TWO SUBGROUPS OF THE LORENTZ GROUP AND THEIR PHYSICAL SIGNIFICANCE

By

L. JÁNOSSY

CENTRAL RESEARCH INSTITUTE FOR PHYSICS
BUDAPEST

(Received 12. II. 1965)

It is shown that the LORENTZ group can be represented as the product of two subgroups. The one subgroup is connected with rotation, the other with translation. The results of the negative relativistic experiments, like the MICHELSON—MORLEY experiment, are connected with the invariance of laws with respect to the rotational subgroup, while the positive relativistic effects, like the change of mass with velocity, are connected with the invariance with respect to the translational subgroup.

§ 1. The LORENTZ transformation can be written*

$$\mathbf{x}' = \mathscr{L}(\mathbf{x}) = \Lambda^{(\mathbf{p})}\mathbf{x} + \lambda\,, \tag{1}$$

where Λ is a fourth order matrix obeying

$$\tilde{\Lambda}^{(\mathbf{p})}\,\Gamma\Lambda^{(\mathbf{p})} = \Theta\,\Gamma \tag{2}$$

with $\Theta > 0\,,\quad \Gamma_{\nu\mu} = \delta_{\nu\mu}\,\gamma_\nu\,,\quad \gamma_1 = \gamma_2 = \gamma_3 = 1,\quad \gamma_4 = -\,c^2\,.$

The index $\mathbf{p}$ stands for the parameters of the transformation. The proper LORENTZ transformations are further restricted as follows:

$$\Theta = 1,\quad \det\Lambda^{(\mathbf{p})} = +\,1,\quad \Lambda^{(\mathbf{p})}_{44} > 0\,. \tag{3}$$

The matrix $\Lambda^{(\mathbf{p})}$ which we shall call a LORENTZ matrix depends on six parameters, explicitly it can be written in the following way:

$$\Lambda^{(\mathbf{p})} = \begin{pmatrix} \mathbf{L} & \mathbf{v}B \\ -\,\mathbf{v}'\,B/c^2 & B \end{pmatrix}, \tag{4}$$

where $\mathbf{v} = v_1,\, v_2,\, v_3$ is a three-component vector with the dimension of a velocity; further

$$\left.\begin{aligned} \mathbf{L} &= \mathbf{O} - (B-1)\,(\mathbf{v}' \circ \mathbf{v})/v^2\,, \\ B &= \frac{1}{\sqrt{1 - v^2/c^2}}\,,\ \ \mathbf{v}' = -\,\mathbf{O}\mathbf{v}, \end{aligned}\right\} \tag{5}$$

* For notation see [1].

and $\mathbf{O}$ is an orthogonal matrix, thus $\mathbf{O}\tilde{\mathbf{O}} = \mathbf{1}$. The relation (5) contains six parameters, i.e. the three components of $\mathbf{v}$ and three parameters in terms of which the orthogonal transformation $\mathbf{O}$ can be expressed.

It can be seen easily that the matrix Λ as given by (4) and (5) obeys indeed (2) and (3) and it can also be shown that any matrix obeying (2) and (3), i.e. any proper LORENTZ matrix, can be brought into the form (4), (5).

The matrix $\Lambda^{(\mathrm{p})}$ can also be written

$$\Lambda^{(\mathrm{p})} = \mathbf{O}^{(4)}\,\Lambda_{\mathrm{v}}\,, \tag{6}$$

where

$$\mathbf{O}^{(4)} = \begin{pmatrix} \mathbf{O} & 0 \\ 0 & 1 \end{pmatrix}, \tag{6a}$$

$$\Lambda_{\mathrm{v}} = \begin{pmatrix} \mathbf{V} & \mathbf{v}B \\ \mathbf{v}B/c^2 & B \end{pmatrix} \quad \text{and} \quad \mathbf{V} = 1 + (B-1)\,(\mathbf{v}\circ\mathbf{v})/v^2\,. \tag{6b}$$

Thus any LORENTZ matrix can be written as the product of an orthogonal transformation matrix of the type $\mathbf{O}^{(4)}$ and a transformation matrix of the type Λ_{v} which does not change the directions of the axes but changes the translational velocity of the system of reference by an amount $\mathbf{v}$.

§ 2. The LORENTZ matrix (4) can be taken as part of a coordinate transformation (1); this transformation leads from a system K to a system K' which moves with a velocity $\mathbf{v}'$ relative to K, the orthogonal matrix defining the directions of the axes of K' relative to K.

Alternatively, a transformation of the form (1) can be taken to describe a LORENTZ deformation. Indeed, consider a physical system $\mathfrak{Q}$. Another system $\mathfrak{Q}^*$ can be produced by replacing the points $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n$ of $\mathfrak{Q}$ by points $\mathfrak{P}_1^*, \mathfrak{P}_2^*, \ldots, \mathfrak{P}_n^*$ making up $\mathfrak{Q}^*$.

Written more explicitly, at the time t the point $\mathfrak{P}_n$ may have coordinates $\mathbf{r}_n(t)$, at a time t^* the corresponding point $\mathfrak{P}_n^*$ then has coordinates $\mathbf{r}_n^*(t^*)$, so that

$$\mathbf{r}_n^*(t^*),\, t^* = \mathbf{x}_n^*\,,$$

and

$$\mathbf{x}_n^* = L_{\mathrm{q}}(\mathbf{x}_n) = \Lambda_{\mathrm{q}}\,\mathbf{x}_n + \lambda\,. \tag{7}$$

In the above consideration $\mathbf{x}_n$ and $\mathbf{x}_n^*$ are the (four-component) coordinates of the points of $\mathfrak{Q}$ and $\mathfrak{Q}^*$ both taken relative to one system of coordinates, K.

We have written Λ_{q} in place of $\Lambda^{(\mathrm{p})}$ to signify that we are considering a transformation that refers to one particular system of reference, K, and describes the change $\mathfrak{Q}\to\mathfrak{Q}^*$ in terms of the coordinates relative to this system of reference. Thus Λ_{q} is the homogeneous part of the transformation (7) which represents a deformation in terms of coordinates relative to K. Λ_q is a tensor and we shall call it the *deformation tensor.*

We may also write symbolically

$$\mathfrak{Q}^* = L_q(\mathfrak{Q}) \tag{7a}$$

and the representation of (7a) relative to K can be written in the form (7). The representation of (7a) relative to another system of coordinates, K', can be written

$$\mathbf{x}_n^{*\prime} = \Lambda_{q'}\mathbf{x}_n' + \lambda' , \tag{8}$$

where

$$\left.\begin{aligned} \mathbf{x}_n^{*\prime} &= \Lambda^{(p)}\mathbf{x}_n^* + \lambda , \\ \mathbf{x}_n' &= \Lambda^{(p)}\mathbf{x}_n + \lambda , \end{aligned}\right\} \tag{9}$$

and $\Lambda^{(p)}$ is the homogeneous part of the transformation leading from K to K'. From (8) and (9) it follows, that

$$\Lambda_{q'} = \Lambda^{(p)}\Lambda_q\Lambda^{(p)-1} . \tag{10}$$

Thus Λ_q, $\Lambda_{q'}$, . . . are the representations of the deformation tensor Λ_q relative to systems of reference K, K', . . .

From (10) it follows that the representations of a deformation tensor Λ_q are all proper LORENTZ matrices if one of the representations is a proper LORENTZ matrix, regardless of whether or not the matrices $\Lambda^{(p)}$ are proper LORENTZ matrices.

§ 3. The representations of a deformation tensor corresponding to the deformation $\mathfrak{Q}\to\mathfrak{Q}^*$ in different systems of reference are given by matrices that are connected by relations of the form (10). From (10) it may be seen that the representations Λ_q, $\Lambda_{q'}$, . . . of Λ_{Λ} are matrices with the same eigenvalues. It was shown elsewhere [2], that the eigenvalues of a LORENTZ matrix can be written

$$e^{i\varphi}, \quad e^{-i\varphi}, \quad \sqrt{\frac{c+v}{c-v}}, \quad \sqrt{\frac{c-v}{c+v}}, \tag{11}$$

i.e. the eigenvalues are characterized by an angle φ and a velocity v. A LORENTZ matrix can be brought into a standard form, this means, that in a suitable representation a matrix Λ_q with eigenvalues (11) obtains the form

$$\Lambda_{q_0} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & B & Bv \\ 0 & 0 & Bv/c^2 & B \end{pmatrix}. \tag{12}$$

The deformation tensor represented by (12) describes turning through an angle φ around the x_3-axis and acceleration by an amount v in the direction of the x_3-axis.

§ 4. We note that the transformation eq. (6a) has the eigenvalues

$$e^{i\varphi},\ e^{-i\varphi},\ 1,\ 1. \tag{13}$$

(We remark that the eigenvalues of the third order orthogonal matrix **O** are $e^{i\varphi}$, $e^{-i\varphi}$, 1 .)

The representations of $\mathbf{O}^{(4)}$ relative to various systems of reference will in general not appear in the form (6a). However, all the representations of $\mathbf{O}^{(4)}$ have eigenvalues of the form (13).

By $\mathbf{O}^{(4)}$ we may denote not only the deformation tensors that appear in the form (6a) but all the deformation tensors with eigenvalues of the form (13). We may call these deformations *rotational* deformations. We see from (13) that the product of two rotational deformation tensors is also a rotational deformation tensor. It can thus be concluded that the rotational deformations form a subgroup of the LORENTZ group.

Similarly, the eigenvalues of the deformation tensors Λ_v are found to be

$$1, 1, \sqrt{\frac{c+v}{c-v}}, \sqrt{\frac{c-v}{c+v}}\,. \tag{14}$$

It follows from the form of the eigenvalues (14) that the product of two matrices with such eigenvalues has also eigenvalues of similar type and therefore the matrices Λ_v form also a subgroup of the LORENTZ group.

§ 5. We see thus that the proper Lorentz group can be built up of two subgroups: one subgroup with elements of the rotational type $\mathbf{O}^{(4)}$, another subgroup with elements of the translational type Λ_v. The elements of the proper Lorentz group can be represented as the products of a rotational element with a translational element.

In order to make this representation unique, we may use the following convention. Of a given Lorentz matrix Λ_q with eigenvalues (11) we determine the normal representation (12) and from this normal form we define the splitting of Λ_q into its two components as follows. Let $\Lambda^{(p)}$ denote the coordinate transformation from the system of reference in which Λ_q appears in the normal form (12) into the system K relative to which we wish to represent Λ_q. We have thus

$$\Lambda_q = \Lambda^{(p)} \Lambda_{q_0} \Lambda^{(p)-1}, \tag{15}$$

and also

$$\Lambda_q = \mathbf{O}^{(4)} \Lambda_v, \tag{15a}$$

with

$$\mathbf{O}^{(4)} = \Lambda^{(\mathbf{p})} \mathbf{O}_{\varphi}^{(4)} \Lambda^{(\mathbf{p})-1}, \quad \Lambda_{\mathbf{v}} = \Lambda^{(\mathbf{p})} \Lambda_{\mathbf{v}} \Lambda^{(\mathbf{p})-1} \tag{15b}$$

and

$$\mathbf{O}_{\varphi}^{(4)} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Lambda_{\mathbf{v}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & B & Bv \\ 0 & 0 & Bv/c^2 & B \end{pmatrix}, \tag{15c}$$

Relations (15), (15a), (15b), (15c) give a unique procedure for the splitting up of a deformation tensor into its rotational and translational parts.

As $\mathbf{O}_{\varphi}^{(4)}$ and $\Lambda_{\mathbf{v}}$ are commutative, their transforms $\mathbf{O}^{(4)}$ and $\Lambda_{\mathbf{v}}$ are also commutative and we find

$$\Lambda_{\mathbf{q}} = \mathbf{O}^{(4)} \Lambda_{\mathbf{v}} = \Lambda_{\mathbf{v}} \mathbf{O}^{(4)} .$$

We see therefore that any element of the Lorentz group can be split into the product of an element of the rotational group and an element of the translational group in a manner in which the factors are commutative.

§ 6. We may make here the following interesting remark on the connection of this splitting up of the Lorentz group with physical phenomena. The theory of relativity is based partly on the negative results of certain experiments like the MICHELSON—MORLEY or the TROUTON—NOBLE experiment. In these negative experiments an arrangement is turned round and no apparent effect is observed.

The turning round of an apparatus corresponds to a LORENTZ deformation of the rotational type. The negative outcome of these experiments can be predicted from the LORENTZ invariance of the laws of nature. However, if the laws of nature were invariant only with respect to the rotational subgroup of the LORENTZ transformation this would be sufficient to account for the negative results of these experiments.

There exist further the so-called positive relativistic effects, like the change of mass with velocity or the perpendicular DOPPLER effect. The latter effects can be understood by supposing that the laws of nature are invariant with respect to the translational group.

We see thus that the invariance of the laws of nature with respect of the two subgroups of the LORENTZ group manifests itself in two distinct groups of experiments. Taking these groups of experiments together, we come to conclude that the laws of nature are invariant against both the translational and the rotational subgroup, and therefore against the whole proper LORENTZ group, as the elements of the whole group can be formed as products of the elements taken from the two subgroups.

From the experimental point of view it may be added that the first type of experiments, i.e. the negative experiments, has been carried out with very great precision, therefore the invariance against the rotational group is very precisely established experimentally.

The experiments concerning the change of mass with velocity are not very accurate (see e.g. [3]) howerer, very good evidence for the invariance with respect to the translational sub-group was obtained by D. C. CHAMPEREY, G. R. ISAAK and A. M. KHAN [4] with the help of the Mössbeuer effect. These measurements seem to be the most precise carried out so far supporting LORENTZ invariance.

Thus the measusements of CHAMPENEY et al. together with the older measurements of the Michelson type provide good evidence for the invariance with respect to both sub-groups and therefore provide evidence for the invariance with respect to the whole group of proper LORENTZ transformations.

REFERENCES

1. L. JÁNOSSY, Acta Phys. Hung., **20**, 81, 1966.
2. L. JÁNOSSY, Acta Phys. Polonica, **27**, 61, 1965,
3. P. FARAGÓ and L. JÁNOSSY, Nuovo Cim., **5**, 1411, 1957.
4. D. C. CHAMPENEY, G. R. ISAAK and A. M. KHAN, Physics Letters, **7**, 241, 1963.

ДВЕ ПОДГРУППЫ ГРУППЫ ЛОРЕНЦА И ИХ ФИЗИЧЕСКИЙ СМЫСЛ

Л. ЯНОШИ

Резюме

В работе показывается возможность представления группы Лоренца в виде произведения двух подгрупп. Первая из подгрупп связана с ротацией, другая — с трансляцией. Отрицательный результат опытов по теории относительности — например опыта Майкельсона—Морли — связан с инвариантностью законов, относящихся к подгруппе вращения, а положительный результат опытов по теории относительности — например, зависимоть массы от скорости тела — связан с инвариантностью по отношению трансляционной подгруппы.