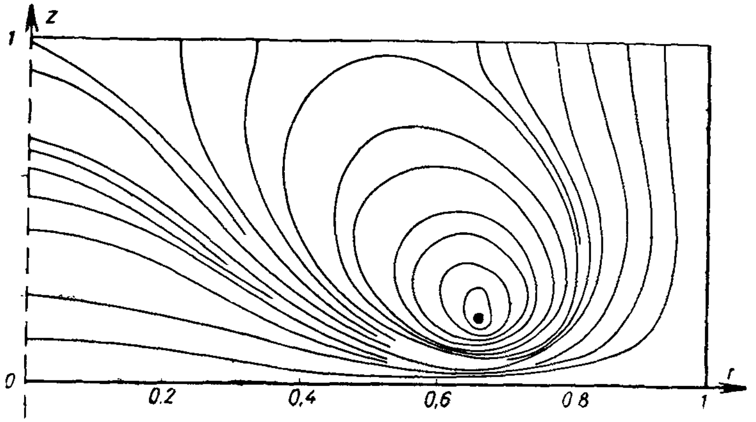

Fig. 2. Relative potential distribution obtained with the method in case of the electrode arrangement of Figure *1a*. Along the r axis measurements are given in R units, whereas along the z axis in $l/_2$ units. The equipotential lines correspond (starting from the cylinder with the value 0% towards the thread corresponding to 100%) to the following relative potential values: 2,5%; 5%; 7.5%; 8.75%; 9.37%; 9.66%; 10%; 11.25%; 11.40%; 12.5%; 13.75%; 15%; 17.5%; 20%; 25%; 30%; 40%; 60%

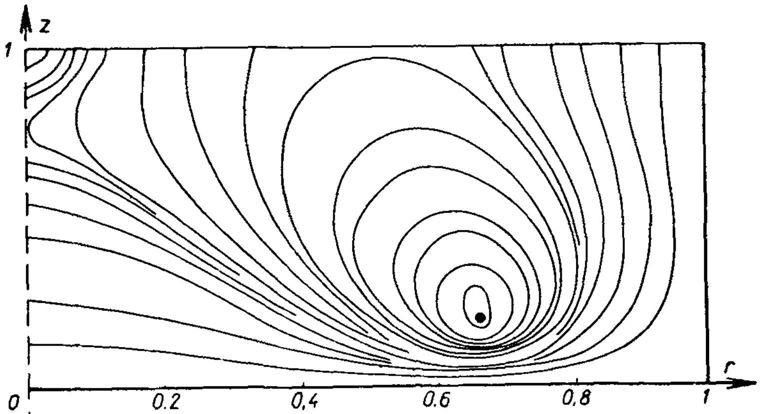

Fig. 3. The relative potential distribution measured in the electrolytic tank in the case of the electrode arrangement of Figure 1*a*. Denotations as in Figure 2. The relative potential values are: 2.5%; 5%; 7.5%; 8.75%; 9.37%; 9.68%; 9.84%; 10%; 11.25%; 12.05%; 13.75%; 15%; 17.50%; 20%; 25%; 30%; 40%; 60%

The problem of the convergence of the series can be dealt with also here in a way analogous to the case 1 by taking into account the asymptotical expressions for the functions *shx* and *chx*.

The potential distributions obtained for the case of electrode arrangements dealt with here were compared with the corresponding potential distributions measured in an electrolytic tank.* Measurements were carried

* We are indebted to Mr. E. KOLTAY pre "candidate" fellow for carrying out the measurements.

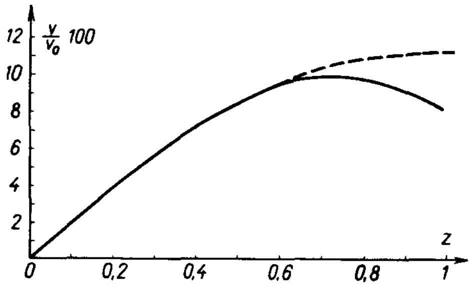

Fig. 4. The dependence of the relative value of the potential on z in case of $r = 0$. The full drawn line corresponds to the potential values obtained experimentally and the dotted to that obtained by the method

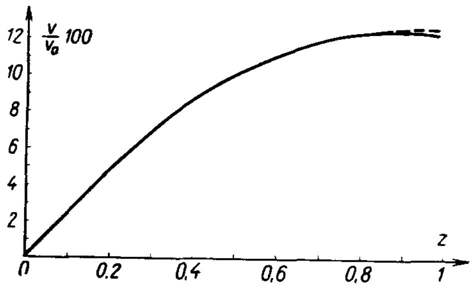

Fig. 5. The dependence of the relative value of the potential on z, in case of $r = 0.22$. Denotations as in Figure 4

out in the so-called tank with tilted bottom. By this modelling procedure the rotational symmetry of the electrode system is used for a simpler realization of the problem. Its drawback is, however, that due to the capillar phenomena appearing in the tank the accuracy of the method strongly decreases near the symmetry axis [6]. The potential distributions obtained by calculation and measurement are presented for case *a)* in Figures 2 and 3. Disregarding the surroundings of the z axis, agreement of the calculated and measured potential distributions within the limit of errors is found. The explanation of the differences observable near the axis may be found in what has been said above about the measuring accuracy. In Figures 4 resp. 5 so as to illustrate the agreement found the dependence of the relative potential value $\frac{V}{V_0}100$ on the z coordinate has been plotted for the values $r = 0$ resp. $r = 0.22$ (V_0 is the potential of the circle). Along the r axis of the Figures values are given in units R along the z axis in units $l/2$, corresponding to the denotations of Figure 1. The experimental curve shown by the full line tends when further away from the axis towards the theoretical curve shown by the dotted line. For small values of z the agreement is good also for small values of r.

3. Now we determine the potential field of the electronoptical two-cylinder lens so important in practical physics. The electrode arrangement

is the following: Two cylindrical electrodes of radius R_1 and length $z_0 - \frac{d}{2}$ are placed along the z axis at a distance d from each other as illustrated in Figure 6. They are surrounded by an earthed metal cylinder of radius R. The cylinders are charged so that their surface charge densities are σ_1 resp.

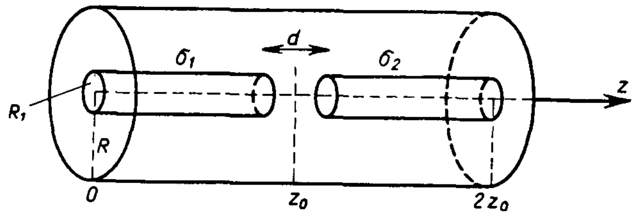

Fig. 6. The scheme of an electrode arrangement of the type of an electronoptical two-cylinder lens

σ_2. The solution of equation (3a) satisfying the boundary conditions is now the complete set of functions

$$\left.\begin{aligned} \Phi_{i0} &= \frac{1}{R J_0'(k_i)\sqrt{z_0}} J_0\left(\frac{k_i}{R} r\right) \\ \Phi_{in}^s &= \sqrt{2}\,\Phi_{i0}(r)\sin\left(n\frac{\pi}{2 z_0} z\right) \\ \Phi_{in}^c &= \sqrt{2}\,\Phi_{i0}(r)\cos\left(n\frac{\pi}{2 z_0} z\right) \end{aligned}\right\} . \tag{10}$$

Hereby the sum of the potentials of rings of surface charge densities σ_1 and σ_2 becomes

$$\begin{aligned} \varphi = \sum_i \Bigg\{ & \frac{4\pi}{z_0 k_i^2 J_0'^2(k_i)} (\sigma_1 + \sigma_2)\left(z_0 - \frac{d}{2}\right) J_0\left(\frac{k_i}{R} R_1\right) J_0\left(\frac{k_i}{R} r\right) + \\ & + \sum_{n=1}^{\infty} \frac{16}{R^2 J_0'^2(k_i)\, n} \frac{J_0\left(\frac{k_i}{R} R_1\right) J_0\left(\frac{k_i}{R} r\right)}{\frac{k_i^2}{R^2} + \left(n\frac{\pi}{2 z_0}\right)^2} \left[\cos\left(n\frac{\pi}{2 z_0} z\right)\left[\sigma_1 \sin n\frac{\pi}{2 z_0}\left(z_0 - \frac{d}{2}\right) - \right.\right. \\ & \left. - \sigma_2 \sin n \frac{\pi}{2 z_0}\left(z_0 + \frac{d}{2}\right)\right] - \sin\left(n\frac{\pi}{2 z_0} z\right)\left[\sigma_1\left(\cos n\frac{\pi}{2 z_0}\left(z_0 - \frac{d}{2}\right) - 1\right) + \right. \\ & \left.\left. + \sigma_2\left((-1)^n - \cos n \frac{\pi}{2 z_0}\left(z_0 + \frac{d}{2}\right)\right)\right]\right]\Bigg\} . \end{aligned} \tag{11}$$

The summation over n can be carried out. We consider the effect of the whole cylinder surface by integrating over the correspondig values of the z coordinate and we obtain the formula

$$\Phi = \sum_i \frac{8\pi R z_0}{k_i} \frac{\Phi_{i0}(r)\Phi_{i0}(R_1)}{\operatorname{sh}\frac{2k_i z_0}{R}} \left\{ \sigma_1 \int_0^{z_0-\frac{d}{2}} \operatorname{ch}\frac{k_i}{R}(2z_0 - |z - z_1|)\,dz_1 + \right.$$

$$\left. + \sigma_2 \int_{z_0+\frac{d}{2}}^{2z_0} \operatorname{ch}\frac{k_i}{R}(2z_0 - |z - z_1'|)\,dz_1' \right\}, \quad \text{if} \quad 0 < |z \pm z_1| < 4z_0 \tag{11a}$$

or in integrated form

$$\Phi = 8\pi R^2 z_0 \sum_i \frac{\Phi_{i0}(r)\Phi_{i0}(R_1)}{k_i^2 \operatorname{sh}\frac{2k_i z_0}{R}} \begin{cases} \sigma_1\left[2\operatorname{sh}\left(\frac{k_i}{R}2z_0\right) - \operatorname{sh}\frac{k_i}{R}(2z_0 - z) - \operatorname{sh}\frac{k_i}{R}\left(z + \right.\right. \\ \quad \left.\left. + z_0 + \frac{d}{2}\right)\right] + \sigma_2\left[\operatorname{sh}\frac{k_i}{R}\left(z + z_0 - \frac{d}{2}\right) - \right. \\ \quad \left. - \operatorname{sh}\left(\frac{k_i}{R}z\right)\right], \quad \text{if} \quad 0 < z < z_0 - \frac{d}{2}, \\ \sigma_1\left[\operatorname{sh}\frac{k_i}{R}\left(3z_0 - z - \frac{d}{2}\right) - \operatorname{sh}\frac{k_i}{R}(2z_0 - z)\right] + \\ \quad + \sigma_2\left[\operatorname{sh}\frac{k_i}{R}\left(z + z_0 - \frac{d}{2}\right) - \operatorname{sh}\left(\frac{k_i}{R}z\right)\right], \\ \quad \text{if} \quad z_0 - \frac{d}{2} < z < z_0 + \frac{d}{2}, \\ \sigma_1\left[\operatorname{sh}\frac{k_i}{R}\left(3z_0 - z - \frac{d}{2}\right) - \operatorname{sh}\frac{k_i}{R}(2z_0 - z)\right] + \\ \quad + \sigma_2\left[2\operatorname{sh}\left(\frac{k_i}{R}2z_0\right) - \operatorname{sh}\frac{k_i}{R}\left(3z_0 - z + \frac{d}{2}\right) - \right. \\ \quad \left. - \operatorname{sh}\left(\frac{k_i}{R}z\right)\right], \quad \text{if} \quad z_0 + \frac{d}{2} < z < 2z_0. \end{cases} \tag{11b}$$

The potential of an infinitely long cylinder capacitor dealt with in 1. is evidently contained in (11b) when $z_0 \to \infty$. As can easily be seen

$$\lim_{z_0 \to \infty} \Phi(r, z)_{z=0} = 8\pi R^2 z_0 \sum_i \frac{\Phi_{i0}(r)\Phi_{i0}(R_1)}{k_i^2} \sigma_1, \quad \text{if} \quad \frac{\sigma_1}{\sigma_2} \neq 0,$$

resp.

$$\lim_{z_0 \leftarrow \infty} \Phi(r, z)_{z=2z_0} = 8\pi R^2 z_0 \sum_i \frac{\Phi_{i0}(r)\, \Phi_{i0}(R_1)}{k_i^2} \sigma_2, \quad \text{if} \quad \frac{\sigma_2}{\sigma_1} \neq 0.$$

From the formula (11b) the potential distribution of the two-cylinder lens used in electron optics is obtained for the case $R \gg R_1$.

The result thus obtained is of interest as — in contrast to any other method applied to the calculation of the field of the electronoptical two-cylinder lens — no stipulation was made during the calculation concerning the width d of the slit. Other methods used for the solution of the problem fail if the width of the slit is of the order of magnitude of the tube diameter.

For the first two cases mentioned as examples in III the formula is exact, whereas in the case of the electronoptical two-cylinder lens it has to be considered as approximative, since for the calculation we started from the assumption that the charge distribution is uniform on the cylinder surfaces. The accuracy of the approximation can be estimated from the equipotential surface running near the cylinder surface.

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ОПРЕДЕЛЕНИЕ ЭЛЕКТРОСТАТИЧЕСКИХ ПОТЕНЦИАЛОВ С ПОМОЩЬЮ РЯДОВ

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Резюме

Мы реформулировали метод рядов для решения уравнения Пуассона, пользуясь решениями задач по собственным значениям, часто встречающихся в разных областях теоретической физики. Если распределение заряда соответствует δ-функции Дирака, то решение получается в форме одного, довольно хорошо сходящегося ряда. В простых случаях наши решения аналитически трансформируемы в хорошо известные решения (напр. в случае III/1). В других случаях (III/2) вычисленное распределение потенциала совпадает с полученным из измерений в электролитическом ванне. В случае электроноптической цилиндрической линзы, (III/3) наш метод дает распределение потенциала для какой-либо ширины щели и соответствующую форму электродов.