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LINEAR RADIAL FLOW OF A VISCOUS LIQUID BETWEEN TWO PARALLEL COAXIAL STATIONARY INFINITE DISKS

By

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An analysis is presented for laminar radial flow due to a linear source between two parallel stationary infinite disks. The source strength varies according to $Q = Q_0\left(\frac{\nu\mathbf{t}}{h^2}\right)$ ($\mathbf{t} > 0$) and the solution is in the form of an infinite series in terms of a reduced Reynolds number $R_a^* = \left(\frac{Q_0}{4\pi\nu h}\right)\Big/\left(\frac{\mathbf{r}}{h}\right)^2$. The results are valid for small values of R_a^* and $t\left(=\frac{\nu\mathbf{t}}{h^2}\right)$. The effect of the parameter R_a^* on the radial velocity distribution, pressure distribution, shear stress at the upper disk at different times is discussed.

Nomenclature

h = half distance between disks
$\mathbf{r}$ = radial coordinate
$r = \frac{\mathbf{r}}{h}$ = dimensionless radial coordinate
$\mathbf{z}$ = axial coordinate
$z = \frac{\mathbf{z}}{h}$ = dimensionless axial coordinate
$\mathbf{t}$ = time
$t = \frac{\nu\mathbf{t}}{h^2}$ = dimensionless time
R = dimensionless radial coordinate of a cross-section in the flow domain
$\mathbf{u}$ = radial velocity
$u = \frac{h\mathbf{u}}{\nu}$ = dimensionless radial velocity
$\mathbf{v}$ = axial velocity
$v = \frac{h\mathbf{v}}{\nu}$ = dimensionless axial velocity
$\mathbf{p}$ = pressure
$p = \frac{\mathbf{p}h^2}{\varrho\nu^2}$ = dimensionless pressure
Q = instantaneous source strength
Q_0 = gradient of source strength
$R_a = \frac{Q_0}{4\pi\nu h}$ = gradient of source Reynolds number
$R_a^* = \frac{R_a}{r^2}$ = gradient of reduced Reynolds number
ϱ = density
μ = viscosity
$\nu = \frac{\mu}{\varrho}$ = kinematic viscosity
$\boldsymbol{\tau}_1$ = shear stress at the upper disk
$\tau_1 = \boldsymbol{\tau}_1\Big/\left(\frac{\mu Q_0}{4\pi h^2\mathbf{r}}\right)$ = dimensionless shear stress at the upper disk

1. Introduction

Unsteady flow is of practical importance in many areas of engineering, e. g. acoustics, biomedical engineering and lubrication. Oscillating radial flow is of primary interest in the design of thrust bearings and radial diffusers.

A system which has received considerable attention is that of unsteady flow in circular tubes, e. g. UCHIDA [1]. Recently ELKOUH [2] has given an analysis for a system in which the flow rate varies sinusoidally about a zero-mean value. His solution is valid for small values of the reduced Reynolds number and all values of the frequency Reynolds number.

In this paper an analysis is presented for laminar flow due to a linear source between two parallel stationary infinite disks. The solution obtained for the motion of the liquid is in the form of an infinite series expansion in terms of a reduced Reynolds number, R_a^*, which signifies the effect of convective inertia. The results are valid for small values of R_a^* and t.

2. Basic equations and their solution

Consider the unsteady axially symmetric flow of a viscous liquid between two parallel stationary infinite disks, which lie in the planes $z = -h$ and $z = +h$ (Fig. 1). The flow through the system shown in Fig. 1 is due to a source, at $r = 0$, whose strength varies according to

$$Q = Q_0\left(\frac{\nu t}{h^2}\right), \quad (t > 0). \tag{2.1}$$

In terms of the dimensionless variables defined, see nomenclature, the Navier-Stokes equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial r} + \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{2.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial z} + \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2}\right), \tag{2.3}$$

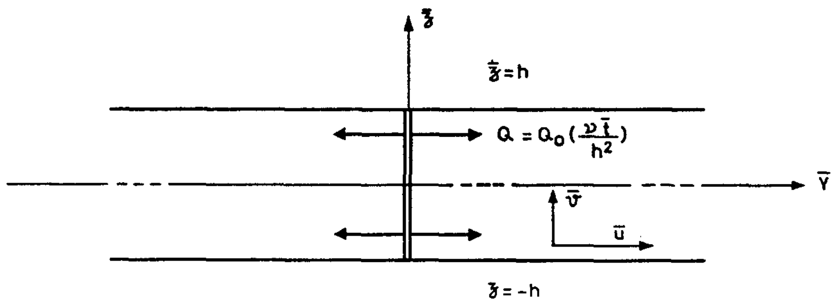

Fig. 1. Flow system and coordinates

and the equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0 . \tag{2.4}$$

The boundary conditions for the flow system under consideration are

$$\begin{gathered} u = 0, \quad v = 0 \quad \text{at} \quad z = \pm 1, \\ \int_{-1}^{+1} u \, dz = \frac{2R_a}{r} t, \qquad (t > 0) \end{gathered} \tag{2.5}$$

where $R_a = \dfrac{Q_0}{4\pi \nu h}$ is the gradient of the source Reynolds number.

The following expansions which are valid for small values of the reduced Reynolds number $R_a^* = \left(\dfrac{R_a}{r^2}\right)$ and away from the source at $r = 0$ are assumed for u, v and p:

$$u = \frac{R_a}{r}\left[f_0'(z, t) + \left(\frac{R_a}{r^2}\right) f_1'(z, t) + \left(\frac{R_a}{r^2}\right)^2 f_2'(z, t) + \ldots\right], \tag{2.6}$$

$$v = \left[2\left(\frac{R_a}{r^2}\right)^2 f_1(z, t) + 4\left(\frac{R_a}{r^2}\right)^3 f_2(z, t) + \ldots\right], \tag{2.7}$$

$$p = h(z, t) + R_a\left[h_0(z, t) \log r + \left(\frac{R_a}{r^2}\right) h_1(z, t) + \left(\frac{R_a}{r^2}\right)^2 h_2(z, t) + \ldots\right], \tag{2.8}$$

where the primes denote partial differentiation with respect to z.

The boundary conditions on the functions $f_n(z, t)$ and their derivatives are

$$\left.\begin{aligned} & f_n'(\pm 1, t) = 0, \; n = 0, 1, 2, \ldots \\ & f_n(\pm 1, t) = 0, \; n = 1, 2, \ldots\ldots \\ \text{and} \qquad & f_0(1, t) - f_0(-1, t) = 2t, \; (t > 0) \end{aligned}\right\} \tag{2.9}$$

which upon choosing

$$f_0(-1, t) = -t$$

gives

$$f_0(1, t) = t .$$

The expressions for the velocity components (2.6) and (2.7) satisfy the continuity equation. Substituting (2.6), (2.7) and (2.8) in (2.2) and (2.3) and equating coefficients of equal powers in r reduces the Navier-Stokes equations to an infinite set of systems of simultaneous linear partial differential equations. For the sake of brevity we record only the first two systems below:

System I

$$\frac{\partial^3 f_0}{\partial z^3} - \frac{\partial^2 f_0}{\partial t\,\partial z} = h_0(z, t)\,, \tag{2.10}$$

$$\frac{\partial h_0}{\partial z} = 0, \quad \text{i. e.} \quad h_0(z, t) = h_0(t) + \text{constant.}$$

The partial differential equation for $h(z, t)$ is

$$\frac{\partial h}{\partial z} = 0, \quad \text{i. e.} \quad h(z, t) = h(t)\,, \tag{2.11}$$

where $h(t)$ is determined from a known pressure at a point in the flow domain.

System II

$$\frac{\partial^3 f_1}{\partial z^3} - \frac{\partial^2 f_1}{\partial t\,\partial z} = -\,2h_1(z, t) - \left(\frac{\partial f_0}{\partial z}\right)^2, \tag{2.12}$$

$$\frac{\partial h_1}{\partial z} = 0, \quad \text{i. e.} \quad h_1(z, t) = h_1(t) + \text{constant.}$$

The solution of (2.10) subject to the boundary conditions (2.9) represents the limiting case when $\left(\frac{R_a}{r^2}\right) \to 0$. The linearity of (2.10) and the form of the boundary conditions suggest a solution of the form

$$f_0(z, t) = F_0(z) + G_0(z)t, \tag{2.13}$$

and

$$h_0(t) = H_0 + P_0\, t. \tag{2.14}$$

Substituting (2.13) and (2.14) in (2.10), we get

$$G_0''' = P_0\,, \tag{2.15}$$

and

$$F_0''' - G_0' = H_0\,. \tag{2.16}$$

The boundary conditions on F_0 and G_0 and their derivatives are

$$\left.\begin{aligned} F_0(\pm 1) &= 0\,, & F_0'(\pm 1) &= 0, \\ G_0(\pm 1) &= \pm 1\,, & G_0'(\pm 1) &= 0\,. \end{aligned}\right\} \tag{2.17}$$

and

The solutions of (2.15) and (2.16) under the boundary conditions (2.17) are

$$F_0(z) = -\frac{(z - 2z^3 + z^5)}{40}, \tag{2.18}$$

$$G_0(z) = \frac{1}{2}(3z - z^3). \tag{2.19}$$

Thus

$$f_0(z, t) = -\frac{(z - 2z^3 + z^5)}{40} + \frac{1}{2}(3z - z^3)\, t, \tag{2.20}$$

$$h_0(t) = -\frac{6}{5} - 3t. \tag{2.21}$$

Substituting for $\left(\frac{\partial f_0}{\partial z}\right)^2$ from (2.20) into the right-hand side of (2.12) will contribute time-independent terms and terms with t and t^2. Taking into account these circumstances we can express the solution of System II in the form

$$f_1(z, t) = F_s(z) + F_1(z)t + G_1(z)t^2, \tag{2.22}$$

and

$$h_1(t) = H_s + H_1 t + P_1 t^2. \tag{2.23}$$

Substituting (2.22), (2.23) and $\left(\frac{\partial f_0}{\partial z}\right)^2$ from (2.20) in (2.12), we get

$$G_1''' = -2P_1 - \frac{9}{4}(1 - 2z^2 + z^4), \tag{2.24}$$

$$F_1''' - 2G_1' = -2H_1 + \frac{3}{40}(1 - 7z^2 + 11z^4 - 5z^6), \tag{2.25}$$

$$F_s''' - F_1' = -2H_s - \frac{1}{1600}(1 - 12z^2 + 46z^4 - 60z^6 + 25z^8). \tag{2.26}$$

The boundary conditions on F_s, F_1, G_1 and their derivatives are

$$\left.\begin{aligned} F_s(\pm 1) &= 0, \quad F_s'(\pm 1) = 0, \\ F_1(\pm 1) &= 0, \quad F_1'(\pm 1) = 0, \\ G_1(\pm 1) &= 0, \quad G_1'(\pm 1) = 0. \end{aligned}\right\} \tag{2.27}$$

The solutions of (2.24), (2.25) and (2.26) subject to the boundary conditions (2.27) are

$$F_s(z) = \frac{8633}{19404000} z - \frac{479}{388080} z^3 + \frac{53}{42000} z^5 - \frac{23}{36750} z^7 + \frac{1}{5600} z^9 - \frac{1}{39600} z^{11}, \quad (2.28)$$

$$F_1(z) = -\frac{97}{11200} z + \frac{191}{8400} z^3 - \frac{23}{1120} z^5 + \frac{3}{400} z^7 - \frac{1}{960} z^9, \quad (2.29)$$

$$G_1(z) = \frac{3}{56} z - \frac{33}{280} z^3 + \frac{3}{40} z^5 - \frac{3}{280} z^7, \quad (2.30)$$

and

$$h_1(t) = -\frac{38}{40425} + \frac{4}{175} t - \frac{27}{35} t^2. \quad (2.31)$$

3. Results and discussion

(a) *Radial velocity distribution*

We now define a dimensionless radial velocity such that

$$u^* = \frac{ur}{R_a}. \quad (3.1)$$

Substituting for $f_0(z, t)$ and $f_1(z, t)$ from (2.13) and (2.22) into (2.6) and neglecting higher order terms, we get

$$u^* = \left[-\frac{(1 - 6z^2 + 5z^4)}{40} + \frac{3}{2}(1 - z^2)\, t\right] + R_a^* \left[\left(\frac{8633}{19404000} - \frac{479}{129360} z^2 + \frac{53}{8400} z^4 - \frac{23}{5250} z^6 + \frac{9}{5600} z^8 - \frac{1}{3600} z^{10}\right) + \left(-\frac{97}{11200} + \frac{191}{2800} z^2 - \frac{23}{224} z^4 + \frac{21}{400} z^6 - \frac{3}{320} z^8\right) t + \left(\frac{3}{56} - \frac{99}{280} z^2 + \frac{3}{8} z^4 - \frac{3}{40} z^6\right) t^2\right]. \quad (3.2)$$

The first term on the right-hand side of (3.2) represents the radial velocity for $R_a^* = 0$, i. e. as r tends to infinity, while the second term represents the effect of the nonlinear inertia. The nonlinear-inertia contribution is in the form of steady and unsteady streamings.

The instantaneous radial velocity distributions for $t = 2$, 3, 6 and for $R_a^* = 0$ and $R_a^* = 0.5$ are shown in Fig. 2. The magnitudes of the nonlinear

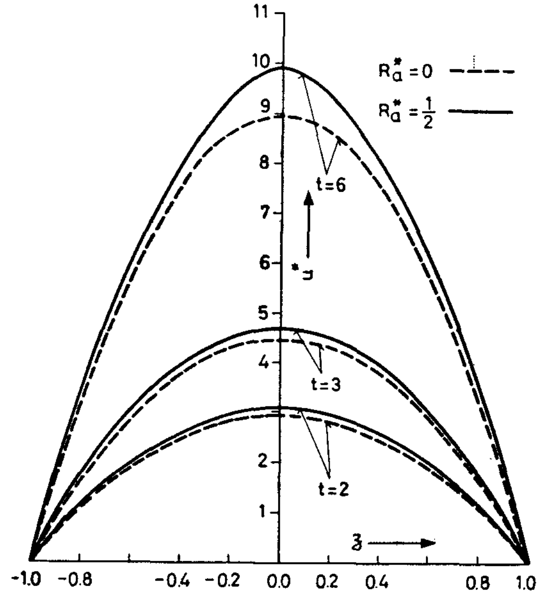

Fig. 2. Variation of u^* with z at different times

inertia contributions to the velocity distributions are very small up to about $t = 2 \forall z \in (-1, 1)$. Fig. 2 also shows that u^* is maximum at $z = 0$ and it decreases monotonically as we move towards the solid boundaries.

(b) *Pressure distribution*

Using (2.11) and neglecting terms of higher order than R_a^*, the pressure distribution is of the form

$$p = h(t) + R_a[h_0(t) \log r + h_1(t) R_a^*], \tag{3.3}$$

where $h(t)$ is determined from a known pressure at some cross-section in the flow domain. Assuming that the pressure is known at $r = R$ and using (2.21) and (2.31), the expression for the pressure distribution is

$$p^* = \frac{p(r,t) - p(R,t)}{R_a}$$

$$= \left(\frac{6}{5} + 3t\right) \log\left(\frac{R}{r}\right) + R_a^*\left(1 - \frac{r^2}{R^2}\right)\left[-\frac{38}{40425} + \frac{4}{175}t - \frac{27}{35}t^2\right]. \quad (3.4)$$

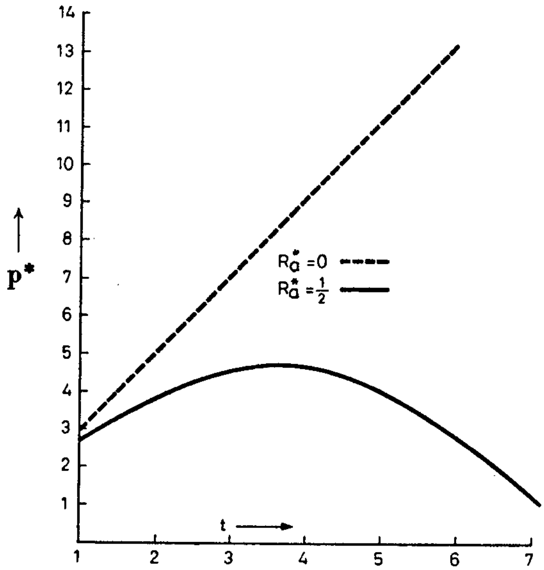

Fig. 3. Variation of p^* with time $\cdot \frac{r}{R} = \frac{1}{2}$

The variation of p^* with time for $\frac{r}{R} = 0.5$, $R_a^* = 0$ and $R_a^* = 0.5$ is shown in Fig. 3. From this Figure it is clear that the effect of the nonlinear inertia is insignificant up to about $t = 2$. For $R_a^* = 0$, the pressure also varies linearly with time.

(c) *Skin friction*

The shear stress at the upper disk is given by

$$\tau_1 = -\mu\left(\frac{\partial \mathbf{u}}{\partial \mathbf{z}}\right)_{z=h} = -\frac{\varrho v^2}{h^2}\left(\frac{\partial u}{\partial z}\right)_{z=1} = -\frac{\mu Q_0}{4\pi h^2 \mathbf{r}}\left[f_0''(1,t) + R_a^* f_1''(1,t)\right].$$

Thus

$$\tau_1 = \left(\frac{1}{5} + 3t\right) - R_a^*\left(\frac{986}{606375} - \frac{6}{175}t + \frac{12}{35}t^2\right). \quad (3.5)$$

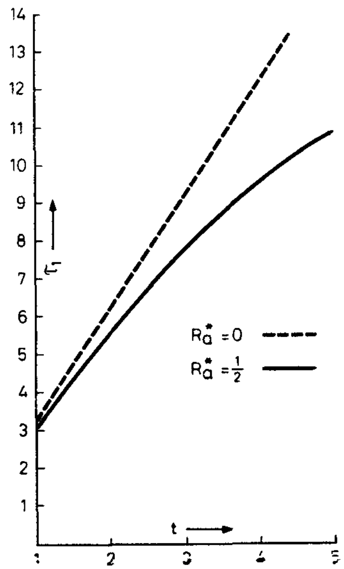

Fig. 4. Variation of τ_1 with time

The variation of τ_1 with time for $R_a^* = 0$ and $R_a^* = 0.5$ is presented in Fig. 4. This Figure indicates that the effect of the nonlinear-inertia is negligible up to about $t = 2$. For $R_a^* = 0$, the shear stress at the upper disk also varies linearly with time.

REFERENCES

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