Acta Physica Academiae Scientiarum Hungaricae, Tomus 31 (1–3), pp. 265–272 (1972)

CALCULATION OF COMPLEX-CONJUGATE PAIRS OF REGGE TRAJECTORIES WITH THE SCALAR BETHE-SALPETER EQUATION

By

K. LADÁNYI

RESEARCH GROUP FOR THEORETICAL PHYSICS OF THE HUNGARIAN ACADEMY OF SCIENCES, BUDAPEST

The BETHE–SALPETER equation of scalar particles is reduced to a form which is tractable numerically. This formalism is applied to the numerical calculation of Regge trajectories. Particular attention is paid to the level mixing effects of the mass difference leading to complex-conjugate pairs of trajectories. The calculations include the imaginary part of the total energy at real values of the angular momenta.

I. Introduction

Considerable interest has recently been attached to the properties of the relativistic Bethe-Salpeter (BS) equations [1, 2] in the ladder approximation. The numerical program initiated by SCHWARTZ [3] and SCHWARTZ and ZEMACH [4] demonstrated that the classic work of WICK [5] and CUTKOSKY [6] may be extended by using conventional computational techniques. Following the first calculations, many results have been obtained both in the bound [7] and scattering regions [8]. This development has been exploited in detailed numerical calculations of Regge trajectories for spinless particles [9–10]. CUTKOSKY and DEO [9] found surprising collision phenomena among the trajectories which indicate the existence of complex branches in particular unequal mass situations.

The calculations of CUTKOSKY and DEO are limited to real angular momenta at real values of s (s is the squared energy). This talk is concerned with extending the numerical calculations to complex energies. The explicit computations were done below elastic threshold at real angular momenta. A detailed report of this work will be published elsewhere [11].

II. Reduction of the BS equation

We consider the homogeneous BS equation of two scalar particles with masses m_1 and m_2 which interact via the exchange of a third scalar particle with nonzero mass $\varkappa$. The BS wave function ψ satisfies the differential equation [4]

$$(L-\lambda V)\,\psi = 0\,, \tag{1}$$

where λ is the coupling strength and V denotes the ladder approximation of the interaction. In the center-of-mass system the Wick-rotated operators L and V are

$$L = \left[-\Box + 2\mu_1 E \frac{\partial}{\partial x_4} - \mu_1^2 s + m_1^2\right] \cdot \left[-\Box - 2\mu_2 E \frac{\partial}{\partial x_4} - \mu_2^2 s + m_2^2\right], \qquad (2)$$

$$V = \frac{4\varkappa}{R} K_1(\varkappa R), \qquad (3)$$

$$\Box = \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2}, \qquad R = \left(\sum_{i=1}^{4} x_i^2\right)^{1/2}, \qquad (4)$$

where $s = E^2$, E is the total c.m. energy, the x_i' s ($i = 1, 2, \ldots, 4$) denote the components of the relative coordinate, and the parameters μ_1 and μ_2 are constrained by the condition $\mu_1 + \mu_2 = 1$. For the K_1 function we used an approximation proposed by VOSKO [12]:

$$yK_1(y) \cong \frac{(1+y)e^{-y} + a(1+2y)\, e^{-2y}}{1+a}, \; a = 0.66746\,. \qquad (5)$$

We can make the usual separation

$$\psi = \psi_l = \Phi_l(R, \theta)\, Y_{lm}(\vartheta, \varphi), \qquad (6)$$

where the angles θ, ϑ, φ are defined by

$$x_1 = R \sin\theta \sin\vartheta \sin\varphi, \; x_2 = R \sin\theta \sin\vartheta \cos\varphi, \; x_3 = R \sin\theta \cos\vartheta,$$
$$x_4 = R\cos\theta.$$

Our starting point is the expansion of ψ_l in terms of the normalized four-dimensional spherical harmonic functions $Y_{nlm}(\theta, \vartheta, \varphi)$:

$$\psi_l = \sum_{q=0}^{\infty} \sum_{h=0}^{\infty} a_{qh}\, \varphi_{lqh}, \qquad (7)$$

$$\varphi_{lqh} = f_{lqh}(R)\, Y_{l+q,\, lm}(\theta, \vartheta, \varphi)\,. \qquad (8)$$

SCHWARTZ [3] suggested the following basis functions:

$$f_{lqh}(R) = R^{l+q+h}\, e^{-\alpha R}, \qquad (9)$$

where α is a nonlinear scale parameter.

Instead of the functions φ_{lqh}, CUTKOSKY and DEO used basis functions

φ'_{lqh} of the form

$$\varphi'_{lqh} = \varphi_{lqh}\, e^{\gamma R \cos\theta} \tag{10}$$

where the parameter $\gamma = \gamma(E)$ was chosen to incorporate the differences in the asymptotic behaviour of the solution when $\cos\theta = \pm 1$. On the other hand, it should be noted that the redefinition

$$\mu_1 \to \mu_1 + \delta, \qquad \mu_2 \to \mu_2 - \delta \tag{11}$$

produces the transformation

$$\psi \to \psi\, e^{\delta E R \cos\theta} \tag{12}$$

in the wave function of the BS equation (1)–(2), but the eigenvalue λ is independent of δ. Thus, the use of the factor exp $[\gamma R \cos\theta]$ (see Eq. (10)) may be avoided by a transformation of the parameters μ_1 and μ_2. In the present calculations we choose the convention $\mu_1 = \mu_2 = 1/2$ and a mass scale in which the external masses are

$$m_1 = 1 + \Delta, \qquad m_2 = 1 - \Delta. \tag{13}$$

It is convenient to introduce a second set of basis-functions χ_{lpk}:

$$\chi_{lpk} = g_{lpk}(R)\, Y_{l+p,\, lm}(\theta, \vartheta, \varphi). \tag{14}$$

In particular, we may choose

$$g_{lpk}(R) = R^{l+p+k} e^{-\beta R}. \tag{15}$$

We now can formulate the BS problem (1)–(2) in the space of states defined by the basis functions φ_{lgh} and χ_{lgk}. One obtains the following system of inhomogeneous linear equations

$$\sum_{q=0}^{Q} \sum_{h=0}^{H} \sum_{\nu=0}^{1} \langle \chi_{lpk} | D_{\mu\nu} - \lambda V_{\mu\nu} | \varphi_{lqh} \rangle\, a_{qh\nu} = 0 \tag{16}$$

for

$$p = 0, 1, \ldots, Q, \quad k = 0, 1 \ldots, H, \quad \mu = 0, 1, \tag{17}$$

and $Q \to \infty$, $H \to \infty$. The matrix elements can be written as

$$\langle \chi_{lpk} | B | \varphi_{lqh} \rangle = \int_0^\infty dR R^3 \int_0^\pi d\theta \sin^2\theta \int_0^\pi d\vartheta \sin\vartheta \int_0^{2\pi} d\varphi \chi^*_{lpk}\, B\, \varphi_{lqh}, \tag{18}$$

$$V_{00} = - V_{11} = V, \quad V_{01} = V_{10} = 0, \tag{19}$$

$$D_{00} = - D_{11} = \left[- \Box + 1 + \Delta^2 - \frac{1}{4} (u^2 - v^2) \right]^2 - (u^2 - v^2) \frac{\partial^2}{\partial x_4^2} - $$
$$- 4 \Delta u \frac{\partial}{\partial x_4} - \left(4 \Delta^2 + \frac{1}{4} u^2 v^2 \right), \tag{20}$$

$$D_{01} = D_{10} = uv \left[- \Box + 1 + \Delta^2 - \frac{1}{4} (u^2 - v^2) \right] + 2uv \frac{\partial^2}{\partial x_4^2} + 4 \Delta v \frac{\partial}{\partial x_4}, \tag{21}$$

with

$$u = 1/2 \, (E + E^*) \quad v = \frac{1}{2_i} (E - E^*). \tag{22}$$

In addition, we have

$$a_{qh6} = \frac{1}{2} (a_{qh} + a_{qh}^*), \qquad a_{qh1} = \frac{1}{2_i} (a_{qh} - a_{qh}^*), \tag{23}$$

where the a_{qh}'s are the coefficients in decomposition (7). The explicit evaluation of the matrix elements is included in [11].

Equations (16)–(17) can be continued in the angular momentum plane, the Regge trajectories correspond to the nontrivial solutions. Approximate solutions of this problem may be calculated by setting finite values of Q and H in Eqs. (16)–(17). The corresponding secular equation is

$$\text{Det} \, |D - \lambda V| = 0, \tag{24}$$

where, of course, the matrix elements

$$\langle \chi_{lpk} | D_{\mu\nu} - \lambda V_{\mu\nu} | \varphi_{lqh} \rangle$$

depend on l, u and v. The solutions of this problem result in the Rayleigh–Ritz approximations [3] by requiring $\chi_{lpk} = \varphi_{lpk}$. On the other hand, we obtain the method of moments if $\chi_{lpk} \neq \varphi_{lpk}$ [13–14]. We emphasize that in these methods unpleasant difficulties may be encountered because the unequal-mass BS problem involves nonhermitean or nondefinite matrices. Our calculations have been carried out by using a generalized least-squares technique. Details of this method can be found in [11].

III. Results and discussion

We computed some Regge trajectories in the region $-4 \leq \text{Re}s \lesssim 3.5$, $-0.5 \lesssim l \lesssim 1.5$ ($l = l^*$). All calculations were done for exchanged mass $\varkappa = 1$ and the coupling strength was adjusted as $\lambda = 16.38$ to place the highest-lying (parent) trajectory of an equal-mass system ($\Delta = 0$) through $l = 1$ at zero energy. The trajectories were calculated by using 14 basis functions φ_{lqh}. Figs. 1–3 do not include results near the threshold and some turning points where the convergence of the results is seen to be poor.

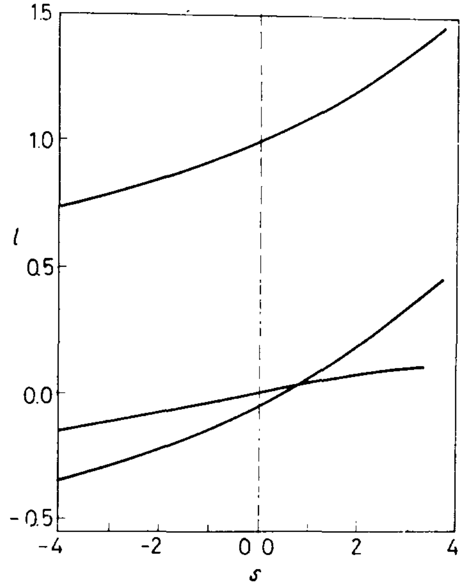

Fig. 1. Regge trajectories for equal-mass systems. The parameters are $\Delta = 0$, $\lambda = 16.38$, $\varkappa = 1$

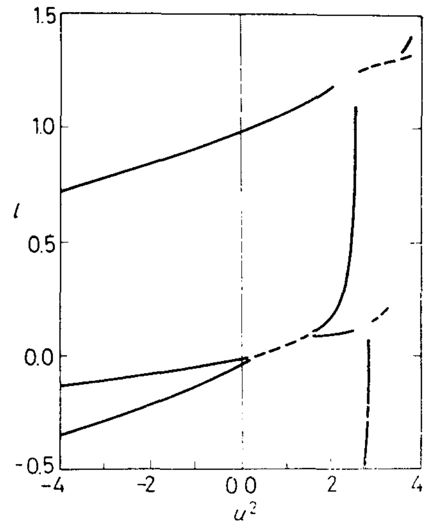

***Fig. 2.* Effect of the mass difference. The Regge trajectories were obtained for $\Delta = 0.3$, $\lambda = 16.38$, $\varkappa = 1$. Solid lines represent real trajectories at real squared energies. Dashed curves are the complex branches**

Fig. 1 shows the equal-mass situation ($\Delta = 0$). We observe two intersecting trajectories which correspond to excited states with opposite time-parity. The effects of the mass difference are summarized in Figs. 2–3 by choosing $\Delta = 0.3$. Consider first the region where trajectories of the equal-mass system are seen to cross ($s \approx 0.75$, $l \approx 0.03$). Since the corresponding states have opposite time parity, the mixing effect of the anti-Hermitean operator $4\Delta u\partial/\partial x_4$ (see Eq. (20)) leads to a complex-conjugate pair of trajectories which connect two colliding real branches according to previous perturbation arguments. Fig. 3 shows the details of this complex branch including the imaginary part of the energy. In addition, Fig. 2 includes two other complex branches of the type already seen.

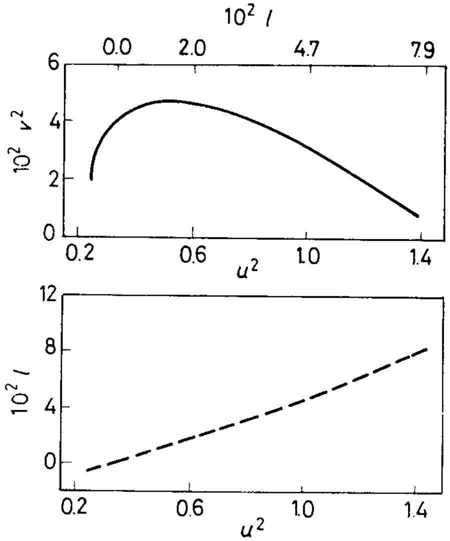

Fig. 3. Details of the first complex branch of Fig. 2 ($\Delta = 0.3$, $\lambda = 16.38$, $\varkappa = 1$)

We conclude that the slopes of Regge trajectories are very model dependent and detailed numerical calculations are necessary before any conclusion can be drawn.

REFERENCES

1. Y. Nambu, Progr. Theor. Phys., **5**, 614, 1950; M. Gell-Mann and F. Low, Phys. Rev., **84**, 350, 1951; J. Schwinger, Proc. Natl. Acad. Sci. U. S., **37**, 452, 1951; E. E. Salpeter and H. A. Bethe, Phys. Rev., **84**, 1232, 1951.
2. For a summary see, e.g., N. Nakanishi, Suppl. Progr. Theor. Phys., **43**, 1, 1969.
3. C. Schwartz, Phys. Rev., **137**, B717, 1965.
4. C. Schwartz and C. Zemach, Phys. Rev., **141**, 1454, 1966.
5. G. C. Wick, Phys. Rev., **96**, 1127, 1954.
6. R. E. Cutkosky, Phys. Rev., **95**, 1135, 1954.
7. K. D. Rothe, Phys. Rev., **170**, 1548, 1968; A. Pagnamenta, Nuovo Cimento, **53A**, 30, 1968; P. Naranayaswamy and A. Pagnamenta, Nuovo Cimento, **53A**, 635, 1968; K. Ladányi, Nuovo Cimento, **56A**, 173, 1968; E. zur Linden and H. Mitter, Nuovo

Cimento, **61B**, 389, 1969; D. Kershaw, H. Snodgrass and C. Zemach, SLAC preprint, 1970; D. Kershaw and C. Zemach, SLAC preprint, 1970; E. zur Linden, MPI preprint, 1970; P. Breitenlohner, MPI preprint, 1970.
8. M. Levine, J. Tjon and J. Wright, Phys. Rev., **154**, 1433, 1967; R. M. Saenger, J. Math. Phys., **8**, 2366, 1967; R. Haymaker, Phys. Rev. Letters, **18**, 968, 1967; Phys. Rev., **165**, 1790, 1968; K. Ladányi, Nuovo Cimento, **61A**, 173, 1969; B. C. McInnis, Phys. Rev., **183**, 1474, 1969.
9. R. E. Cutkosky and B. B. Deo, Phys. Rev. Letters, **19**, 1256, 1967.
10. E. zur Linden, Nuovo Cimento, **63A**, 181, 1969; W. B. Kaufmann, Phys. Rev., **187**, 2051, 1969; R. N. Madan, R. W. Haymaker and R. Blankenbecler, Phys. Rev., **172**, 1788, 1968; R. Gatto and P. Menotti, CERN preprint, 1970.
11. K. Ladányi, Nuovo Cimento, **70A**, 405, 1970.
12. S. H. Vosko, J. Math. Phys., **1**, 505, 1960.
13. E.g., L. V. Kantorovich and V. I. Krylov, Approximation Methods in Higher Analysis (New York, 1958), p. 150.
14. For applications in the nonrelativistic quantum mechanics see, e.g., É. Szondy and T. Szondy, Acta Phys. Hung., **20**, 253, 1966.

РАСЧЕТ КОМПЛЕКСНО-СОПРЯЖЕННЫХ ПАР ТРАЕКТОРИЙ РЕДЖЕ С ПОМОЩЬЮ СКАЛЯРНОГО УРАВНЕНИЯ БЕТЕ—СОЛПИТЕРА

К. ЛАДАНИ

Резюме

Уравнение Бете—Солпитера для скалярных частиц приведено к упрощенному виду, допускающему численное решение. Этот формализм применен к расчету траекторий Редже. Особое внимание уделено эффектам смешивания уровней при различных массах, приводящим к комплексно сопряженным парам траекторий. Также рассчитана мнимая часть полной энергии при вещественных значениях углового момента.