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POSSIBLE RELATIONS BETWEEN CURRENT ALGEBRA AND MESON POLE DOMINANCE

By

B. RENNER

INSTITUTE 2 FOR THEORETICAL PHYSICS, THE UNIVERSITY, HAMBURG
GERMAN FEDERAL REPUBLIC

The suggestion is made there is an overlap among the information obtained from current algebra and from meson pole dominance principles. In simple models it is shown that current algebra results can be rederived from meson pole dominance principles alone, up to an unknown scale of the weak axial current. A non-compact alternative to the usual $SU_2 \times SU_2$ current algebra is shown to be inconsistent with meson pole dominance.

Introduction

In this lecture I will report on work in progress. Its interpretation is still tentative, its further development is yet uncertain. For the steps taken so far in the present program, the credit is to be shared with J. ELLIS and J. DE AZCARRAGA, who have contributed many of the arguments I will present in the following.

To motivate the program, we recall some steps in the development of current algebra. When the theory was proposed by GELL-MANN, it was presented in terms of basic quantum mechanical principles: an assumption of simple commutators for simple observables. Owing to a lack of experimental data on photon and neutrino reactions it remained essentially untestable for over two years, until ADLER and WEISBERGER utilized the approximation of pion pole dominance for the divergence of the weak axial current to derive their celebrated sum rule. In the subsequent developments which include in successive stages soft-pion theorems, hard pion calculations and chiral Lagrangians, the principle of pion pole dominance gained a central role, and Nambu's original interpretation as partial conservation of the weak axial vector current was revived indicating the proximity of a chiral $SU_2 \times SU_2$ symmetry limit as the pion mass is taken to zero.

So far, pion pole dominance has been used mainly in an auxiliary capacity to test the current algebra commutators. To my knowledge, it was MANDELSTAM who first explicitly turned the argument round and demonstrated the implications of pion pole dominance on current algebra. In some generality he showed that in the zero pion-mass limit the commutator of two weak axial currents $A^i_\mu(x)$

$$[\int A^i_0(x)\,(dx)^3,\ A^j_\mu(y)] = ie^{ijk}\, V^k_\mu(y) \tag{1}$$

gives rise to a conserved vector current as a consequence of Adler zeros.

This statement becomes trivial in theories where the limit of massless pions implies conservation of the axial current, because the commutator of a conserved charge with a conserved current necessarily produces a conserved current. Later, DASHEN and DASHEN and WEINSTEIN argued that it is only in such theories with approximate chiral symmetry where the use of pion pole dominance is plausible. To illustrate their point of view, let us consider the Goldberger–Treiman relation

$$(2m_N)(g_A/g_V) \approx -(\sqrt{2}\,F_\pi)(\sqrt{2}\,G_{NN\pi}, \tag{2}$$

which connects the axial vector coupling constant (g_A/g_V) in nucleon decay with the pion decay constant $(\sqrt{2}\,F_\pi)$ and the charged pion–nucleon Yukawa coupling constant $(\sqrt{2}\,G_{NN\pi})$. The most popular, though perhaps not most considerate derivation starts by assuming an unsubtracted dispersion relation in momentum transfer for the form factor of the weak axial divergence

$$\langle p\,|\partial A^W|\,n\rangle = i(2\,m_N)\bar{U}_P\gamma_5 G(\Delta^2)U_n \tag{3}$$

with

$$G(0) = (g_A/g_V)$$

and

$$2m_N\,G(\Delta^2) = \frac{(\sqrt{2}\,F_\pi\,m_\pi^2)\,(\sqrt{2}\,G_{NN\pi})}{\Delta^2 - m_\pi^2} + \frac{1}{2\pi i}\int_{9m_\pi^2}^{\infty}\frac{\operatorname{disc} G(s)}{s - \Delta^2} \tag{4}$$

and then retains only the pion pole contribution for $\Delta^2 \approx 0$, disregarding the three-pion and higher cuts (and any anomalous thresholds) which start at $\Delta^2 = 9m_\pi^2$ and above. The 10% disagreement of the Goldberger–Treimann relation is then sometimes quoted as an illustration of the principle that the influence of singularities can be estimated by the inverse of their distance from the point of comparison. Such a principle, however, would entirely disregard the possibility of the singularities having different strengths, i.e. different sizes of pole residues and cut discontinuities. This is particularly relevant in the present case, as the pion pole residue contains the factor m_π^2 which is to be considered small to the same extent as the pole denominator is considered small. To maintain pole dominance, we require that the cut discontinuity be similarly small, i.e. of the order $0(m_\pi^2)$ like the pole residue. To see how stringent a requirement this is, we consider a model for some typical cut contributions to the Goldberger–Treiman relation.

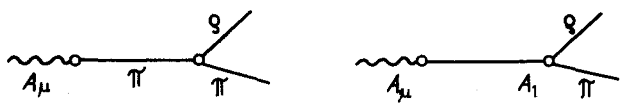
From among the contributions to the three-pion cut

$$\operatorname{disc} G(s) \sim \sum_N \delta(m_N^2 - s) < 0\,|\partial A^W|\,N\rangle\ \langle N\,|\,n\bar{p}\rangle \tag{5}$$

we select the intermediate states $|\varrho\pi\rangle$. Just to estimate orders of magnitude, let us suppose that the strong interaction matrix element $\langle\varrho\pi\,|\,n\bar{p}\rangle$ is adequately characterized as being of the same order of magnitude as the pion–nucleon coupling constant $G_{NN\pi}$; we then have to demand that $\langle 0 \mid \delta A \mid \varrho\pi\rangle$ should be of order $0(m_\pi^2)$, to make the Goldberger–Treiman relation plausible. To illustrate that this is not obvious, we rewrite

$$\langle 0 \mid \delta A \mid \varrho\pi\rangle = (i)\ (p^\varrho + p^\pi)_\mu\ \langle 0 \mid A^\mu \mid \varrho\pi\rangle \tag{6}$$

and we make a simple model for the terms $\langle 0 \mid A^\mu \mid \varrho\pi\rangle$ of two Feynman graphs (Fig. 1):



Substituting contemporary estimates for the coupling constants, we find that each one of the contributions exceeds considerably the required order $0(m_\pi^2)$, and the Goldberger–Treiman relation can only be plausibly maintained if there is a near-cancellation among them. The only understood mechanism — to our present knowledge — to provide such a cancellation is the proximity of our world to a chiral symmetry limit with all matrix elements of ∂A being proportional to m_π^2. Indeed, we find the required cancellation if we substitute the coupling constants and their signs as prescribed by the appropriate chiral Lagrangians. The details are presented in [8].

In summary we find an interesting situation: pion-pole dominance may be exactly valid only in the limit of chiral symmetry. The same effects which are responsible for the non-conservation of the axial current, also limit the application of pion-pole dominance to its divergence. Turning this argument round, we may arrive at a working hypothesis which generalizes MANDELSTAM's result (see above) to theories with finite pion mass: To the same accuracy as we accept pion-pole dominance, we are bound to find the vector current in the commutator of two axials [Eq. (1)] conserved, or alternatively: any possible non-conservation of the vector current in the commutator of two axials is associated with corrections to pion-pole dominance and cannot be seen in the tests where pion-pole dominance is used as an approximation.

This is an example of how statements on current algebra and meson-pole dominance can become interrelated. We suggest that there is a certain overlap of information. Although current algebra by itself may be formulated in a variety of theories with or without pion-pole dominance, the principle of pion-pole dominance for the axial divergence — which is not obvious by itself — implies definite information on possible current algebra structures. Our objective will be to study this information in definite models and also to extract further constraints following from the related hypothesis of vector and axial vector meson dominance for the transversal components of the currents.

There is one piece of information, however, characteristic of current algebra, which we can never hope to extract from meson-pole dominance or related dynamical principles: the specification of the scale of the weak axial current. This is because we have always been treating weak interactions exacty. Recalling that on this level weak couplings enter the unitarity relations only in a linearized form, we see that we can never determine their scale.

The best we can, therefore, hope to deduce from meson-pole dominance orl related principles is a statement that, within the model considered, we can make the same deductions as if we had assumed the following current algebra:

$$[\int \hat{A}_0^i(x)\,(dx)^3,\ \int \hat{A}_0^j(x)\,(dx)^3] = ie^{ijk}(\alpha^2)\int \hat{V}_0^k(x)\,(dx)^3. \tag{7}$$

We have denoted the currents of the model by a caret, to distinguish them from the physical currents for which we have $\alpha = 1$, according to GELL-MANN. Obviously, once we have Eq. (7), we can always replace $\hat{A}_\mu^i$ by $\alpha\hat{A}_\mu^i = A_\mu^i$ by merely rescaling weak coupling constants. To first order, this does not cause any inconsistencies.

At present, we are unable to make any further statements on general grounds, and we turn to the study of specific models.

1. Tree-graph model for pion amplitudes

Most of the credit for this Section is due to J. ELLIS.

The prediction of pion scattering amplitudes has been a very fruitful field of application for current algebra. Using chiral Lagrangians, we obtain a tree graph model, with the one-pion irreducible parts taken to second order in the momenta. This prediction is proposed to be valid for low energies: $p^2 \ll m_\varrho$.

We want to see to what extent we can reproduce such a model without assuming current algebra. The only assumption we will make is the postulate of ADLER zeros: a pion amplitude is required to vanish wherever we extrapolate one pion to zero energy-momentum and leave to the others on mass-shell. As it is well known, ADLER zeros are a consequence of pion-pole dominance for the divergence of the axial current.

$$\langle f|\pi(P), i\rangle = \frac{(p^2-m_\pi^2)}{F_\pi\, m_\pi^2}\langle f|\partial A|i\rangle = \frac{(p^2-m_\pi^2)}{F_\pi\, m_\pi^2}\cdot(iP)^\mu\langle f|A_\mu|i\rangle, \tag{8}$$

$$\to 0 \text{ as } p \to 0.$$

As a preliminary, we study the four-pion amplitude. To second order in the momenta, on and off-shell, its most general form has been given by WEINBERG

$$\langle \pi_\alpha \pi_\beta | \pi_\gamma \pi_\delta \rangle = \delta_{\alpha\beta}\delta_{\gamma\delta}(A + B(p_\alpha + p_\beta)^2 + C\left((p_\alpha - p_\gamma)^2 + (p_\alpha - p_\delta^2)\right) + \\ + \text{(permutations in isospin indices)}. \tag{9}$$

ADLER zeros give the constraint

$$A + (B + 2C)m_\pi^2 = 0. \tag{10}$$

From the scale of the axial current [Eq. (1)], WEINBERG deduces

$$(C - B) = 1/F_\pi^2. \tag{11}$$

To fully determine the amplitude, yet another input is needed: an assumption on the mode of chiral symmetry breaking. Excluding σ terms of isospin two from the four-pion amplitude, WEINBERG derives $C = O$. This, however, is not a deduction from current algebra alone, and any value of C can in fact be reproduced from a standard chiral Lagrangian by making a suitable assumption on the mode of chiral symmetry breaking.

We see that for the four-pion amplitude, Eq. (11) is the only deduction from current algebra, not implied in meson pole dominance [Eq. (1)]. Through the size of the pion decay constant F_π, it fixes the scale of the weak axial current in terms of strong interaction parameters, as discussed above, yet it contains a further piece of information: the fact that $(C - B)$ is positive. This is necessary for a compact $SU_2 \times SU_2$ current algebra to hold, rather than a non-compact SO (3, 1) or E_3. As J. ELLIS has shown in detail, the construction of chiral Lagrangians can be performed without difficulties with these generalized current algebras, and ADLER zeros can be guaranteed by a suitable choice of the pion interpolating field

$$\partial A^i = F_\pi^2 m_\pi^2 \varphi_\pi^i. \tag{12}$$

Nevertheless, there is a striking physical difference between theories with compact and non-compact current algebras, which cannot be removed just by rescaling pure weak-interaction quantities. Consider, for instance, the ADLER WEISBERGER relation for $(\pi\pi)$ scattering which would follow from a generalized current algebra:

$$\left[\int A_0^i(x)\,(dx)^3, \int A_0^j(y)\,(dy)^3\right] = ie^{ijk} \cdot \alpha \cdot \int V_0^k(x)\,(dx)^3. \tag{13}$$

$$\alpha = (F_\pi)^2 \int_{(2m_\pi)^2}^{\infty} \frac{ds}{(s - m_\pi^2)^2} \left(\sigma_{\pi^+\pi^-}^{\text{tot}}(s) - \sigma_{\pi^+\pi^+}^{\text{tot}}(s)\right) \frac{2P_\pi^L m_\pi}{\pi}. \tag{14}$$

Obviously $\alpha \leq 0$ would demand $\sigma^{tot}_{\pi^+\pi^+}s \geq \sigma^{tot}_{\pi^+\pi^-}$, at least in average, which is totally at odds with our present understanding. By assuming absence of exotic states, the non-compact current algebra can be excluded fairly trivially (since noncompact algebras cannot have finite dimensional unitary representations), but we will see in the next section that we can also exclude them through vector meson dominance — which may be less trivial.

To complete the discussion of the pion model, we ought to see whether the present balance of information continues to hold for many-pion amplitudes: given any three-graph model for pion interactions with second order polynomials in the momenta to approximate the one-pion irreducible parts and with ADLER zeros, we can always find a generalized chiral Lagrangian which reproduces the model, utilizing three sources of indeterminacy:

1. the scale of the axial current (α) in terms of Eq. (13);
2. the compactness property of the algebra (sign α in terms of Eq. (13));
3. the mode of chiral symmetry breaking.

For the four-pion amplitude, this statement follows trivially from the proceding discussion: for the $(2N)$ pion amplitude let us use an inductive argument to balance the degrees of freedom.

Assuming that we have chosen the amplitudes for $(2N-2)$ pions in accordance with ADLER zeros, we have no longer any freedom in constructing the reducible parts of the $(2N)$ pion amplitudes, and new information may come only from an irreducible "contact" contribution. Assuming that there are two possible choices, say C_1 and C_2, for this contact term, we realize that the difference $(C_1 - C_2)$ must have all the ADLER zeros by itself. As J. ELLIS has shown in detail, there is only one such form for $(2N) \geq 6$ pions:

$$(C_1 - C_2) = \gamma_N \left\{ \left(\sum_{i=1}^{2N} p_2^i \right) - (2N-1) m_\pi^2 \right\} \tag{15}$$

with an unspecified multiplicate constant γ_N. We now realize the restrictive power of the requirement of ADLER zeros: only N constants are left free in the construction of a $(2N)$-pion amplitude.

The same can be seen to be the case for (generalized) chiral Lagrangians. We concentrate here on the case of $SU_2 \times SU_2$; the cases of $SO(2,2)$ and E_3 are similar. Let us recall some details in the construction of chiral Lagrangians.

Once the pion field has been chosen according to Eq. (11) to guarantee ADLER zeros, and the scale of the axial current has been set by identifying the pion decay constant F_π, the kinetic part of the chiral Lagrangian is uniquely determined in terms of covariant derivatives, and the only freedom left is in the construction of the chiral symmetry breaking generalized pion mass term

$$H^{(B)} = \sum_{n=1}^{\infty} a_n H_n, \tag{16}$$

where the operators H_n are the isoscalar parts of the $SU_2 \times SU_2$ operator multiplets $(n/2, n/2)$. Their construction in terms of pion fields

$$H_n = \sum_{i=0}^{\infty} b_n^i (\varphi^2)^i \tag{17}$$

is uniquely fixed, once Eq. (11) has been imposed: however, their relative weights are unconstrained.

Collecting terms of second order in the pion fields we set

$$\frac{m_\pi^2}{2} = \sum_{n=1}^{\infty} a_n\, b_n^1 . \tag{18}$$

Terms of N^{th} order give just one unspecified contact term

$$C_N = \sum_{n=1}^{\infty} a_n\, b_n^N \tag{19}$$

which can be given any value, to match the indeterminacy of γ_N in Eq. (15) by suitably adjusting the parameters a_n. This completes the present demonstration; more detailed arguments can be found in [8] and [9].

Before leaving the subject, let us recall that despite the indeterminacies left after imposing ADLER zeros in pion amplitudes, MANDELSTAM's suggestion has been verified in the present model: the theory with meson-poles dominanc is always equivalent to one where the commutator of two axial currenaeproduces a conserved vector current.

2. The $\langle$A, A, V$\rangle$ vertex

As a first step to include effects of vector and axial vector meson dominance, we investigate a classic object: the vertex of one vector and two axial currents. After many attempts, its structure has been clarified by SCHNITZER and WEINBERG with the joint use of current algebra and a specific form of meson-pole dominance. As regards meson-pole dominance, we shall make the same assumptions as SCHNITZER and WEINBERG did: the only singularities we allow will be poles of A_1, ϱ and π-mesons and the irreducible vertices will be constrained to be low-order polynomials in the momenta in the problem.

As regards current algebra, we continue to assume the isospin algebra and the commutators of axial vector currents with vector currents, but we do not make any specific assumption about the commutator of two axial currents

$$[\textstyle\int A_0^i(x)\,(dx)^3,\, A_\mu^j(y)] \stackrel{\text{def}}{=} i\widetilde{X}_\mu^{ij}(y) . \tag{20}$$

$\widetilde{X}^{ij}_{\mu}(y)$ is taken to be some unspecified vector current; only its isospin-one component enters the problem.

It can easily be seen that the commutator of two axial currents contributes only in one Ward identity:

$$\begin{aligned} 0 = ip^{\mu} \cdot \iint (dx)^4 (dy)^4 e^{ipx} e^{ipy} \langle 0 | T^{x} \{A^{i}_{\mu}(x), A^{j}_{\nu}(y), V^{k}_{\lambda}(0)\} | 0 \rangle + \\ + \iint (dx)^4 (dy)^4 e^{ipx} e^{ipy} \langle 0 | T^{x} \{\partial A^{i}(x), A^{j}_{\nu}(y), V^{h}_{\lambda}(0)\} | 0 \rangle + \\ + i \iint (dx)^4 e^{i(p+q)x} \langle 0 | T^{x} \{\widetilde{X}^{ij}_{\nu}(x), V^{k}_{\lambda}(0)\} | 0 \rangle \\ - i e^{ijk} \iint (dy)^4 e^{iqy} \langle 0 | T^{x} \{A^{j}_{\nu}(y), A^{i}_{\lambda}(0)\} | 0 \rangle . \end{aligned} \tag{21}$$

In the third term, only intermediate states of total spin-one can contribute, since for all others $< 0 \mid V^{k}_{\lambda} \mid s >$ would vanish. So only the transversal components of $\widetilde{X}^{y}_{\nu}$ enter the calculation; longitudinal components, if at all possible, cannot contribute. So, within the framework of the present calculation, $\widetilde{X}^{ij}_{\nu}$ acts like a conserved current of isospin one

$$\widetilde{X}^{ij}_{\mu} = e^{ijk} X^{k}_{\mu}. \tag{22}$$

Using now our specific meson pole structure, we see that the ϱ-meson is the only possible intermediate state in the third term of Eq. (21) to cause a singularity in $(p + q)^2$. This has the effect of confining all the information about X_{μ} to a single coupling constant:

$$\langle 0 \mid X^{k}_{\mu} \mid \varrho \rangle = \alpha \langle 0 \mid V^{k}_{\mu} \mid \varrho \rangle = \langle 0 \mid \alpha V^{k}_{\mu} \mid \varrho \rangle \tag{23}$$

with the parameter a being defined through Eq. (23). So, we find again the same predictions from meson pole dominance alone as if we had assumed in addition:

$$[\int A^{i}_{0}(x)\, (dx)^3, A^{j}_{\mu}(y)] = ie^{ijk} \alpha\, V^{k}_{\mu}(y) . \tag{24}$$

The non-trivial aspect of this model is the fact that there we can demonstrate that α has to be positive for consistency. The details of the algebra are unfortunately somewhat long; they have been presented in a paper by J. DE AZCARRAGA and the present author. The important fact is that, for consistency of the solution of the Ward identity in terms of reduced vertices, the first Weinberg sum rule has to be satisfied, which reads in the present case:

$$F^{2}_{\pi} + \frac{f^{2}_{A}}{m^{2}_{A}} = \alpha \frac{f^{2}_{\varrho}}{m^{2}_{\varrho}} . \tag{25}$$

Positivity requires $\alpha \geq 0$; a non-compact alternative to the usual $SU_2 \times SU_2$ current algebra is not compatible with vector meson dominance.

3. Hopes and conclusions

It is fairly difficult to judge what has been achieved so far. The outcome will lie somewhere between two extremes:

1. It may turn out that the systems studied so far have reproduced the current algebra structures from meson pole dominance because of their limited degrees of freedom. As soon as we study more complicated systems, new sources of indeterminacy will open up. Even in this case we would attach some value to the recognition that the examples so far studied, which have been chosen from among the standard applications of current algebra, have not tested the theory to any greater extent than to a scale factor. We would have to call even more intensely for neutrino data to properly test its validity.

2. It may also turn out that a very considerable amount of current algebra structures may be deduced from a dynamical hypothesis like meson pole dominance. Dominance of single mesons in certain channels is not the most realistic principle, however, in our contemporary understanding of strong-interaction dynamics. Both the recent successes of duality theories with infinite recurrences and the still unexplained dipole fits to nucleon electromagnetic form factors clearly demonstrate the need for a different dynamics than based on nearby singularities. The hope remains, however, that the present model studies may lead to the formulation of new concepts which may eventually allow us to search for relations between current algebra and more realistic dynamical theories, like those based on duality.

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О ВОЗМОЖНЫХ СВЯЗЯХ МЕЖДУ АЛГЕБРОЙ ТОКОВ И АППРАКСИМАЦИЕЙ МЕЗОННЫМИ ПОЛЮСАМИ

Б. РЕННЕР

Резюме

Предложена идея, согласно которой между информациями, получаемыми из алгебры токов и принципов аппраксимации мезонными полюсами имеется частичное совпадение. На простых моделях показано, что результаты алгебры токов с точностью до неизвестной калибровки слабого аксиального тока могут быть получены для мезонов из одних принципов аппраксимации мезонными полюсами. Доказывается, что некомпактный вариант обычной $SU_2 \times SU_2$ алгебры токов является несовместимым с аппраксимацией мезонными полюсами.