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SOME REMARKS ON "ENERGY-DEPENDENT" REPRESENTATIONS

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After explaining the meaning of energy dependent representation, we sketch how it can be obtained for the case of the SL(2, C) group. Some physical applications are also treated.

Some time ago we examined the problem how one can expand a general two-particle—two-particle scattering amplitude in terms of Lorentz group representations at any s and t values [1]. To do this, first one has to define the scattering amplitude as a function on the group in question. As we noticed in [1], a possible and in some sense desirable way is the following:

$$f_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t) = \langle p_1 s_1 \lambda_1, p_2 s_2 \lambda_2 | T | p_3 s_3 \lambda_3, p_4 s_4 \lambda_4 \rangle = \\ = \langle \tilde{P} = \tilde{p}_1 + \tilde{p}_2, \tilde{p}_1 - \tilde{p}_2, s_1 \lambda_1 s_2 \lambda_2 | \Lambda T | \tilde{P}'_1 \tilde{p}_3 - \tilde{p}_4, s_3 \lambda_3 s_4 \lambda_4 \rangle = f_H(\Lambda) \tag{1}$$

where

$$\tilde{P} = \left(\frac{m_1+m_2}{2\sqrt{m_1 m_2}} \sqrt{s-(m_1-m_2)^2},\ 0,\ 0, \frac{m_1-m_2}{2\sqrt{m_1 m_2}} \sqrt{s-(m_1+m_2)^2} \right) = \qquad \text{if } s \geqslant (m_1+m_2)^2$$

$$= \left(\frac{m_1+m_1}{2\sqrt{m_1 m_2}} \sqrt{(m_1+m_2)-s},\ 0,\ 0, \frac{m_1-m_2}{2\sqrt{m_1 m_2}} \sqrt{(m_1-m_2)^2-s} \right) \qquad \text{if } s \leqslant (m_1-m_2)^2 .$$

Λ is a Lorentz transformation acting on two-particle states, its detailed form together with $\tilde{p}_1—\tilde{p}_2$, $\tilde{p}_3—\tilde{p}_4$ can be found in [1], $\tilde{P}' = \tilde{P}(1 \leftrightarrow 3,\ 2 \leftrightarrow 4)$. If we now perform the expansion using the ordinary $|j_0 \sigma j m\rangle$ basis* of the Lorentz group, we get

$$f_H(\Lambda) = \sum \mathfrak{D}^{j_0\sigma}_{jmj'm'}(\Lambda) T_H(s, j_0\sigma \ldots). \tag{2}$$

For continuous variables, like σ, integration is meant in Eq. (2). The expansion coefficients, i.e. the T functions in Eq. (2) are quite complicated due to $\langle \tilde{P} \ldots / j_0 \sigma j m \ldots \rangle$ type coefficients in it. The main problem with the

* In the $|j_0 \sigma j m\rangle$ basis, j_0 and σ characterize an irreducible representations. To label the vectors of a representation space one chooses a subgroup of the Lorentz group; generally it is the rotation group. The $|jm\rangle$ states are representations of the rotation group. For further details, see, e.g. [3].

latter is that the little group of $\tilde{P}$ is different from that chosen to label vectors in irreducible Lorentz representation spaces. To simplify the expansion we have to make the two groups identical. Since the little group is given, this is what we want to introduce as "basis labelling" group. This problem, together with finding the representation matrix elements, was solved in [2].

As the first step, let us write $\tilde{P}$ in the following way:

$$\tilde{P} = f(s)\left(\frac{1+v}{2}, 0, 0, \frac{1-v}{2}\right), \tag{3}$$

then $\tilde{P}^2 = s = f^2(s)\, v$. Comparing Eq. (3) with Eq. (2):

$$f(s) = [(m_1 + m_2)\sqrt{s - (m_1 - m_2)^2} + (m_1 - m_2)\sqrt{s - (m_1 + m_2)}\,]^2/2\sqrt{m_1 m_2} \qquad \text{if } s \geqslant (m_1 + m_2)^2 \tag{4}$$

and a similar expression below the pseudothreshold, hence

$$v = s/f^2(s) \tag{5}$$

is a nice function of s.

Let M_i and N_i be the generators of the Lorentz group commuting as

$$[M_i, M_j] = i\varepsilon_{ijk}M_k, \; [M_i, N_j] = i\varepsilon_{ijk}N_k, \; [N_i, N_j] = -i\varepsilon_{ijk}M_k.$$

Here M_i generates the rotations, N_i the boosts. It is not hard to find out that the little group of P is generated by

$$S_1 = \frac{1+v}{2}M_2 + \frac{1-v}{2}N_1, \quad S_2 = -\frac{1+v}{2}M_1 + \frac{1-v}{2}N_2 \qquad S_3 = M_3 \tag{6}$$

commuting as

$$[S_1, S_2] = ivS_3, \; [S_1, S_3] = -iS_2, \; [S_2, S_3] = iS_1. \tag{7}$$

The Casimir operator of this subgroup, what we call sometimes interpolating group, IG, is $S_1^2 + S_2^2 + vS_3^2$. The structure of this interpolating group and the corresponding algebra as well as its basis depend on the parameter v which is proportional to s. One can check that in the $s > 0$ region it is isomorphic to SU(2), at $s = 0$ to E(2), in the $s < 0$ region to SU(1, 1) — if the masses are unequal. As s and, together with it, v vary, this group interpolates between them smoothly.

However, if the masses are equal, $v = s/|s|$, hence going to $s = 0$ from above, we always have SU(2), and coming from below always SU(1, 1). Now, since at $s = 0$ both SU(2) and SU(1, 1) are little groups, their minimal extension is a little group too, which is SL(2, C). This fact is well-known, but it is amusing to recover it in this way.

If we choose the interpolating group as basis labelling group, then the basis of the Lorentz group will also depend on s, as well as its representations; and in this sense we shall work with energy-dependent representations.
The need for energy dependent representations appears not only in this case. Examination of the dynamical symmetries of a charged spinless harmonic oscillator in a constant magnetic field presents a similar problem [4]. Such representations are also useful when treating the H-atom problem [5].

In the following, we shall briefly sketch how one can find the explicit basis functions. The details can be found in [2]. We start with the basic formula for SL(2, C) representation [6]:

$$U_g \Phi(z) = (\beta z+\delta)^{J_0+\sigma-1} \, \overline{(\beta z+\delta)}^{\,-J_0+\sigma-1} \, \Phi\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right), \tag{8}$$

where $\Phi(z)$ are infinitely differentiable functions of one complex variable,

$$g = \begin{pmatrix} \alpha\, \beta \\ \gamma\, \delta \end{pmatrix} \in \mathrm{SL}(2, \mathrm{C}), \ \alpha\delta - \beta\gamma = 1.$$

Relation (8) does not specify the functions $\Phi(z)$. To do this, we can set additional equations.

First of all we derive from Eq. (8) the form of the generators as differential operators acting on $\Phi(z)$. If we form out of them the operators $M_i N_i$ and $M^2 - N^2$, we get simple numbers, so the function space of Φ is irreducible.

To define basis functions we form the S_i operators and set

$$(S_1^2 + S_2^2 + vS_3^2)\, \Phi = vj(j+1)\, \Phi, \ S_3\Phi = m\Phi. \tag{9}$$

If we require that Φ should be one valued and regular at $z = 0$, we obtain Φ uniquely. From Eq. (9) we can learn that the Φ functions serve not only as basis functions for SL(2, C) representations, but at the same time they are representation functions of the interpolating group.

From our basic relation it is not hard to see that if we restrict ourselves to the SU(1, 1) subgroup of the Lorentz group, the z plane breaks up to two disjoint regions since

$$\begin{gathered} |z'| > 1 \text{ if } |z| > 1 \text{ and } |z'| < 1 \text{ if } |z| < 1, \\ z' = \frac{\alpha z + \bar\beta}{\beta z + \bar\alpha} \quad \begin{pmatrix} \alpha & \bar\beta \\ \bar\beta & \bar\alpha \end{pmatrix} \in \mathrm{SU}(1,1) \subset \mathrm{SL}(2,\mathrm{C}). \end{gathered} \tag{10}$$

This means that the SU(1, 1) type basis will appear with multiplicity two.

We do not think it is necessary to give here the detailed form of the Φ functions, as they are rather complicated. The inquirer can found it in [2]. But we emphasize again that though the structure of the algebra changes radically with v, the Φ functions change smoothly, without the appearing of any type of singularities.

One may ask whether our representation is unitary or not. This depends on whether one can or one cannot introduce a scalar product into the space of the functions. It turns out that it is possible only for special j_0 and σ values; for the others, the positive definiteness condition cannot be maintained; hence we have to work with some generalization of the scalar product what is called invariant bilinear functional [6]. Using this, we were able to calculate the normalization of the basis, the finite group representation matrix elements and overlap functions between different bases [2].

Finishing our discussion we make some remarks about the application of this formalism for generalized partial wave analysis.

If in the expansion of the scattering amplitude we use energy dependent representations instead of $|j_0\sigma jm>$ basis, the T functions of Eq. (2) will be simple reduced matrix elements, which, because of the Wigner—Eckart theorem depend only on the Casimirians. As the scattering amplitude is expanded in terms of Lorentz representations, one tends to write Lorentz Casimirians into T. However, the scattering amplitude has a larger symmetry group, the Poincaré one, and our group is its subgroup. Hence the reduced matrix elements depend only on Poincaré Casimirians, e.g. on s and W^2.

Since T does not depend on j_0 and σ, we can perform the summation on it in Eq. (2); in this way we get an expansion of the scattering amplitude in terms of the interpolating group. This result seems to be trivial, but the whole procedure is not useless.

We can conclude that the generalization of the ordinary partial wave analysis is not singular at $s = 0$, whereas in the unequal case the old fashioned one is; consequently, the introduction of Toller or Lorentz poles is not the only theoretical way-out. Secondly, as the reduced matrix elements do not depend on the SL(2, C) Casimirians in general, the introduction of Lorentz poles is a bit artificial. However, as generally at $s = 0$ a large number of Regge poles contribute with almost equal weight, the SL(2, C) expansion can be useful to handle them since it can correlate them, at least in some sense. The detailed description of these and further results will be published elsewhere [1b, 7].

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НЕКОТОРЫЕ ЗАМЕЧАНИЯ ОТНОСИТЕЛЬНО ПРЕДСТАВЛЕНИЙ ЗАВИСЯЩИХ ОТ ЭНЕРГИИ

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Резюме

После объяснения понятия «представление зависящее от энергии» рассматривается, как оно может быть получено для случая групп SL (2, C). Описываются некоторые физические применения.