Acta Physica Academiae Scientiarum Hungaricae, Tomus 31 (1–3), pp. 251–258 (1972)

CURRENT COMMUTATORS AT SMALL TIME DIFFERENCES

By

I. Farkas and G. Pócsik

INSTITUTE FOR THEORETICAL PHYSICS, ROLAND EÖTVÖS UNIVERSITY, BUDAPEST

Using the equal-time current algebra and the divergence conditions, we calculate the current commutators for small time differences. It is shown that the commutators are explicitly model-dependent and the contributions of the symmetry-breaking terms do not drop out. The physical content of the current commutators of non-equal time is discussed in terms of new sum rules. We point out that the disconnected contributions are necessary for the consistency of the sum rules. The sum rules favour the field algebra.

I. Introduction

In spite of the enormous successes of the equal-time current algebra, extension of the current commutators for non-equal times is an urgent and difficult task. Up to now, the motivation for generalized current algebra is twofold.[1] In order to clarify the meaning of the infinite momentum method, attempts have been made for an extension of the current algebra near the light cone [1, 2]. 2. The light cone commutators control the high energy behaviour, therefore theories of generalized current algebras were developed on the light cone [3, 4, 5] to study inelastic processes at high energies.

These generalized current algebras are, however, postulated with a certain arbitrariness, as far as the compatibility with the divergence conditions (*PCAC*, *CVC*, etc.) and the presence of the symmetry-breaking terms are concerned. In Section II we show that the equal-time current algebra and the divergence conditions determine the extension to non-equal times, at least in a small strip along the space-axis. It is explicitly seen that the generalized commutators are model-dependent and, in general, the contributions of the symmetry-breaking terms do not drop out.

The next question concerns the physical content of the generalized current commutators (Section III). In the case of conserved vector currents we describe a simple method leading to sum rules. The new system of sum rules becomes inconsistent if one neglects the disconnected contributions. The sum rules favour the field algebra.

II. Non-equal-time current commutators

We confine ourselves to $SU_3 \times SU_3$ algebra and write [6]

$$[j_0^a(x), J_\mu^b(y)]_{x_0=y_0} = if_{abc}\,\delta(\bar{x}-\bar{y})\,[jJ]_\mu^c(x) + S(j_0, J_\mu). \tag{1}$$

Here j_μ^a, J_μ^a are vector or axial vector currents, $S(j, J)$ means possible gradient terms and $[j\ J]$ is a suitable vector or axial vector current, $[VA] = [AV] = A$, $[AA] = [VV] = V$. We suppose that the current divergences are given (*CVC, PCAC*)

$$\partial_\mu j_a^\mu = d_a, \quad \partial_\mu J_a^\mu = D_a. \tag{2}$$

To calculate the current commutators at small time differences, we subject $j_0^a(x)$ to a small time translation; then our task is reduced to the calculation of

$$\left[\frac{\partial_0^n j_0^a(\bar{x}, y_0)}{\partial y_0^n}, J_\mu^b(y)\right]$$

and this can be done for the first few derivatives.

Case $n = 1$. First, let us take $\mu = 0$ and substitute (1) and (2); we get

$$\begin{aligned}[\partial_0 j_0^a(\bar{x}, y_0), J_0^b(y)] &= [d^a(\bar{x}, y_0), J_0^b(y)] + \\ &+ i f_{abc}\,[j\,J]_k^c(y)\,\partial_k\,\delta(\bar{x}-\bar{y}) + \partial_k\,S(j_k, J_0),\ k = 1, 2, 3.\end{aligned} \tag{3}$$

The commutator on the right-hand side can be taken from the localized version of GELL-MANN, OAKES and RENNER's charge-current divergence commutator [7], so, this term carries the symmetry breaking and it is not compensated by other terms.

The case $\mu = k$ is more difficult because of the unknown (although small) $[d^a, J_k^b]$ [8]:

$$\begin{aligned}[\partial_0 j_0^a(\bar{x}, y_0), J_k^b(y)] &= [d^a(\bar{x}, y_0), J_k^b(y)] + \\ &+ i\,\alpha f_{abc}\,[j\,J]_0^c(y)\,\partial_k\,\delta(\bar{x}-\bar{y}) + \alpha\,\partial_l\,S(j_l J_k),\end{aligned} \tag{4}$$

where $\alpha = 0$ for field algebra and $\alpha = 1$ for quark algebra. For conserved SU_3 vector currents the term $[d^a, J_k^b]$ is absent.

Case $n = 2$. The corresponding commutator remains, in general, unknown (except for special cases) due to the presence of the symmetry-breaking terms.

For $\mu = 0$ one gets [9]:

$$\begin{aligned}[\partial_0^2 j_0^a(\bar{x}, y_0), J_0^b(y)] &= [\partial_0\,d^a(\bar{x}, y_0), J_0^b(y)] - \\ &- [\partial_l j_l^a(\bar{x}, y_0), D^b(y)] + if_{abc}\,\partial_0\,[jJ]_k^c(y)\,\partial_k\,\delta(\bar{x}-\bar{y}) - \\ &- i\alpha f_{abc}\,\partial_k^y(\partial_k\,\delta(\bar{x}-y)\,[jJ]_0^c(y)] + \partial_k\,\partial_0\,S(j_k, J_0) - \alpha\,\partial_l^y\,\partial_k\,S(j_k, J_l).\end{aligned} \tag{5}$$

The first two commutators on the right-hand side are absent in the case of SU_3-symmetric vector currents, $d^a = D^a = 0$, while, in general, they are determined by

$$\partial_0[d^a, J_0^b] - [d^a, D^b] - [d^a, \partial_k J_k^c] - [\partial_k j_k^a, D^b]. \tag{6}$$

We again take the first term from [7]; the second term vanishes in some simple cases, for example, by requiring canonical equal-time commutation relations for pion and kaon-field operators, etc.

In case of $\mu = k$

$$[\partial_0^2 j_0^a(\bar{x}, y_0), J_k^b(y)] = [\partial_0 d^a(\bar{x}, y_0), J_k^b(y)] + \partial_l^x[\partial_0 j_l^a(\bar{x}, y_0), J_k^b(y)]. \tag{7}$$

When $d^a \neq 0$, the first term of the right-hand side is non-vanishing [8]; the second term is known at present only in field algebra.

Finally, let us remark that the commutator $n = 3$ is completely known in field algebra for SU_3-symmetric vector currents. At the same time, in the case $n = 4$, only the commutator $\mu = 0$ is known.

Summarizing, we see that the current commutator of non-equal time depends on the model from which the equal-time commutator of the space components is calculated, and, it depends also on the value of the divergence of the current.

III. Sum rules

For the sake of simplicity we shall deal with SU_3-symmetric vector currents, $d_a = 0$, and consider $[V_0^a(x), V_l^b(y)]$, $l = 1, 2, 3$ for $x_0 = y_0$ small enough. From Section II one writes

$$\left[V_0^a\left(\frac{1}{2}x\right), V_l^b\left(-\frac{1}{2}x\right)\right] = i f_{abc}\,\delta(\bar{x})\,V_l^c\left(-\frac{1}{2}x\right) S(V_0, V_l) +$$
$$+ \alpha x_0 \partial_k \delta(\bar{x})\left[\delta_{kl} f_{abc}\, V_4^c\left(-\frac{1}{2}x\right) - i\varepsilon_{klm}\times\right. \tag{8}$$
$$\left.\times\left(\sqrt{\frac{2}{3}}\,\delta_{ab}\, A_m^0\left(-\frac{1}{2}x\right) + d_{abc}\, A_m^c\left(-\frac{1}{2}x\right)\right)\right],$$

provided x_0 is small.

Let us take (8) between spinless single particle states of the same mass and show how one gets sum rules if an expansion like (8) is given.

We introduce the following notation

$$P_\mu = \frac{1}{2}(p_{1\mu}+p_{2\mu}),\ Q_\mu = \frac{1}{2}(q_{1\mu}+q_{2\mu}),$$

$$\Delta_\mu = p_{2\mu}-p_{1\mu} \equiv q_{1\mu}-q_{2\mu},\ \nu = PQ,\ t = \Delta^2, \tag{9}$$

$$t_{0l}^{ab}(p_1;p_2;Q) = \int_{-\infty}^{\infty} e^{iQx}\left\langle p_2\left|\left[V_0^a\left(\frac{1}{2}x\right), V_l^b\left(-\frac{1}{2}x\right)\right]\right|p_1\right\rangle d^4x \tag{10}$$

and

$$g(z) = \frac{e^{iz\delta}-1}{iz}, \tag{11}$$

where δ is a small parameter, and we choose $p_1 \neq p_2$. Assuming $S(V_0, V_l)$ is a c-number, from (8) and (10) we arrive at the sum rule:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} dQ_0\, t_{0l}^{ab}(p_1;p_2;Q)\, g(k_0-Q_0) = \\ - if_{abc} P_l F^c(t)\, g\left(k_0-\frac{1}{2}\Delta_0\right) - i\alpha f_{abc} F^c(t) P_0 \times \\ \times \left(Q_l-\frac{1}{2}\Delta_l\right)\frac{dg(z)}{dz}\bigg|_{z=k_0-\frac{1}{2}\Delta_0}\end{aligned} \tag{12}$$

The axial vectors A_m^0, A_m^c and $S(V_0, V_l)$ could not give any contributions to Eq. (12).

The form factor $F^c(t)$ is defined by the equation

$$\langle p_2 \mid V_\mu^c(0) \mid p_1 \rangle = F^c(t) P_\mu. \tag{13}$$

Since in (8) we have kept only the first derivative, the sum rule (12) contains terms of order δ and δ^2, everything else has to be neglected. It is, however, important that this must be done after having calculated the integral (12), otherwise extra convergence problems could appear.

The matrix element $t_{\mu\nu}^{ab}(p_1;\ p_2;\ Q)$ can be decomposed into its invariant parts in the following way:

$$\begin{aligned}t_{\mu\nu}^{ab}(p_1;p_2;Q) = a_1 P_\mu P_\nu + a_2 P_\mu \Delta_\nu + a_3 P_\mu Q_\nu + a_4 \Delta_\mu P_\nu + a_5 \Delta_\mu \Delta_\nu + \\ + a_6 \Delta_\mu Q_\nu + a_7 Q_\mu P_\nu + a_8 Q_\mu \Delta_\nu + a_9 Q_\mu Q_\nu + a_{10} g_{\mu\nu}\end{aligned} \tag{14}$$

with invariants a_i dependent on ν, t, q_1^2, q_2^2. Substituting (14) into (12) we get:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} dQ_0(a_1P_0 + a_4\Delta_0 + a_7Q_0)g(k_0 - Q_0) &= \\ &= if_{abc}\,F(t)\left[\delta + \frac{1}{2}i\delta^2\left(k_0 - \frac{1}{2}\Delta_0\right)\right],\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} dQ_0(a_2P_0 + a_5\Delta_0 + a_8Q_0)g(k_0 - Q_0) &= \\ &= -\frac{1}{2}\alpha f_{abc}P_0F^c(t)\frac{1}{2}\delta^2,\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} dQ_0(a_3P_0 + a_6\Delta_0 + a_9Q_0)g(k_0 - Q_0) &= \\ &= \alpha f_{abc}F^c(t)P_0\frac{1}{2}\delta^2.\end{aligned} \tag{17}$$

As we have remarked, the left-hand sides of Eqs. (15)–(17) must be considered in the order δ^2 after integration. If we would expand $g(k_0 - Q_0)$ in δ, the sum rules obtained immediately from the equal-time commutators $[\partial_0^n j_0^a, J_\mu^b]$ would be exactly recovered. Keeping the terms of order x_0^2 in (8), even the terms of order δ^3 would be exact in Eqs. (15)–(17).

The advantage of the equations such as (15)–(17) is that they include information coming from the usual current algebra and the time derivatives (in the present example $n = 1$, see Eq. (4)). At the same time, they show the usual dependence on the reference system [10, 11] and also a model-dependence through α.

In what follows we derive the $P_0 \to \infty$ sum rules. First let us assume that the limit $P_0 \to \infty$ may be taken under the integrals (15)–(17); then we are led to a contradiction. To see this, we first define the $P_0 \to \infty$ system [11]

$$\begin{gathered}P^\mu = (\gamma\sqrt{P^2},\ 0,\ 0,\ -\gamma\sqrt{P^2}),\ \Delta_\mu = (0,\ \Delta_1,\ \Delta_2,\ 0),\\ Q_\mu = (Q_3,\ Q_1,\ Q_2,\ Q_3),\ dQ_0 = \frac{d\nu}{\gamma\sqrt{P^2}},\\ \gamma \to \infty.\end{gathered} \tag{18}$$

q_1^2, q_2^2 become independent of ν. Then, we get from Eq. (15) in the limit $P_0 \to \infty$

$$\frac{1}{2\pi}g(k_0 - Q_3)\int_{-\infty}^{\infty} d\nu\, a_1 = if_{abc}F^c\left[\delta + \frac{1}{2}i\delta^2k_0\right]; \tag{19}$$

since $Q_3 \neq 0$, this could be satisfied only by a vanishing form factor. In the same way, Eqs. (16) and (17) are consistent only if $\alpha F^c = 0$, that is in field algebra.

So, we conclude that, in general, the infinite mass contributions are necessary for the consistency of the $P_0 \to \infty$ sum rules [11]. That is, interchanging the limit $P_0 \to \infty$ and the integration over Q_0, the contributions of the disconnected diagrams must be separately taken into account. Let us denote by a_i^d the disconnected contributions to the invariants a_i; then the contributions of the disconnected graphs to the integrals (15)–(17) in the limit $P_0 \to \infty$ are in turn:

$$\lim_{P_0 \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dQ_0 (a_1^d P_0 + a_7^d Q_0)\, g(k_0 - Q_0) = A_1 \delta + \frac{1}{2} i\delta^2 A_2 , \tag{20}$$

$$\lim_{P_0 \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dQ_0 (a_2^d P_0 + a_8^d Q_0)\, g(k_0 - Q_0) = B_1 \delta + \frac{1}{2} i\delta^2 B_2 , \tag{21}$$

$$\lim_{P_0 \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dQ_0 (a_3^d P_0 + a_9^d Q_0)\, g(k_0 - Q_0) = C_1 \delta + \frac{1}{2} i\delta^2 C_2 , \tag{22}$$

where

$$Q_0 = \frac{\nu}{\gamma \sqrt{P^2}} - \beta Q^3 , \quad \gamma = (1-\beta^2)^{-\frac{1}{2}} . \tag{23}$$

The functions $A_1, \ldots, C_2$ are defined by Eqs. (20)–(22), their detailed forms depend on the dynamics. In pole approximation we know that the disconnected contributions are finite and we have seen that at least one of A_1 and A_2 (B_1 and B_2, C_1 and C_2 in the quark model) is non-vanishing.

Finally, comparing the coefficients of δ and δ^2 in (15)–(17), we get the following sum rules for $P_0 \to \infty$:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu a_1 + A_1 = i f_{abc} F^c , \tag{24}$$

$$\frac{Q_3}{2\pi} \int_{-\infty}^{\infty} d\nu a_1 + k_0 A_1 = A_2 , \tag{25}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu a_2 + B_1 = 0 , \tag{26}$$

$$(Q_3 - k_0)\, B_1 + B_2 = \frac{1}{2} i\alpha P_0 f_{abc} F^c , \tag{27}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu a_3 + C_1 = 0 , \tag{28}$$

$$(Q_3 - k_0) C_1 + C_2 = - i\alpha P_0 f_{abc} F^c . \tag{29}$$

Here (24), (26), (28) correspond to the usual current algebra sum rules [12], while the others are new ones. For $\alpha = 1$ the right-hand sides of Eqs. (27) and (29) become infinite, since B_2 and C_2 cannot compensate these infinities; only the field algebra ($\alpha = 0$) is consistent with the $P_0 \to \infty$ sum rules [13]. It is, however, important to emphasize that at this conclusion we assume the existence of the integrals of a_2 and a_3 in Eqs. (26) and (28).

IV. Discussion

In the present paper we have shown that the equal-time current algebra and the divergence conditions determine the non-equal-time current commutators in a small strip along the space-axis. The model-dependence and the symmetry-breaking terms in the generalized current commutators were shown. The purpose of Section III was to present such a method which expresses the non-equal-time commutators in terms of sum rules. These generalized sum rules contain also the information coming from the time derivatives [14] and they are significant, for instance, in connection with the theory of high-energy inelastic scattering.

In the specific example based on Eq. (8) we deal with conserved vector currents and get the sum rules (15)–(17) and (24)–(29), respectively, as restrictions for the scattering of scalar and vector particles. Although they verify our general statements above, their immediate evaluation cannot be carried out because of the lack of the experimental data concerning the scattering of spinless and spin-one octet particles. Nevertheless, it is important to establish the crucial role of the disconnected diagrams in (24)–(29) and that the sum rules are compatible with the field algebra provided (26) and (28) converge.

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КОММУТАТОРЫ ТОКА ПРИ МАЛЫХ ИНТЕРВАЛАХ ВРЕМЕНИ

И. ФАРКАШ и Г. ПОЧИК

Резюме

Используя алгебру токов при равенстве времен, и условия расходимости, рассчитаны коммутаторы тока для малых интервалов времени. Показано, что коммутаторы явно зависят от модели, и вклады от членов, нарушающих симметрию не отнадают. Рассмотрено физическое содержание коммутаторов тока в терминах новых правил сумм. Показано, что для последовательности правил сумм необходимо наличие вкладов несвязанных диаграмм. Правила сумм дают предпочтение алгебре поля.