Acta Physica Academiae Scientiarum Hungaricae, Tomus 30 (4), pp. 351–364 (1971)

ON THE TRACE OF THE PRODUCT OF PAULI AND DIRAC MATRICES

By

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(Received 5. V. 1970)

Several methods for the determination of the trace of the product of an arbitrary number of Pauli matrices are established. Formulae are derived for the evaluation of various types of products of two traces when terms of the type $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_n}$ occur in both of them. Expressions are found for the product of two different traces and the square of the trace of an arbitrary number of Pauli matrices. Similar formulae are obtained when Dirac matrices occurring as $\sum_{i=1}^{5}\gamma_i A_i$ are considered instead of Pauli matrices. From this all previous results in which γ_5 has been considered separately are recovered. A useful identity for traces involving either Pauli or Dirac matrices is given.

Introduction

One of the purposes of this paper is to reduce the problem of the calculation of the trace of the product of any odd or even number of Pauli matrices, to one involving a smaller number — in the final stage, two or three — of Pauli matrices. First the formulae for the determination of various types of products of two traces when $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_n}$ occur in both traces (summation over the dummy suffixes i_r is implied) are derived and then expressions for the product of two different traces and the square of the trace of an arbitrary number of Pauli matrices. Next, the five Dirac matrices γ_1, γ_2, γ_3, γ_4 and γ_5 are considered simultaneously instead of the Pauli matrices and similar expressions are obtained. More explicitly, the Dirac matrices occur in the trace as product of an arbitrary number of elements like $\sum_{i=1}^{5} A_i\gamma_i$. From the results obtained in this second stage all the results of the author's previous paper [1], in which Dirac matrices occur in the form $\sum_{i=1}^{4} A_i\gamma_i$ and γ_5 may occur separately, can be reproduced. An identity for traces involving either Pauli or Dirac matrices has been established. This is found to be useful in the reduction of the formulae and in demonstrating the equivalence of some of the results in our deduction. In this connection it should be mentioned that CHISHOLM [2] has evaluated the sums $\sum_{r=i}^{3}\sigma_r\sigma_a\sigma_b\ldots\sigma_d\sigma_r$ and $\sum_{r=i}^{3}\ldots\sigma_r\ldots Sp(\sigma_r\sigma_a\sigma_b\ldots\sigma_d)$. CHISHOLM [2] has also solved the same problem for Dirac matrices. CAIANIELLO and FUBINI [3] and KAHANE [4] have investigated various aspects of the problem of evaluating the trace of the product of Dirac matrices. The calculations are set out in three sections: Section I deals with the Pauli matrices whereas Sections II and III with Dirac matrices. I have generally followed the notation and method of [1].

I

All the formulae derived here are based on the following algebraic properties of Pauli matrices:

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\,\delta_{ij}, \tag{1}$$

$$\sigma_i \sigma_j = \epsilon_{ijk}\,\sigma_k + \delta_{ij}; \tag{2}$$

where ϵ_{ijk} is the Levi Civita tensor of the third rank. Let us use the abbreviated notation

$$\text{trace} \quad (A_1 A_2 A_3 \ldots A_n) = (A_1 A_2 A_3 \ldots A_n), \tag{3}$$

$$A = A_i\,\sigma_i, \tag{4}$$

$$A \cdot B = A_i B_i\,. \tag{5}$$

Summation over $i = 1, 2, 3$ is to be done. (Throughout this paper summation is implied whenever repeated suffixes occur.)

Let us denote $(A_1 A_2 A_3 \ldots A_n)$ by S, i.e.

$$S = (A_1 A_2 A_3 \ldots A_n). \tag{6}$$

When n is even we already know the result

$$S = \sum_{i=2}^{n} (-1)^i A_1 \cdot A_i (A_2 A_3 \ldots A_{i-1} A_{i+1} \ldots A_n). \tag{7}$$

For n odd we shall develop other methods for the evaluation of S, some of which may be applicable for even n also.

We can write

$$\begin{aligned} S &= \sum_{i,j} A_{1i}\, A_{2j}\; (\sigma_i \sigma_j A_3 \ldots A_n) = \\ &= A_1 \cdot A_2 (A_3 A_4 \ldots A_n) + i\epsilon_{ijk} A_{1i} A_{2j} (\sigma_k A_3 A_4 \ldots A_n) = \\ &= A_1 \cdot A_2 (A_3 A_4 \ldots A_n) - \frac{1}{2} \sum_{r \geq 3}^{n} (-1)^r (A_1 A_2 A_r) \\ &\qquad (A_3 A_4 \ldots A_{r-1} A_{r+1} \ldots A_n). \end{aligned} \tag{8}$$

This formula is valid when n is odd.

S can also be written in the form

$$S = \sum_{i,j,k} A_{1i} A_{2j} A_{3k} \ (\sigma_i \sigma_j \sigma_k A_4 A_5 \ldots A_n). \tag{9}$$

The summation here can be split up in the following manner:

$$\sum_{i,j,k} = \sum_{i \neq j \neq k} + \sum_{i=j \neq k} + \sum_{i=k \neq j} + \sum_{i \neq j = k} + \sum_{i=j=k} \tag{10}$$

Application of Eq. (10) provides the third formula for S, which is valid for any n, even or odd:

$$\begin{aligned} S = A_1 \cdot A_2 (A_3 A_4 \ldots A_n) + A_2 \cdot A_3 (A_1 A_4 \ldots A_n) - \\ - A_1 \cdot A_3 (A_2 A_4 \ldots A_n) + \frac{1}{2} (A_1 A_2 A_3)(A_4 A_5 \ldots A_n). \end{aligned} \tag{11}$$

Let us now discuss some relations involving the product of two traces. We have

$$\begin{aligned} (\sigma_i A_1 A_2 \ldots A_n)(\sigma_i A_1' A_2' \ldots A_m') = \sum_{i=1}^{n} (-1)^{i+1} (A_1 A_2 \ldots A_{i-1} A_{i+1} \ldots A_n) \cdot \\ \cdot (A_i A_1' A_2' \ldots A_m'). \end{aligned} \tag{12}$$

Here n is odd and m may be either even or odd. Knowing that

$$\sum_{i,j} = \sum_{i=j} + \sum_{i \neq j} . \tag{13}$$

We obtain with the help of Eqs. (1) and (2)

$$\begin{aligned} T_2 = (\sigma_i \sigma_j A_1 A_2 \ldots A_n)(\sigma_i \sigma_j A_1' A_2' \ldots A_m') = \\ = 3 (A_1 A_2 \ldots A_n)(A_1' A_2' \ldots A_m') - 2 (\sigma_k A_1 A_2 \ldots A_n)(\sigma_k A_1' A_2' \ldots A_m'). \end{aligned} \tag{14}$$

By repeated application of Eq. (14) we get

$$\begin{aligned} T_3 = (\sigma_i \sigma_j \sigma_k A_1 A_2 \ldots A_n)(\sigma_i \sigma_j \sigma_k A_1' A_2' \ldots A_n') = \\ = -6 (A_1 A_2 \ldots A_n)(A_1' A_2' \ldots A_m') + 7 (\sigma_i A_1 A_2 \ldots A_n)(\sigma_i A_1' A_2' \ldots A_m'). \end{aligned} \tag{15}$$

In general, if we write

$$\begin{aligned} T_{m+1} = (\sigma_{i1} \sigma_{i2} \ldots \sigma_{im+1} A_1 A_2 \ldots A_n) \cdot (\sigma_{i1} \sigma_{i2} \ldots \sigma_{im+1} A_1' A_2' \ldots A_m') = \\ = \alpha_m (A_1 A_2 \ldots A_n)(A_1' A_2' \ldots A_m') + \beta_m (\sigma_i A_1 A_2 \ldots A_n)(\sigma_i A_1' A_2' \ldots A_m') \end{aligned} \tag{16}$$

then

$$\alpha_{m+1} = 3\beta_m , \qquad \beta_{m+1} = \alpha_m - 2\beta_m , \tag{17}$$

and we easily obtain

$$\beta_m = \sum_{r=0}^{m-2} (-3)^r - 2(-3)^{m-1} \tag{18}$$

and

$$\alpha_m = 1 - \beta_m . \tag{19}$$

In Eqs. (14), (15) and (16) both n and m can be either even or odd. With the help of Eq. (12) we obtain

$$\begin{aligned}(\sigma_i \sigma_j A_1 A_2 \ldots A_n)(\sigma_i \sigma_j A_1' A_2' \ldots A_m') = 3(A_1 A_2 \ldots)(A_1' A_2' \ldots) - \\ - 2 \sum_{r>s} (-1)^{r+s} (A_1 A_2 \ldots)'' [(A_r A_s A_1' A_2' \ldots) - A_r A_s (A_1' A_2' \ldots)] .\end{aligned} \tag{20}$$

Here n must be even but m can be either even or odd.

In Eq. (20) the notation of double primes over the trace $(A_1 A_2 \ldots)''$ implies that the two unprimed Pauli matrices (in this case A_r and A_s) which are now present in the other term, are now absent from the trace $(A_1 A_2 \ldots)$. This notation in the general form, with any number of primes over the notation of trace, will be widely used in this paper.

Comparing Eqs. (14) and (20) we have

$$\begin{aligned}(\sigma_i A_1 A_2 \ldots A_n)(\sigma_i A_1' A_2' \ldots A_m') = \sum_{r>s} (-1)^{r+s} (A_1 A_2 \ldots)'' \times \\ \times [(A_r A_s A_1' A_2' \ldots) - A_r \cdot A_s (A_1' A_2' \ldots)] .\end{aligned} \tag{21}$$

When n and m are both even, Eq. (12) is not applicable and must be replaced by Eq. (21).

A particular case of Eq. (21) is

$$(\sigma_i A_1 A_2)(\sigma_i A_1' A_2' \ldots A_m') = 2(A_1 A_2 A_1' A_2' \ldots A_m') - 2 A_1 \cdot A_2 (A_1' A_2' \ldots A_m') . \tag{22}$$

If we use the relation (7) in the left hand side of Eq. (22), taking m to be odd, we obtain Eq. (8) for determining S. With the help of Eq. (1) we can write

$$\begin{aligned}(A_1 A_2 \ldots A_n) = 2 \sum_{i=2}^{n-1} (-1)^i A_1 \cdot A_i (A_2 A_3 \ldots A_{i-1} A_{i+1} \ldots A_n) - \\ - (A_2 A_3 \ldots A_1 A_n)\end{aligned} \tag{23}$$

where n is odd.

From Eq. (23) we obtain the identity

$$\sum_{i=2}^{n} (-1)^i A_1 \cdot A_i (A_2 A_3 \ldots A_{i-1} A_{i+1} \ldots A_n) = 0 . \tag{24}$$

This identity proves to be very useful in reducing the number of terms occurring in the expansion of some traces.

Through successive application of Eq. (12) we can write

$$(\sigma_i \sigma_j \sigma_k A_1 A_2 \ldots A_n)(\sigma_i \sigma_j \sigma_k A_1' A_2' \ldots A_m') = 7(\sigma_i A_1 A_2 \ldots A_n) \tag{25}$$
$$(\sigma_i A_1' A_2' \ldots A_m') + \sum_{i>j>k} (-1)^{i+j+k} (A_1 A_2 \ldots A_n)''' \sum_P \delta_P P(A_i A_j A_k A_1' A_2' \ldots A_m').$$

In Eq. (25) and elsewhere, P is any permutation of the suffixes (in this case i, j and k), and $\delta_p = \pm 1$ depending on whether P is an even or odd permutation. The notation $\sum_P$ denotes summation over all the possible permutations. Now it can be shown that

$$\sum_P \delta_P P(A_i A_j A_k A_1' A_2' \ldots) = \tag{26}$$
$$= 6([A_i A_j A_k - A_i \cdot A_j A_k + A_i \cdot A_k A_j - A_j \cdot A_k A_i] A_1' A_2' \ldots).$$

Eq. (25) combined with Eq. (15) enables us to establish that

$$(A_1 A_2 \ldots A_n)(A_1' A_2' \ldots A_m') = -\frac{1}{6} \sum_{i>j>k} (-1)^{i+j+k} (A_1 A_2 \ldots A_n)''' \cdot \tag{27}$$
$$\cdot \sum_P \delta_P P(A_i A_j A_k A_1' A_2' \ldots A_m').$$

With the help of Eqs. (27) and (26), and taking $n = 3$ and m as either even or odd, we obtain formula (11) for the determination of S.

Eq. (27) can also be rewritten in the following alternative form:

$$(A_1 A_2 \ldots A_n)(A_1' A_2' \ldots A_m') = \sum_{i>j>k} (-1)^{i+j+k} (A_1 A_2 \ldots A_n)''' \cdot \tag{28}$$
$$\cdot \sum_{r>s} (-1)^{r+s} (A_i \cdot A_r' A_j \cdot A_s' - A_j \cdot A_r' A_i \cdot A_s')(A_k A_1' A_2' \ldots A_m')'' \ldots$$

$$= - \sum_{i>j>k} (-1)^{i+j+k} (A_1 A_2 \ldots A_n)''' \cdot \tag{29}$$
$$\cdot \sum_{r>s>t} (-1)^{r+s+t} (A_1' A_2' \ldots A_m')''' \sum_P \delta_P P[A_i \cdot A_r' A_j \cdot A_s' A_k \cdot A_t'].$$

The general relation (29) leads to the particular relations

$$(A_1 A_2 A_3)(A_1' A_2' \ldots A_m') = \tag{30}$$
$$= -2 \sum_{r>s>t} (-1)^{r+s+t} \sum_P \delta_P P[A_3 \cdot A_r' A_2 \cdot A_s' A_1 \cdot A_t'](A_1' A_2' \ldots A_m')''',$$

$$(A_1A_2A_3)(A'_rA'_sA'_t) = -4\sum_P \delta_P P[A_3\cdot A'_tA_2\cdot A'_sA_1\cdot A'_r]. \tag{31}$$

$$(A_1A_2\ldots A_n)^2 = \sum_{i>j>k}(-1)^{i+j+k}(A_1A_2\ldots A_n)'''\cdot$$
$$\cdot\sum_{r>s>t}(-1)^{r+s+t}\sum_P \delta_P P\,[A_i\cdot A_rA_j\cdot A_sA_k\cdot A_t]\,(A_1A_2\ldots A_n)'''. \tag{32}$$

In Eqs. (29)—(32) permutation P of the suffixes r, s and t is implied.

By putting $A_1, A_2, A_3 = \sigma_1, \sigma_2, \sigma_3$ in relation (30) we get another formula for the determination of S:

$$(A_1A_2\ldots A_n) = \frac{1}{2}\sum_{r>s>t}(-1)^{r+s+t}(A_tA_sA_r)(A_1A_2\ldots A_n)'''. \tag{33}$$

In Eqs. (28)—(33), both n and m must be odd.

We have now established four equations (7), (8), (11) and (33) for determining $(A_1A_2\ldots A_n)$.

For $n = 5$, using identity (24), we get according to both Eqs. (8) and (11)

$$S_5 = (A_1A_2A_3A_4A_5) = A_1\cdot A_2(A_3A_4A_5) + A_2\cdot A_3(A_1A_4A_5) - \tag{34}$$
$$- A_1\cdot A_3(A_2A_4A_5) + A_4\cdot A_5(A_1A_2A_3)\,.$$

Eq. (33) gives 10 terms for S_6, which can be reduced to four terms with the help of the identity given by Eq. (24). For other odd values of n, Eqs. (8) and (33) give more terms than Eq. (11).

S_n for even values of n are found from Eqs. (7) and (11). For $n = 6$ we get from Eq. (11)

$$S_6 = (A_1A_2A_3A_4A_5A_6) = A_1\cdot A_2(A_3A_4A_5A_6) + A_2\cdot A_3(A_1A_4A_5A_6) - \tag{35}$$
$$- A_1\cdot A_3(A_2A_4A_5A_6) + \frac{1}{2}(A_1A_2A_3)(A_4A_5A_6)\,.$$

Eq. (11) gives as expression for S_n shorter than that of Eq. (7) and is thus most convenient for determining S_n for both odd and even values of n.

II

The notation and method of Section I are mostly followed in Sections II and III also. For Dirac matrices, the anticommutation relation (1) is replaced by

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\,\delta_{ij}\,, \tag{36}$$

where the suffixes of γ matrices can be 1, 2, 3, 4 and 5. We use the abbreviated notation given by Eq. (3), except that here

$$A = A_i \gamma_i \,, \tag{37}$$

$$A \cdot B = A_i B_i \,, \tag{38}$$

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 \,. \tag{39}$$

Summation over $i = 1, 2, 3, 4$ and 5 is to be done.

With the help of relation (10) we can write

$$\begin{aligned} S = (A_1 A_2 \ldots A_n) &= \sum_{i,j,k} A_{1i} A_{2j} A_{3k} (\gamma_i \gamma_j \gamma_k A_4 A_5 \ldots A_n) = \\ &= A_1 \cdot A_2 (A_3 A_4 \ldots) + A_2 \cdot A_3 (A_1 A_4 \ldots) - A_1 \cdot A_3 (A_2 A_4 \ldots) - \\ &\quad - \frac{1}{2} \epsilon_{ijklm} A_{1i} A_{2j} A_{3k} (\gamma_l \gamma_m A_4 A_5 \ldots A_n) \,, \end{aligned} \tag{40}$$

where ϵ_{ijklm} is a Levi Civita tensor.

It can be easily shown that the last term of Eq. (40) is

$$-\frac{1}{4} \sum_{s>r} (-1)^{r+s} (A_1 A_2 A_3 A_r A_s)(A_4 A_5 \ldots A_n)'' .$$

This is the $\Sigma_{i \neq j \neq k}$ part of $\Sigma_{i,j,k}$ and can be rewritten in the form (as in [1])

$$\begin{aligned} \sum_{i \neq j \neq k} &= \sum_{i \neq j \neq k} A_{1i} A_{2j} A_{3k} A_{4l} (\gamma_i \gamma_j \gamma_k \gamma_l A_5 A_6 \ldots A_n) = \\ &= \sum_{i \neq j \neq k \neq l} + \sum_{\substack{i \neq j \neq k \\ l=i}} + \sum_{\substack{i \neq j \neq k \\ l=j}} + \sum_{\substack{i \neq j \neq k \\ l=k}} \end{aligned} \tag{41}$$

Now

$$\sum_{i \neq j \neq k \neq l} = \epsilon_{ijklm} A_{1i} A_{2j} A_{3k} A_{4l} (\gamma_m A_5 A_6 \ldots A_n) . \tag{42}$$

With the help of Eqs. (41) and (42) we obtain the second formula for S:

$$\begin{aligned} S = ([&A_1 \cdot A_2 A_3 A_4 - A_1 \cdot A_3 A_2 A_4 + A_2 \cdot A_3 A_1 A_4 + \\ &+ A_1 \cdot A_4 A_2 A_3 - A_2 \cdot A_4 A_1 A_3 + A_3 \cdot A_4 A_1 A_2] \\ &A_5 A_6 \ldots A_n) - \frac{1}{4} (A_1 A_2 A_3 A_4) \cdot (A_5 A_6 \ldots A_n) - \\ &- \frac{1}{4} \sum_{r \geq 5}^{n} (-1)^r (A_1 A_2 A_3 A_4 A_r)(A_5 A_6 \ldots A_{r-1} A_{r+1} \ldots A_n) . \end{aligned} \tag{43}$$

Proceeding in a way similar to that used in deriving Eq. (41), we obtain

$$\sum_{i \neq j \neq k \neq l} = \sum_{i \neq j \neq k \neq l \neq m} + \sum_{\substack{i \neq j \neq k \neq l \\ m=i}} + \sum_{\substack{i \neq j \neq k \neq l \\ m=j}} + \sum_{\substack{i \neq j \neq k \neq l \\ m=k}} + \sum_{\substack{i \neq j \neq k \neq l \\ m=l}} \tag{44}$$

Applying relations (44) and (10) we have the third formula for S:

$$
\begin{aligned}
S = A_{1i} A_{2j} A_{3k} A_{4l} A_{5m} (\gamma_i \gamma_j \gamma_k \gamma_l \gamma_m A_6 A_7 \ldots A_n) = \\
= ([A_1 \cdot A_2 A_3 A_4 - A_1 \cdot A_3 A_2 A_4 + A_2 \cdot A_3 A_1 A_4 + A_1 \cdot A_4 A_2 A_3 - \\
- A_2 \cdot A_4 A_1 A_3 + A_3 \cdot A_4 A_1 A_2] A_5 A_6 \ldots) - \\
- \frac{1}{4} (A_1 A_2 A_3 A_4)(A_5 A_6 \ldots) + ([-A_1 \cdot A_5 \{A_2 A_3 A_4\} + \\
+ A_2 \cdot A_5 \{A_1 A_3 A_4\} - A_3 \cdot A_5 \{A_1 A_2 A_4\} + A_4 \cdot A_5 \{A_1 A_2 A_3\}] \times \\
\times A_6 A_7 \ldots) + \frac{1}{4} (A_1 A_2 A_3 A_4 A_5)(A_6 A_7 \ldots).
\end{aligned}
\tag{45}
$$

In this equation terms of the type $\{A_2 A_3 A_4\}$ stand for

$$\{A_2 A_3 A_4\} = A_2 A_3 A_4 - A_2 \cdot A_3 A_4 + A_2 \cdot A_4 A_3 - A_3 \cdot A_4 A_2 . \tag{46}$$

Using relation (10) we arrive at the following result for the product of two traces:

$$
\begin{aligned}
(\gamma_i \gamma_j \gamma_k A_1 A_2 \ldots A_n)(\gamma_i \gamma_j \gamma_k A_1' A_2' \ldots A_m') = \\
= 3 (\gamma_i \gamma_j A_1 A_2 \ldots A_n)(\gamma_i \gamma_j A_1' A_2' \ldots A_m') + \\
+ 13 (\gamma_i A_1 A_2 \ldots A_n)(\gamma_i A_1' A_2' \ldots A_m') - 15 (A_1 A_2 \ldots A_n) \cdot \\
\cdot (A_1' A_2' \ldots A_m') .
\end{aligned}
\tag{47}
$$

By repeatedly applying Eq. (47) we obtain

$$
\begin{aligned}
(\gamma_i \gamma_j \gamma_k \gamma_l A_1 A_2 \ldots A_n)(\gamma_i \gamma_j \gamma_k \gamma_l A_1' A_2' \ldots A_m') = \\
= 22 (\gamma_i \gamma_j A_1 A_2 \ldots A_n)(\gamma_i \gamma_j A_1' A_2' \ldots A_m') + \\
+ 24 (\gamma_i A_1 A_2 \ldots A_n)(\gamma_i A_1' A_2' \ldots A_m') - 45 (A_1 A_2 \ldots A_n) \cdot \\
(A_1' A_2' \ldots A_m') ,
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
(\gamma_i \gamma_j \gamma_k \gamma_l \gamma_m A_1 A_2 \ldots A_n)(\gamma_i \gamma_j \gamma_k \gamma_l \gamma_m A_1' A_2' \ldots A_m') = \\
= 90 (\gamma_i \gamma_j A_1 A_2 \ldots A_n)(\gamma_i \gamma_j A_1' A_2' \ldots A_m') + \\
+ 241 (\gamma_i A_1 A_2 \ldots A_n) \cdot (\gamma_i A_1' A_2' \ldots A_m') - \\
- 330 (A_1 A_2 \ldots A_n)(A_1' A_2' \ldots A_m') .
\end{aligned}
\tag{49}
$$

In general, proceeding in this way we can evaluate

$$(\gamma_{i1} \gamma_{i2} \ldots \gamma_{in} A_1 A_2 \ldots)(\gamma_{i1} \gamma_{i2} \ldots \gamma_{in} A_1' A_2' \ldots) .$$

For odd values of n we have a relation similar to Eq. (12):

$$(\gamma_i A_1 A_2 \dots A_n)(\gamma_i A_1' A_2' \dots A_m') = \\ = \sum_{i=1}^{n} (-1)^{i+1}(A_1 A_2 \dots A_{i-1} A_{i+1} \dots A_n)(A_i A_1' A_2' \dots A_m'). \tag{50}$$

Corresponding to Eqs. (20) and (25) we have the relations

$$(\gamma_i \gamma_j A_1 A_2 \dots A_n)(\gamma_i \gamma_j A_1' A_2' \dots A_m') = \\ = 5(A_1 A_2 \dots A_n)(A_1' A_2' \dots A_m') - 2 \sum_{i>j} (-1)^{i+j}(A_1 A_2 \dots A_n)'' \times \\ \times [(A_i A_j A_1' A_2' \dots A_m') - A_i \cdot A_j (A_1' A_2' \dots A_m')] \tag{51}$$

and

$$(\gamma_i \gamma_j \gamma_k A_1 A_2 \dots A_n)(\gamma_i \gamma_j \gamma_k A_1' A_2' \dots A_m') = \\ = 13(\gamma_i A_1 A_2 \dots A_n)(\gamma_i A_1' A_2' \dots A_m') + \\ + \sum_{i>j>k} (-1)^{i+j+k}(A_1 A_2 \dots A_n)''' \sum_P \delta_P P(A_i A_j A_k A_1' A_2' \dots A_m'). \tag{52}$$

From Eqs. (52) and (47) we have for odd values of n and m

$$(\gamma_i \gamma_j A_1 A_2 \dots A_n)(\gamma_i \gamma_j A_1' A_2' \dots A_m') = \\ = \frac{1}{3} \sum_{i>j>k} (-1)^{i+j+k}(A_1 A_2 \dots A_n)''' \sum_P \delta_P P(A_i A_j A_k A_1' A_2' \dots A_m') \\ + 5(A_1 A_2 \dots A_n)(A_1' A_2' \dots A_m'). \tag{53}$$

Relation (51) can be applied when either n or m is even. In a similar manner we obtain

$$(\gamma_i \gamma_j \gamma_k \gamma_l A_1 A_2 \dots A_n)(\gamma_i \gamma_j \gamma_k \gamma_l A_1' A_2' \dots A_m') = \\ = -44 \sum_{i>j} (-1)^{i+j}(A_1 A_2 \dots A_n)''([A_i A_j - A_i \cdot A_j] A_1' A_2' \dots A_m') + \\ + \sum_{i>j>k>l} (-1)^{i+j+k+l}(A_1 A_2 \dots A_n)'''' \times \\ \times \sum_P \delta_P P(A_i A_j A_k A_l A_1' A_2' \dots A_m') + \\ + 65(A_1 A_2 \dots A_n)(A_1' A_2' \dots A_m') \tag{54}$$

and

$$
\begin{aligned}
&(\gamma_i \gamma_j \gamma_k \gamma_l \gamma_m A_1 A_2 \ldots A_n)(\gamma_i \gamma_j \gamma_k \gamma_l \gamma_m A_1' A_2' \ldots A_m') = \\
&\quad = 24 (\gamma_i \gamma_j A_1 A_2 \ldots A_n)(\gamma_i \gamma_j A_1' A_2' \ldots A_m') + \\
&\quad + 241 (\gamma_i A_1 A_2 \ldots A_n)(\gamma_i A_1' A_2' \ldots A_m') - 120 (A_1 A_2 \ldots A_n) \times \\
&\quad \times (A_1' A_2' \ldots A_m') + 22 \sum_{i>j>k} (-1)^{i+j+k} (A_1 A_2 \ldots A_n)''' \times \\
&\quad \times \sum_P \delta_P P (A_i A_j A_k A_1' A_2' \ldots A_m') - \\
&\quad - \sum_{i>j>k>l>m} (-1)^{i+j+k+l+m} (A_1 A_2 \ldots A_n)''''' \times \\
&\quad \times \sum_P \delta_P P (A_i A_j A_k A_l A_m A_1' A_2' \ldots A_m') .
\end{aligned} \tag{55}
$$

Comparing Eqs. (48) and (54), we have for even values of n and m the result

$$
\begin{aligned}
&(\gamma_i A_1 A_2 \ldots A_n)(\gamma_i A_1' A_2' \ldots A_m') = \\
&\quad = \frac{1}{24} \sum_{i>j>k>l} (-1)^{i+j+k+l} (A_1 A_2 \ldots A_n)'''' \times \\
&\quad \times \sum_P \delta_P P (A_i A_j A_k A_l A_1' A_2' \ldots A_m').
\end{aligned} \tag{56}
$$

For this case where n and m are both even relation (50) is not applicable In Eqs. (52), (54) and (55) permutation P of the suffixes i, j, k, l and m is implied.

Similarly from Eqs. (49) and (55) we obtain for odd values of n and m

$$
\begin{aligned}
&(A_1 A_2 \ldots A_n)(A_1' A_2' \ldots A_m') = \\
&\quad = - \frac{1}{120} \sum_{i>j>k>l>m} (-1)^{i+j+k+l+m} (A_1 A_2 \ldots A_n)''''' \times \\
&\quad \times \sum_P \delta_P P (A_i A_j A_k A_l A_m A_1' A_2' \ldots A_m') .
\end{aligned} \tag{57}
$$

Putting $n = m$ and $A_i' = A_i$ we can obtain from Eq. (57) a relation for $(A_1, A_2 \ldots A_n)$ [2]:

Taking $A_1, A_2, A_3, A_4, A_5 = \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ and $n = 5$, we obtain from Eq. (57)

$$
\begin{aligned}
S &= (A_1 A_2 \ldots A_n) = \\
&= - \frac{1}{4} \sum_{i>j>k>l>m} (-1)^{i+j+k+l+m} (A_i A_j A_k A_l A_m)(A_1 A_2 \ldots A_n)''''' .
\end{aligned} \tag{58}
$$

In a manner given in Section I, taking $n = 3$ and m even, we obtain from Eqs. (52) and (47) the first relation (40) for S. Similarly, taking $n = 4$, m odd in Eqs. (54), (48) and $n = 5$, m arbitrary in Eqs. (55) and (49) we derive the second and third relations (43) and (45), respectively, for S.

Eqs. (40) and (43) both give the same number of terms in the expansion of S for $n = 7$ and 9. On the other hand, (45) gives 11 terms for $n = 7$. But these can be reduced to 9 terms by using the identity (24), which is also valid when A stands for $\sum_{i=1}^{5} A_i \gamma_i$. For all values of $n > 13$ Eq. (45) gives the smallest number of terms.

III

From the results of Section II we shall now go on to derive similar results to those obtained in [1], in which A stands for $\sum_{i=1}^{4} A_i \gamma_i$ and the dummy suffixes in terms like

$$(\gamma_{i1} \gamma_{i2} \dots \gamma_{in} A_1 A_2 \dots)(\gamma_{i1} \gamma_{i2} \dots \gamma_{in} A_1' A_2' \dots)$$

are restricted to the values 1, 2, 3 and 4. To do this we split the summation over the dummy suffixes occurring in Section II in the following manner:

$$\sum_{i=1}^{5} = \sum_{i=1}^{4} + \text{terms corresponding to } i = 5 \text{ only.}$$

By following this prescription we obtain from Eq. (40), taking $A_1 = \gamma_5$

$$S' = (\gamma_5 A_2 A_3 \dots A_n) = A_2 \cdot A_3 (\gamma_5 A_4 A_5 \dots A_n) - \\ - \frac{1}{4} \sum_{s>r} (-1)^{r+s} (\gamma_5 A_2 A_3 A_r A_s)(A_4 A_5 \dots A_n)'' . \tag{59}$$

Assuming that in Eq. (43) A_4 only involves γ_5 we can have the relation

$$S' = (\gamma_5 [A_1 \cdot A_2 A_3 - A_1 \cdot A_3 A_2 + A_2 \cdot A_3 A_1] A_4 A_5 \dots) \\ + \frac{1}{4} \sum_{i \geq 4} (-1)^i (\gamma_5 A_1 A_2 A_3 A_i)(A_4 A_5 \dots A_{i-1} A_{i+1} \dots) . \tag{60}$$

Similarly, assuming that in Eqs. (43) and (45) A_5 only involves γ_5, we obtain third formula for S':

$$S' = (\gamma_5 [A_1 \cdot A_2 A_3 A_4 - A_1 \cdot A_3 A_2 A_4 + A_1 \cdot A_4 A_2 A_3 + \\ + A_2 \cdot A_3 A_1 A_4 - A_2 \cdot A_4 A_1 A_3 + A_3 \cdot A_4 A_1 A_2] \times \\ \times A_5 A_6 \dots) - \frac{1}{4} (A_1 A_2 A_3 A_4)(\gamma_5 A_5 A_6 \dots) + \\ + \frac{1}{4} (\gamma_5 A_1 A_2 A_3 A_4)(A_5 A_6 \dots) . \tag{61}$$

Taking n and m to be even and odd successively, we get from Eq. (47)

$$(\gamma_5 \gamma_i \gamma_j A_1 A_2 \ldots A_n)(\gamma_5 \gamma_i \gamma_j A_1' A_2' \ldots A_m') = \\ = 4(\gamma_5 A_1 A_2 \ldots)(\gamma_5 A_1' A_2' \ldots) - 4(A_1 A_2 \ldots)(A_1' A_2' \ldots) + \\ + (\gamma_i \gamma_j A_1 A_2 \ldots)(\gamma_i \gamma_j A_1' A_2' \ldots), \tag{62}$$

and

$$(\gamma_i \gamma_j \gamma_k A_1 A_2 \ldots A_n)(\gamma_i \gamma_j \gamma_k A_1' A_2' \ldots A_m') = \\ = 6(\gamma_5 \gamma_i A_1 A_2 \ldots)(\gamma_5 \gamma_i A_1' A_2' \ldots) + \\ + 10(\gamma_i A_1 A_2 \ldots)(\gamma_i A_1' A_2' \ldots). \tag{63}$$

Similarly from Eq. (48) we obtain

$$(\gamma_i \gamma_j \gamma_k \gamma_l A_1 A_2 \ldots A_n)(\gamma_i \gamma_j \gamma_k \gamma_l A_1' A_2' \ldots A_m') = \\ = 10(\gamma_i \gamma_j A_1 A_2 \ldots)(\gamma_i \gamma_j A_1' A_2' \ldots) + \\ + 6(\gamma_5 \gamma_i \gamma_j A_1 A_2 \ldots)(\gamma_5 \gamma_i \gamma_j A_1' A_2' \ldots) \tag{64}$$

and

$$(\gamma_5 \gamma_i \gamma_j \gamma_k A_1 A_2 \ldots A_n)(\gamma_5 \gamma_i \gamma_j \gamma_k A_1' A_2' \ldots A_m') = \\ = 10(\gamma_5 \gamma_i A_1 A_2 \ldots)(\gamma_5 \gamma_i A_2' A_1' \ldots) + \\ + 6(\gamma_i A_1 A_2 \ldots)(\gamma_i A_1' A_2' \ldots). \tag{65}$$

In an identical manner Eq. (49) yields

$$(\gamma_5 \gamma_i \gamma_j \gamma_k \gamma_l A_1 A_2 \ldots A_n)(\gamma_5 \gamma_i \gamma_j \gamma_k \gamma_l A_1' A_2' \ldots A_m') = \\ = 40(\gamma_5 A_1 A_2 \ldots)(\gamma_5 A_1' A_2' \ldots) - 40(A_1 A_2 \ldots)(A_1' A_2' \ldots) + \\ + 16(\gamma_i \gamma_j A_1 A_2 \ldots)(\gamma_i \gamma_j A_1' A_2' \ldots) \tag{66}$$

and,

$$(\gamma_i \gamma_j \gamma_k \gamma_l \gamma_m A_1 A_2 \ldots)(\gamma_i \gamma_j \gamma_k \gamma_l \gamma_m A_1' A_2' \ldots) = \\ = 120(\gamma_5 \gamma_i A_1 A_2 \ldots)(\gamma_5 \gamma_i A_1' A_2' \ldots) + \\ + 136(\gamma_i A_1 A_2 \ldots)(\gamma_i A_1' A_2' \ldots). \tag{67}$$

In general

$$(\gamma_{i1} \gamma_{i2} \ldots \gamma_{in} A_1 A_2 \ldots)(\gamma_{i1} \gamma_{i2} \ldots \gamma_{in} A_1' A_2' \ldots)$$

and

$$(\gamma_5 \gamma_{i1} \gamma_{i2} \ldots \gamma_{in-1} A_1 A_2 \ldots)(\gamma_5 \gamma_{i1} \gamma_{i2} \ldots \gamma_{in-1} A_1' A_2' \ldots)$$

can be easily evaluated, in a similar way, from the expression for

$$(\gamma_{i1} \gamma_{i2} \ldots \gamma_{in} A_1 A_2 \ldots)(\gamma_{i1} \gamma_{i2} \ldots \gamma_{in} A_1' A_2' \ldots)$$

in which case A's and summation over dummy suffixes are defined in the manner of Section II.

These general results an not treated in [1].

Relation (50) of Section II remains unaltered in this Section. Eq. (51) yields

$$(\gamma_i \gamma_j A_1 A_2 \ldots)(\gamma_i \gamma_j A_1' A_2' \ldots) = 4 (A_1 A_2 \ldots)(A_1' A_2' \ldots) - 2 \sum_{i>j} (-1)^{i+j} (A_1 A_2 \ldots)'' [(A_i A_j A_1' A_2' \ldots) - A_i \cdot A_j (A_1' A_2' \ldots)]. \quad (68)$$

From Eq. (53) the following relation is obtained:

$$(\gamma_5 \gamma_i A_1 A_2 \ldots)(\gamma_5 \gamma_i A_1' A_2' \ldots) = = \frac{1}{6} \sum_{r>s>t} (-1)^{r+s+t} (A_1 A_2 \ldots)''' \sum_P \delta_P P(A_r A_s A_t A_1' A_2' \ldots). \quad (69)$$

Eq. (56) leads to

$$(\gamma_5 A_1 A_2 \ldots A_n)(\gamma_5 A_1' A_2' \ldots) = \sum_{i>j>k>l} (-1)^{i+j+k+l} (A_1 A_2 \ldots)'''' \times \times \sum_{r>s>t>u} (-1)^{r+s+t+u} (A_1' A_2' \ldots)'''' \times \sum_P \delta_P P[A_i \cdot A_r' A_j \cdot A_s' A_k \cdot A_t' A_l \cdot A_u']. \quad (70)$$

In Eqs. (69) and (70), permutation P of the suffixes r, s, t and u is implied.

Putting $n = 4$ and $A_1, A_2, A_3, A_4 = \gamma_1, \gamma_2, \gamma_3, \gamma_4$ in Eq. (70), we obtain

$$S' = (\gamma_5 A_1 A_2 \ldots) = \frac{1}{4} \sum_{i>j>k>l} (-1)^{i+j+k+l} (\gamma_5 A_i A_j A_k A_l)(A_1 A_2 \ldots)''''. \quad (71)$$

Taking $A_i' = A_i$ and $n = m$, we can obtain from Eq. (70) an expression for the square of S'.

In identity (24), where in general $A = \sum_{i=1}^{5} A_i \gamma_i$ for Dirac matrices, if we assume that A_n only involves γ_5, then Eq. (24) reduces to the following identity:

$$\sum_{i=2}^{n} (-1)^i A_1 \cdot A_i (A_2 A_3 \ldots A_{i-1} A_{i+1} \ldots A_n \gamma_5) = 0. \quad (72)$$

In this Section all the relations except that given by Eqs. (59) and (67) rea derived in [1] in a different manner. It is found that among the various formulae for determining S', that given by Eq. (61) is the most convenient for $n > 8$.

Acknowledgment

The author wishes to thank Dr. N. C. Sil for stimulating discussions.

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О СЛЕДЕ ПРОИЗВЕДЕНИЯ МАТРИЦ ПАУЛИ И ДИРАКА

С. САРКАР

Резюме

Предложено несколько методов для определения следа произведения любого числа матриц Паули. Даны формулы для вычисления произведений двух следов различного типа в случае, когда в обоих имеются члены типа $\sigma_{i_1} \sigma_{i_2} \ldots\ldots \sigma_{i_n}$. Выведены выражения для произведений двух следов и квадрата следа в случае произвольного числа матриц Паули. Подобные формулы получены и для матриц Дирака, построенны как $\sum_{i=1}^{5} \gamma_i A_i$ вместо матриц Паули. Из них также можно получить все наши предыдущие результаты, когда же γ_5, были рассмотрены в отдельности.

Выведено очень полезное тождество для следов, содержащих либо матрицы Паули, либо матрицы Дирака.