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THE LORENTZ PRINCIPLE AND THE GENERAL THEORY OF RELATIVITY

PART II

INHOMOGENEOUS PROPAGATION OF LIGHT

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The generalization of the Lorentz transformation to regions where light is propagated inhomogeneously is given and it is shown that the Lorentz principle can be maintained in its ordinary form provided the Lorentz transformation is taken in its more general form formulated for such regions. The well-known equations for the geodetic lines in a gravitational field are obtained from the Lorentz principle thus generalized.

Generalized definition of the Lorentz transformation

§ 1. In this section we shall formulate the Lorentz principle for regions of space where light is propagated inhomogeneously. We shall assume, however, that even if the propagation at large is inhomogeneous still in sufficiently small regions the propagation remains homogeneous. Thus we suppose that a light signal starting from a point P with coordinates $\mathbf{r}$ at the time t arrives in a point Q with coordinate vector $\mathbf{r} + \boldsymbol{\rho}$ at the time $t + \tau$ so that

$$\boldsymbol{\xi}\mathbf{g}(\mathbf{x})\boldsymbol{\xi} = 0\,, \tag{1}$$

where

$$\mathbf{x} = \mathbf{r}, t \qquad \text{and} \qquad \boldsymbol{\xi} = \boldsymbol{\rho}, \tau$$

provided the components of $\boldsymbol{\xi}$ are sufficiently small so that the change of $\mathbf{g}(\mathbf{x})$ while $\mathbf{x}$ changes by $\boldsymbol{\xi}$ should be negligible.

As a first step we generalize the Lorentz transformation to the case of inhomogeneous propagation of light.

Let us consider to start with an arbitrary transformation of coordinates. Suppose the coordinates $\mathbf{x}$ and $\boldsymbol{\xi}$ refer to a system K. We introduce a system K' in which the four coordinate vectors are given by

$$\mathbf{x}' + \boldsymbol{\xi}' = \mathbf{f}(\mathbf{x} + \boldsymbol{\xi})\,, \tag{2}$$

where $\mathbf{f}$ has four components f_ν, $\nu = 1, 2, 3, 4$ and all four components are supposed to be slowly varying functions of their argument. More precisely we shall consider only such values of $\boldsymbol{\xi}$ for which we can write in a good approximation

$$\mathbf{f}(\mathbf{x}+\boldsymbol{\xi}) = \mathbf{f}(\mathbf{x}) + \boldsymbol{\xi}\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}.$$

Writing more explicitly

$$\frac{\partial f_\nu(\mathbf{x})}{\partial x_\mu} = S_{\nu\mu} \qquad \nu,\mu = 1,2,3,4\,.$$

The transformation (2) can also be written as

$$\boldsymbol{\xi}' = \mathbf{S}\boldsymbol{\xi} \qquad \text{and} \qquad \mathbf{x}' = \mathbf{x} + \boldsymbol{\mu}\,, \tag{3}$$

where

$$\boldsymbol{\mu} = \mathbf{f}(\mathbf{x}) - \mathbf{x}\,. \tag{3a}$$

The transformation (2) should possess a unique inverse and therefore we suppose

$$\det \mathbf{S} \neq 0. \tag{4}$$

§ 2. The propagation of light in the vicinity of the point P can thus be expressed relative to K' expressing (1) in terms of the transformed variables. Neglecting small terms we find thus

$$\boldsymbol{\xi}'\,\mathbf{g}'(\mathbf{x}')\,\boldsymbol{\xi}' = 0\,, \tag{5}$$

where

$$\mathbf{g}'(\mathbf{x}') = \widetilde{\mathbf{S}}^{-1}\,\mathbf{g}(\mathbf{x})\,\mathbf{S}^{-1}. \tag{6}$$

There exist coordinate transformations which leave the components of $\mathbf{g}(\mathbf{x})$ unchanged. We consider these transformations as the generalized Lorentz transformations. Thus a generalized Lorentz transformation $\mathscr{M}(\mathbf{x},\boldsymbol{\mu})$ is expressed with the help of a shift $\boldsymbol{\mu}$ and a matrix $\mathbf{M}$ such that

$$\mathbf{x}' = \mathbf{x} + \boldsymbol{\mu} \tag{7a}$$

and

$$\widetilde{\mathbf{M}}\mathbf{g}(\mathbf{x}+\boldsymbol{\mu})\,\mathbf{M} = \mathbf{g}(\mathbf{x})\,. \tag{7b}$$

It must be emphasized that the transformation $\mathscr{M}$ does not change the components of $\mathbf{g}(\mathbf{x})$ in the fixed point $\mathbf{x}$ but it may change the values $\mathbf{g}(\mathbf{x}+\boldsymbol{\xi})$ in the vicinity of $\mathbf{x}$ and therefore it may change the derivatives of $\mathbf{g}(\mathbf{x})$ in $\mathbf{x}$ (see Parts IV and V)

The transformations defined by (7a) and (7b) form a structure with the following properties. Consider two transformations $\mathscr{M}(\mathbf{x}+\boldsymbol{\mu}_1)$ and $\mathscr{N}(\mathbf{x}+\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)$, we have thus

$$\left.\begin{aligned} &\widetilde{\mathbf{M}}\mathbf{g}(\mathbf{x}+\boldsymbol{\mu}_1)\,\mathbf{M} = \mathbf{g}\,, && \text{(a)}\\ &\widetilde{\mathbf{N}}\mathbf{g}(\mathbf{x}+\boldsymbol{\mu}_1+\boldsymbol{\mu}_2)\,\mathbf{N} = \mathbf{g}(\mathbf{x}+\boldsymbol{\mu}_1)\,. && \text{(b)}\end{aligned}\right\} \tag{8}$$

Multiplying (8b) from the left by $\tilde{\mathbf{M}}$ and from the right by $\mathbf{M}$ we find

$$(\tilde{\mathbf{N}}\tilde{\mathbf{M}})\,\mathbf{g}(\mathbf{x}+\mu_1+\mu_2)\,\mathbf{N}\mathbf{M}=\mathbf{g}(\mathbf{x})\;.$$

Thus $\mathbf{NM}$ and $\mu_1+\mu_2$ define a transformation $\mathcal{M}(\mathbf{x},\mu_1+\mu_2)$ and $\mathbf{NM}$ is thus itself a Lorentz matrix.

Thus Lorentz transformations consecutively applied give again Lorentz ransformations, however, a given Lorentz transformation refers to the fixed point, say $\mathbf{x}$ and produces a shift to another point, say $\mathbf{x}'$. Therefore applying a Lorentz transformation which produces a shift $\mathbf{x}\to\mathbf{x}'$ we can apply on the transformed quantities only such further transformations which produce shifts from $\mathbf{x}'\to\mathbf{x}''$. Therefore, if $\mathbf{x}\neq\mathbf{x}'$, then the two transformations are of different categories. These transformations fulfill the postulates of a partial algebraic structure and may be denoted a semi-group.* In the case of a homogeneous propagation of light the dependence of the transformation on the coordinates of the points upon which it is to be applied disappears and so the semi-group degenerates into an ordinary group — in this way the semi-groups of Lorentz transformations defined for the inhomogeneous case degenerate into the Lorentz group if the inhomogeneity disappears.

§ 3. The generalized Lorentz transformation can also be interpreted (like the more special transformation) to give not a coordinate transformation but to describe a deformation of some physical system $\mathfrak{Q}$. Suppose thus $\mathfrak{Q}$ to be a physical system in the vicinity of $\mathbf{x}=\mathbf{r},t$; various points $\mathfrak{P}_1,\mathfrak{P}_2,\ldots$ of $\mathfrak{Q}$ can be described by four vectors

$$\mathbf{x}+\xi_\varkappa\quad \varkappa=1,2,\ldots\,.$$

Thus the point $\mathfrak{P}_\varkappa$ as represented in K moves along an orbit which at a time $t+\tau$ has a distance $\rho(\tau)$ from $\mathbf{r}$.

The deformed system $\mathfrak{Q}^*$ consists of points $\mathfrak{P}_1^*,\mathfrak{P}_2^*,\ldots$ with coordinate vectors

$$\mathbf{x}^*+\xi_\varkappa^*\quad \varkappa=1,2,\ldots\,,$$

* The expression semi-group is used somewhat loosely. In the usual sense the structure we use is that known as a BRANDT gruppoid with unit element, i.e. a special type of partial algebraic structure. If an algebraic structure is partial, then the product ab does not exist for an arbitrary pair ab of its elements. In our case of the semi-group, if a, b, c are any three elements of it and $ab=c$ holds, then any of the elements a, b, c is uniquely determined by the other two. If ab and bc exist, the product abc may be written without parenthesis, thus the associativity law holds. Although in the case of BRANDT gruppoid every element has uniquely determined right and left unit elements, and conversely for two unit elements e_1, e_2 there is an element whose right and left unit elements are e_1 and e_2, in our case every element in the gruppoid has the same left and right unit elements. The existence of the inverse element is needed too. Gruppoid was introduced by BRANDT. (H. BRANDT: "Über die Axiome des Gruppoids". Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, LXXXV [1940], 95—104.)

I am greatly indebted to Mr. J. DÉNES for having put at my disposal the above information.

where

$$\mathbf{x}^* = \mathbf{x} + \mu \, , \tag{9a}$$

and

$$\xi_\varkappa^* = \mathbf{M}\xi_\varkappa \quad \varkappa = 1, 2, 3, \ldots \tag{9b}$$

and $\mathbf{M}$ and μ are the parameters of a transformation $\mathbf{M}(\mathbf{x}, \mu)$ which gives the deformation $\mathfrak{Q} \rightarrow \mathfrak{Q}^*$ in terms of representation in K.

The Lorentz principle can now be formulated for regions with inhomogeneous propagation of light as follows. *The laws of nature possess such forms that provided $\mathfrak{Q}$ is a real system obeying certain laws, then any Lorentz deformed system $\mathfrak{Q}^*$ obeys the same laws.*

Furthermore we may add: *if a system is accelerated adiabatically then it changes its configuration as a result of the acceleration into a Lorentz deformed configuration $\mathfrak{Q}^* = \mathbf{M}(\mathfrak{Q})$.*

The above formulation of the Lorentz principle is identic in form to former formulation, however, its contents are enlarged as it is supposed to be valid to the generalized family of Lorentz transformations $\mathbf{M}(\mathbf{x}, \mu)$. We show in the following that the latter form of the principle leads to results which are obtained usually from the general theory of relativity.

§ 4. It may appear as a deficiency of the Lorentz transformation as defined above, that it can be applied to small systems only, i.e. to systems which occupy such parts of space in which effects of the inhomogeneity of propagation of light can be neglected. However, this apparent deficiency is not a real one, it simply reflects upon material properties of physical systems.

Indeed, considering a system which is so large that the propagation of light inside the space occupied by the system is inhomogeneous to a noticeable extent, then gravitational stresses will appear in the system and its state of equilibrium will be determined partly by the gravitational field, but also by the material properties (compressibility, rigidity, etc.) of the system. If we shift such a system to different parts of space, then it will readjust itself to the field of the new surroundings and the change of configuration which thus arises depends very much on the actual physical properties of the system. If the Lorentz transformation depends only upon the distribution of the gravitational field, then it cannot possibly describe the material changes of a large system the changes of which depend — apart from the gravitational field — also upon the material properties of the system. We see thus, that it would be unreasonable to expect the existence of a general transformation which describes the changes of large physical systems when moved about in gravitational fields — since changes thus arising depend very much on the actual material properties of the system.

The fact that Lorentz transformations are suitable to express the changes small physical systems suffer when transported adiabatically into regions in which the gravitational field differs, shows that the reaction of micro-structures upon gravitational field obey general laws.

We have here an analogy of the circumstance that the Lorentz deformations described by the special theory of relativity are independent of the material properties of the systems provided the interferences causing the deformations are adiabatic.

§ 5. The Lorentz transformations $\mathbf{M}(\mathbf{x}, \mu)$ can be divided into two kinds: 1) transformations with $\mu = 0$, the latter may be denoted *local transformations*, as they produce no immediate shift of the system $\mathfrak{Q}$. 2) We may consider transformations $\mathbf{M}_0(\mathbf{x}, \mu)$ which produce a parallel shift, i.e. a shift, with as little changes apart from the parallel displacement, as it is possible.

Concerning the local transformations we find from (9a, b) that they contain matrices $\mathbf{M}$ obeying

$$\tilde{\mathbf{M}}\mathbf{g}(\mathbf{x})\,\mathbf{M} = \mathbf{g}(\mathbf{x})\,. \tag{10}$$

Thus the matrices of the local transformations are exactly those which are obtained for the case of homogeneous propagation of light, these matrices were considered in Part I — we explained there that the Lorentz principle can be supposed to be valid for such transformations.

The local transformations in distant points have, however, different forms. Consider a number of locations $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$. Let us denote

$$\mathbf{g}(\mathbf{x}_k) = \mathbf{g}_k\,. \tag{10a}$$

Further we write $\mathbf{M}_k$ for the transformation matrix relating to transformations in $\mathbf{x}_k$. Thus we suppose

$$\tilde{\mathbf{M}}_k\,\mathbf{g}_k\,\mathbf{M}_k = \mathbf{g}_k\,, \qquad k = 1, 2, 3. \tag{11}$$

The matrices $\mathbf{M}_1, \mathbf{M}_2, \ldots$ define local transformations near $\mathbf{x}_1, \mathbf{x}_2, \ldots$ A connection between the matrices $\mathbf{M}_k$ for different k can be found. Denote

$$\mathbf{g}_l^{-1/2}\,\mathbf{g}_k^{1/2} = \mathbf{S}_{lk}\,, \tag{12}$$

or *alternatively*, if $\mathbf{S}_{lk}$ thus defined possessed complex elements, then (12) can be replaced by matrices defined in equations (9a), or by those defined by (9b); the latter have real elements only and behave algebraically similar to the matrices (12).

We may put

$$\mathbf{M}_k = \mathbf{S}_{lk}^{-1}\,\mathbf{M}_l\,\mathbf{S}_{lk}\,. \tag{13}$$

Introducing (13) into (11) we find with the help of (12)

$$\tilde{\mathbf{M}}_l\,\mathbf{g}_l\,\mathbf{M}_l = \mathbf{g}_l\,.$$

Thus relation (13) can be taken as the transformation formula between matrices of local transformation in different locations.

§ 6. The matrices $\mathbf{S}_{lk}$ can be taken to define parallel shifts. The $\mathbf{S}_{lk}$ are matrices corresponding to Lorentz transformations producing shifts from $\mathbf{x}_k \rightarrow \mathbf{x}_l$. Indeed with the help of (12) we obtain

$$\widetilde{\mathbf{S}}_{lk}\,\mathbf{g}_l\,\mathbf{S}_{lk} = \mathbf{g}_k\,. \tag{14}$$

Remembering the definition (10a) and comparing (14) with (7b) we see that $\mathbf{S}_{lk}$ is indeed a matrix producing the shift $\mathbf{x}_k \rightarrow \mathbf{x}_l$.

We note that according to (12) the matrices $\mathbf{S}_{lk}$ obey the relation

$$\mathbf{S}_{lk}\,\mathbf{S}_{km} = \mathbf{S}_{lm} \tag{14a}$$

and also

$$\mathbf{S}_{lm}^{-1} = \mathbf{S}_{ml}\,. \tag{14b}$$

If we make three parallel shifts which compensate each other, i.e. $\mathbf{x}_k \rightarrow \mathbf{x}_l$, $\mathbf{x}_l \rightarrow \mathbf{x}_m$ and finally $\mathbf{x}_m \rightarrow \mathbf{x}_n$, then the corresponding matrix is found to be

$$\mathbf{S}_{kl}\,\mathbf{S}_{lm}\,\mathbf{S}_{mk} = \mathbf{1}\,. \tag{15}$$

If instead of (12) alternative definitions of S in accordance with (9a) or (9b) of Part I are taken, then relations (14a), (14b) and (15) remain valid. Thus if we carry out a number of adiabatic parallel shifts with a system $\mathfrak{Q}$ such that we return in the end to the original positions, then the configuration of the system $\mathfrak{Q}$ also returns to its original form.

The latter statement in this form has, however, no real physical content. Indeed, a shift $\mathbf{x}_k \rightarrow \mathbf{x}_l$ takes some time to carry out and therefore we have necessarily $x_{k4} < x_{l4}$. When carrying out a series of shifts we cannot arrive back to the first position $\mathbf{x}_k$ from which we started.

However, relation (14) expresses the real physical fact; it follows from (14) that shifting $\mathfrak{Q}$ first from $\mathbf{x}_k \rightarrow \mathbf{x}_l$ and then from $\mathbf{x}_l \rightarrow \mathbf{x}_m$ we obtain the same result as if we had carried out directly a parallel shift $\mathbf{x}_k \rightarrow \mathbf{x}_l$. I.e. the parallel shift here defined is a true parallel shift and *the result of such a shift does not depend on the path along which the shift is carried out as long as the end points are kept fixed.*

§ 7. The most general form of the Lorentz transformation is obtained by combining a local transformation and a parallel shift. We may put

$$\mathbf{M}_{lk} = \mathbf{S}_{lk}\,\mathbf{M}_k \tag{16a}$$

or inserting for $\mathbf{M}_k$ the expression (13), the identical relation

$$\mathbf{M}_{lk} = \mathbf{M}_l\,\mathbf{S}_{lk}\,. \tag{16b}$$

The transformations $\mathbf{M}\,(\mathbf{x}_k;\ \mathbf{x}_l - \mathbf{x}_k)$ possess matrices each of which can be written in the form (16a) respectively in the form (16b). We see thus that any transformation $\mathbf{M}\,(\mathbf{x}, \boldsymbol{\mu})$ can be taken as to consist of a local transformation $\mathbf{M}_k$ at $\mathbf{x}_k$ and a parallel shift μ — but it can also be represented by a parallel shift $\mathbf{M}_l$ in the final position $\mathbf{x}_l$. The connection between the local transformations $\mathbf{M}_k$ and $\mathbf{M}_l$ which lead to the same final result is given by relation (13).

Small displacements

§ 8. Let us consider that approximation of the Lorentz transformation which is valid in the case of small shifts. We consider as a small shift one which might be very much larger than the dimensions of the system subjected to the shift, but which is small on a cosmical scale, i.e. a shift μ such that we have in a good approximation

$$\mathbf{g}(\mathbf{x} + \mu) = \mathbf{g}(\mathbf{x}) + \mu \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}. \tag{17}$$

A small shift in general can be expressed by a matrix

$$\mathbf{S} = \mathbf{1} + \boldsymbol{\sigma} \boldsymbol{\mu}, \tag{18}$$

where we suppose

$$\boldsymbol{\sigma}\boldsymbol{\mu} = \sum_{\varkappa} \boldsymbol{\sigma}^{(\varkappa)} \mu_{\varkappa}. \tag{19}$$

Introducing (18) into (7b) and neglecting terms of higher order we find for the condition that S should be a Lorentz matrix

$$\tilde{\boldsymbol{\sigma}}^{(\varkappa)} \mathbf{g} + \mathbf{g} \boldsymbol{\sigma}^{(\varkappa)} = - \frac{\partial \mathbf{g}}{\partial \mathbf{x}_{\varkappa}}.$$

Thus we find

$$\boldsymbol{\sigma}^{(\varkappa)} = - \frac{1}{2} \mathbf{g}^{-1} \left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}_{\varkappa}} + \mathbf{A}^{(\varkappa)} \right), \tag{20}$$

where $\mathbf{A}^{(\varkappa)}$ (for any value of $\varkappa$) is an arbitrary antisymmetric matrix, i.e. a matrix obeying

$$\tilde{\mathbf{A}}^{(\varkappa)} = - \mathbf{A}^{(\varkappa)}. \tag{20a}$$

A small shift is thus produced by a transformation containing the matrix

$$\mathbf{S} = \mathbf{1} - \frac{1}{2} \mathbf{g}^{-1} \left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}} \mu + \mathbf{A} \right). \tag{21}$$

The parallel shift is obtained for $\mathbf{A} = 0$. For $\mathbf{A} \neq 0$ we obtain a parallel shift superimposed on a local transformation which differs from unity only by terms of the order of μ.

Geodetic orbits

§ 9. In regions where the propagation of light is inhomogeneous there exists, apart from the parallel shift described above, another type of transformation.

Indeed, a physical system $\mathfrak{Q}$ even if no outside interference takes place may drift away if it has an initial velocity $\mathbf{v}$ and thus it moves into regions in which the tensor $\mathbf{g}$ differs from that in the original position. The question arises what changes occur due to the changing environment of the system?

It may be supposed that the changes which take place in the course of the free motion of a system can be described also by Lorentz transformations. We give presently an argument as the result of which the latter type of Lorentz transformation can be determined.

In the homogeneous case a closed system may move with some constant velocity and suffers no changes, therefore the transformation describing this motion corresponds to $\mathbf{M} = 1$ and a displacement $\mu = \mathbf{m}, t$ with $\mathbf{m} = \mathbf{v}t$.

If we transform the coordinates as described in § 1, we obtain in the new representation a tensor $\mathbf{g}'(\mathbf{x}')$ which depends on $\mathbf{x}'$ and thus the propagation of light appears relative to K' inhomogeneous.

Conversely, if in the representation K the propagation appears inhomogeneous the question arises whether it is possible by means of a suitable coordinate transformations to obtain a new representation in which the propagation appears to be homogeneous.

Considering thus the vicinity of a fixed four vector $\mathbf{x}$ we ask whether it is possible to find a transformation

$$\mathbf{x}' + \boldsymbol{\xi}' = \mathbf{f}(\mathbf{x} + \boldsymbol{\xi}) \tag{22}$$

such that

$$\tilde{\mathbf{S}}(\mathbf{x} + \boldsymbol{\xi})\, \mathbf{g}'\, \mathbf{S}(\mathbf{x} + \boldsymbol{\xi}) = \mathbf{g}(\mathbf{x} + \boldsymbol{\xi}), \tag{23a}$$

where $\mathbf{g}'$ has constant components and describes the homogeneous propagation of light relative to K', further

$$S_{\nu\mu}(\mathbf{x} + \boldsymbol{\xi}) = \frac{\partial f_\nu(\mathbf{x} + \boldsymbol{\xi})}{\partial \xi_\mu}. \tag{23b}$$

In the above relations we have not neglected the terms of higher order in $\boldsymbol{\xi}$ as we wanted to define the transformation which leads from the representation of an apparently inhomogeneous region to a representation in which the region appears homogeneous.

Equations (23a) and (23b) give a system of ten partial differential equations to the four unknown functions f_ν and thus the system is as a rule overdetermined. We may therefore consider those cases where the equations (23a) and (23b) have solutions as exceptional cases and we may regard them as representing the cases where light is truly propagated homogeneously. Thus we may suppose that a propagation tensor $\mathbf{g}(\mathbf{x})$ if it possesses a representation $\mathbf{g}' =$ constant, then $\mathbf{g}(\mathbf{x})$ represents homogeneous propagation, only the representation is given in terms of curved coordinates. We note if in the above case we were to construct coordinates according to the methods described in Part I the latter method would automatically lead to a representation in which the propagation appeared to be homogeneous.

§ 10. In general it is impossible to "transform away" the inhomogeneity of propagation of light which appears in a given representation K. It is, however, possible to find by transformation of $\mathbf{g}(\mathbf{x})$ a representation $\mathbf{g}'(\mathbf{x}')$ the first derivatives of which are zero, thus a representation in which

$$\frac{\partial \mathbf{g}'(\mathbf{x}' + \boldsymbol{\xi}')}{\partial \boldsymbol{\xi}'} = 0 \quad \text{for} \quad \boldsymbol{\xi}' = 0 . \tag{24}$$

In the latter representation the propagation of light appears as near as possible to homogeneous propagation.

We note that if there exists a transformation of the form (23) which leads to a transformed $\mathbf{g}'$ satisfying (24), then there exists also a transformation such that the transformed quantities satisfying apart from (24) also

$$\mathbf{x}' = \mathbf{x} , \qquad \mathbf{g}' = \mathbf{g}(\mathbf{x}). \tag{25}$$

Relations (25) are satisfied if the transformation functions obey

$$f_\nu(\mathbf{x}) = x_\nu \quad \text{and} \quad S_{\nu\mu}(\mathbf{x}) = \left(\frac{\partial f_\nu(\mathbf{x} + \boldsymbol{\xi})}{\partial \xi_\mu} \right)_{\xi = 0} = \delta_{\nu\mu} . \tag{26}$$

Differentiating (23c) into $\xi_\varkappa$ we find in the limit $\boldsymbol{\xi} = 0$ using (24) and (25)

$$\frac{\partial \mathbf{S}}{\partial \xi_\varkappa} \mathbf{g} + \mathbf{g} \frac{\partial \mathbf{S}}{\partial \xi_\varkappa} = \frac{\partial \mathbf{g}}{\partial \xi_\varkappa} \quad \text{for } \boldsymbol{\xi} = 0. \tag{27}$$

The above equations admit solutions

$$\frac{\partial \mathbf{S}}{\partial \xi_\varkappa} = \frac{1}{2} \mathbf{g}^{-1} \left(\frac{\partial \mathbf{g}}{\partial \xi_\varkappa} + \mathbf{A}^{(\varkappa)} \right) \qquad \boldsymbol{\xi} = 0, \tag{28}$$

where $\mathbf{A}^{(\varkappa)}$ for $\varkappa = 1, 2, 3, 4$ are antisymmetric matrices obeying

$$\widetilde{\mathbf{A}}^{(\varkappa)} = - \mathbf{A}^{(\varkappa)} . \tag{28a}$$

The latter transformation is of the form given in § 7, equs. (20) and (20a), however, in the present case the matrices $\mathbf{A}^{(\varkappa)}$ can be determined. Indeed, differentiating (23b) into $\xi_\varkappa$ we find

$$\frac{\partial S_{\nu\mu}}{\partial \xi_\varkappa} = \frac{\partial^2 f_\nu(\mathbf{x} + \boldsymbol{\xi})}{\partial \xi_\mu \, \partial \xi_\varkappa}$$

Thus we find interchanging μ and $\varkappa$ in the above relation

$$\frac{\partial S_{\nu\mu}}{\partial x_\varkappa} = \frac{\partial S_{\nu\varkappa}}{\partial x_\mu} . \tag{29}$$

Comparing (28a) and (29) we find

$$A_{\nu\mu}^{(\varkappa)} = \frac{\partial g_{\nu\varkappa}}{\partial x_\mu} - \frac{\partial g_{\mu\varkappa}}{\partial x_\nu} .$$

We may write in place of (22)

$$\frac{\partial \mathbf{S}}{\partial \xi_\varkappa} = - \frac{1}{2} \mathbf{g}^{-1} \mathbf{C}^{(\varkappa)} , \tag{30}$$

where

$$C_{\nu\mu}^{\varkappa} = - \frac{1}{2} \left(\frac{\partial g_{\nu\mu}}{\partial x_\varkappa} + \frac{\partial g_{\nu\varkappa}}{\partial x_\mu} - \frac{\partial g_{\mu\varkappa}}{\partial x_\varkappa} \right) , \tag{30a}$$

thus the $C_{\nu\mu}^{(\varkappa)}$ are equal to the well-known Christoffel brackets

$$C_{\nu\mu}^{(\varkappa)} = \begin{bmatrix} \nu\mu \\ \varkappa \end{bmatrix} \qquad \nu, \mu, \varkappa = 1, 2, 3, 4. \tag{30b}$$

Furthermore using the usual notation

$$\frac{\partial S_{\nu\mu}}{\partial \xi_\varkappa} = \begin{Bmatrix} \nu\,\mu \\ \varkappa \end{Bmatrix} \tag{30c}$$

we shall also use the following notation

$$\frac{\partial \mathbf{S}}{\partial \xi_\varkappa} = \mathscr{C}^{(\varkappa)} \qquad \varkappa = 1, 2, 3, 4. \tag{30d}$$

§ 11. So as to return to the transformation describing the free drift of a system we remark the following. Consider some four vector $\mathfrak{B}$ with a representation **B** relative to K which describes a feature of $\mathfrak{Q}$. The four-vector may be the four distance $\xi_k - \xi_\lambda$ between two points of $\mathfrak{Q}$ — but it might describe alternatively a velocity, an electromagnetic potential, etc.

If the propagation tensor $\mathfrak{g}$ as represented relative to K' is constant, i.e. if $\mathbf{g}' =$ constant, then in the representation K' the system $\mathfrak{Q}$ drifts freely and denoting the configuration of $\mathfrak{Q}$ after it has drifted some distance μ' by $\mathfrak{Q}^*$ we find

$$\mathbf{B}'^* = \mathbf{B}' \tag{31}$$

for the representations of the four-vector $\mathfrak{B}$ appearing in $\mathfrak{Q}$ respectively of $\mathfrak{B}^*$ appearing in $\mathfrak{Q}^*$ the shifted system. In the original representation K we have, however,

$$\mathbf{B} = \mathbf{S}^{-1}(\mathbf{x}')\,\mathbf{B}', \qquad \mathbf{B}^* = \mathbf{S}^{-1}(\mathbf{x}' + \mu')\,B'^*.$$

From (26) we find

$$\mathbf{S}(\mathbf{x}') = \mathbf{S}^{-1}(\mathbf{x}') = \mathbf{1} \quad \text{and} \quad \frac{\partial \mathbf{S}}{\partial \mathbf{x}} = -\frac{\partial \mathbf{S}^{-1}}{\partial \mathbf{x}} \quad \text{for} \quad \xi = 0$$

and therefore since neglecting higher order terms we may put $\mu' = \mu$, $\mathbf{B}' = \mathbf{B}$

$$\delta \mathbf{B} = \mathbf{B}^* - \mathbf{B} = \mu \frac{\partial \mathbf{S}^{-1}}{\partial \mathbf{x}} \mathbf{B}\,.$$

Thus with the help of (30d) we find

$$\delta \mathbf{B} = -\Sigma\, \mu_\varkappa\, \mathfrak{C}^{(\varkappa)}\, \mathbf{B}\,, \tag{32a}$$

we may also write explicitly

$$\delta B_\nu = -\sum_{\mu\varkappa} \left\{ {\mu\varkappa \atop \nu} \right\} B_\mu\, \mu_\varkappa\,. \tag{32b}$$

§ 12. Relations (32a) or (32b) give the change of the measures of the components of a vector **B** in the course of the drift in the particular case where the system $\mathfrak{Q}$ is drifting in a homogeneous region and therefore in the proper representation K' (where $\mathbf{g}' =$ constant) the vector $\mathfrak{B}$ does not change at all because of the drift. The change $\delta\mathbf{B}$ reflects on changes of measures of the components of **B** which appear because the coordinates in K must be taken to be curved coordinates.

Thus relations (32a) or (32b) express only the result of a coordinate transformation if applied to a region in which the propagation is truly homogeneous in the sense of § 8. Following the idea of EINSTEIN, we may suppose that (32) remains valid whether or not the propagation of light is truly homogeneous. Thus it may be assumed that the free drift of a system $\mathfrak{Q}$ is characterized so that in a representation K' in which

$$\frac{\partial \mathbf{g}'(\mathbf{x}' + \xi')}{\partial \xi'} = 0 \qquad \text{for } \xi' = 0,$$

i.e. in the representation K' where the propagation appears as near to homogeneous as possible — in that representation the change of a vector is characterized by

$$\delta \mathbf{B}' = 0$$

or more precisely

$$\delta \mathbf{B}' = \text{order of } \mu^2 .$$

If the latter assumption is made then we are led to relation (32) irrespective of the true mode of propagation of light.

As we suppose that the inhomogeneity of propagation of light is connected with the gravitational field, we may thus suppose that (32) describes the changes which occur in a system moving freely in a gravitational field, i.e. the changes occurring in a free falling system.

§ 13. It is very important that the change although spontaneous should take place adiabatically. If a system would be subjected to a sudden impact through some sudden change of gravitational field, then it might very well deform non-adiabatically. Thus relation (32) can be taken only to be valid for sufficiently slow changes. We have here a complete analogy with the limitations of the adiabatic principle attached to the Lorentz principle in the case of homogeneous propagation of light.

Relation (32) describes the deformations a free falling system suffers, while the parallel shifts discussed in § 5 and which are given (in the case of small shifts) by (18) arise if a system is shifted adiabatically in such a manner that the gravitational action is compensated by outside forces and thus the system is not allowed to fall but is made to move with some small velocity.

From the above remark it becomes clear that a system, which is not allowed to fall freely but is brought adiabatically with small velocity from one position into another, when thus treated will take up in its final position a configuration which is independent of the path along which it was brought there.

On the contrary if a system $\mathfrak{Q}$ falls freely from $\mathbf{x}_k \to \mathbf{x}_m$ then it will arrive in $\mathbf{x}_m$ with a velocity which it acquired in the course of its fall. However, if the system is made first to fall from $\mathbf{x}_k \to \mathbf{x}_l$ then to fall from $\mathbf{x}_l \to \mathbf{x}_m$ then it must receive in $\mathbf{x}_m$ the intermediate position $\mathbf{x}_l$ an impact which make it to change its direction so as to proceed towards $\mathbf{x}_m$. Because of this impact the system will arrive in $\mathbf{x}_m$ with a different velocity when it travelled via $\mathbf{x}_l$ than in the case of the direct journey. The difference of velocity of $\mathfrak{Q}$ when it arrives directly from $\mathbf{x}_k$ or when it arrives on a round about way $\mathbf{x}_m$ can be represented by a local Lorentz transformation, namely the one which corresponds to the change between the two velocities.

From the above consideration we see clearly that the analogy to the conventional parallel shift of a system is not the free fall but the parallel shift where gravitational effects are compensated by outside forces.

We note, that in the usual relativistic terminology the shift as a result of free falling is denoted "parallel shift" and therefore the parallel shift so defined depends on the path in a manner as explained further above. If we define alternatively the parallel shift as a shift which takes place while the gravitational action is compensated by outside forces, then we obtain a type of parallel shift independent of the orbit. Here we use this latter definition.

Adiabatic orbits

§ 14. With the help of relation (32) it is possible to determine the orbit of a free falling system. Consider thus a system the centre of which can be described by some vector $\mathbf{x}(p)$, i.e. we suppose that at the time

$$t = x_4(p) \tag{33}$$

its coordinates are given by

$$\mathbf{r}(t) = x_1(p),\ x_2(p),\ x_3(p). \tag{34}$$

The motion of the centre of the system is thus given in a parameter representation. The velocity of the system can be written

$$\mathbf{v}(p) = \dot{\mathbf{r}}(p)/\dot{x}_4(p)\,, \tag{35}$$

where the dot denotes derivation into p. Further the acceleration is given by

$$\mathbf{a}(p) = \frac{d\mathbf{v}(p)}{dt} = \dot{\mathbf{v}}(p)/\dot{x}_4(p). \tag{36}$$

With the help of (35) and (36) we have also

$$\mathbf{a}(p) = \frac{\ddot{\mathbf{r}}(p) - \mathbf{v}(p)\,\ddot{x}_4(p)}{\dot{x}_4^2(p)}. \tag{37}$$

A system left on its own will thus move in first approximation with a constant velocity. In a time

$$\delta t = \dot{x}_4(p)\,\delta p\,,$$

it will shift by

$$\delta \mathbf{r} = \dot{\mathbf{r}}(p)\,\delta p\,,$$

and we may thus suppose that it will shift by

$$\mu = \delta \mathbf{r}, \delta t\,. \tag{38}$$

We may introduce (38) into (32) and introducing $\dot{\mathbf{x}}(p)$ in place of $\mathbf{B}$ we find

$$\delta \dot{x}_\nu(p) = -\sum_{\varkappa\lambda} \begin{Bmatrix} \varkappa\lambda \\ \nu \end{Bmatrix} \dot{x}_\varkappa(p)\,\dot{x}_\lambda(p)\,\delta p$$

or writing $\ddot{x}_\nu(p)$ for $\delta\,\dot{x}_\nu(p)/\delta p$ we have

$$\ddot{x}_\nu(p) + \sum_{\varkappa\lambda} \begin{Bmatrix} \varkappa\lambda \\ \nu \end{Bmatrix} \dot{x}_\varkappa(p)\,\dot{x}_\lambda(p) = 0\,. \tag{39}$$

The above relation is the well-known equation of the so-called four dimensionat geodetic line. We see that supposing a system if left on its own suffers Lorentzl deformations of the particular form (32) we are led to equation (39) for the orbit of a free particle or of a free closed system.

Since relation (39) contains no specific quantity of the moving system this leads to the conclusion that any small closed system left on its own, will move on the same orbit (determined only by initial conditions). Thus the fact that relation (39) contains only the coordinates of the moving system and their derivatives reflects the general law of the equivalence of inertial and gravitational masses.

§ 15. It is well known that the equation of motion (39) can also be derived from a variational principle. It can be shown that (39) are the Euler equations

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} \left(\frac{ds}{dp}\right)^2 dp = 0, \tag{40}$$

where

$$\left(\frac{ds}{dp}\right)^2 = \dot{\mathbf{x}}\mathbf{g}\dot{\mathbf{x}}\,.$$

Multiplying (40) with the mass m_0 of the particle we may write if we choose the parameter p equal to x_4 with the help of notation used in Part I,

$$m_0\left(\frac{ds}{dp}\right)^2 = m_0(\mathbf{v}+\mathbf{V})\,\mathbf{G}(\mathbf{v}+\mathbf{V}) - m_0\,\Phi\,.$$

The first term can be regarded as a kind of kinetic energy (the velocity of the particle being taken to the aether drifting with a velocity $-\mathbf{V}$). The second term is a kind of negative potential energy, thus relation (40) is reminiscent of the Lagrange equation

$$\delta\int L\,dt = 0\,,$$

with

$$L = K - U\,.$$

Furthermore it can be shown that (39) can also be derived from the following variational principle

$$\delta\int_{\mathbf{x}_1}^{\mathbf{x}_2} ds = 0\,. \tag{41}$$

The latter principle requires that the orbit of the system should be a four dimensional geodetic line.

From the physical point of view, we prefer the derivation of the equation of motion (39) through the generalization of the Lorentz principle and thus to consider the tensor $\mathbf{g}(\mathbf{x})$ as characteristic for the propagation of light in the vicinity of $\mathbf{x}$. However, the fact that (39) gives mathematically the solution of a variational problem is very important from another point of view. In deriving (39) we have used approximations and have considered shifts μ small on a cosmical scale. A larger shift can be built up from the succession of a number of small shifts, but it is not immediately obvious that the small errors committed considering the small steps do not accumulate.

From the fact that (39) is the solution of a variational problem one concludes that the orbits obtained as a solution of (39) are independent of the particular choice of coordinates. Therefore one is inclined to take equation (39) to be exact — or at least to be strictly independent of the choice of coordinates.

In the case of homogeneous propagation of light we have $d\mathbf{g}/d\mathbf{x} = 0$ and therefore all the Christoffel brackets vanish. In the latter case we find $\ddot{\mathbf{x}}(p) = 0$ and thus

$$\mathbf{x} = \boldsymbol{\alpha}\,p + \boldsymbol{\beta}\,,$$

or eliminating p we have

$$\mathbf{r} = \mathbf{v}t + \mathbf{b}\,, \tag{42}$$

where $\mathbf{v}$ has components $v_k = \alpha_k/\alpha_4$, $k = 1, 2, 3$. Thus in a region with homogeneous propagation of light NEWTON's first law appears to be valid. If we consider the same region in terms of transformed coordinates $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$, then in the latter the Christoffel brackets will not vanish, but we obtain the exact representation of the translational motion in terms of the transformed coordinates. This result, however, is only a check of consistency of one assumption as the concept of motion with constant velocity in the space free of gravitation was made use of in the derivation of the equations of motion.

For physical application it is of course necessary to establish the connection between the tensor $\mathbf{g}(\mathbf{x})$ and the gravitational field or more exactly it is necessary to obtain $\mathbf{g}(\mathbf{x})$ from the distribution of gravitating matter. The latter problem was solved by EINSTEIN, we give certain aspects of the problem in Part III of this series.

ПРИНЦИП ЛОРЕНЦА И ОБЩАЯ ТЕОРИЯ ОТНОСИТЕЛЬНОСТИ
Часть II.

Л. ЯНОШИ

Резюме

Дано обобщение преобразования Лоренца для областей где свет распространяется неоднородно, и показано, что принцип Лоренца может быть сохранён в своей обычной форме если преобразование Лоренца взять в своем более общем виде, формулированном для таких областей. Известные уравнения геодезических линий в гравитационном поле выводятся из обобщенного таким образом принципа Лоренца.