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SOME PROBLEMS OF INVESTIGATING PERIODICITIES OF COSMIC RAY TIME SERIES*

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A generalization of the method of the analysis of variance is given to investigate the existence and the shape of a periodicity with given length of period. Allowance is made for slow variations of the intensity of cosmic rays as well as for meteorological effects. In addition to the exact test of the existence of the periodicity, maximum likelihood estimates both of the constants characterizing the shape of the periodicity and of the mean square amplitude of the periodic function are given, together with their respective statistical errors, in the case of an arbitrary number of meteorological factors affecting the intensity of the cosmic radiation.

Disadvantages in applying the Fourier method when investigating a periodicity with given length of period are pointed out as well as the fact that the determination of meteorological coefficients, if done statistically, must not be separated from the analysis of the periodicity.

I. Introduction

§ 1. It is known long since that the intensity of cosmic radiation shows periodic variations. Three kinds of periodic variations have, up to now, been demonstrated without doubt. The lengths of periods of these are one solar day, about 27 days, and about 11 years, respectively. The shapes of these periodicities are not constant, especially large variations may be observed in the case of the shape of the 27 day variation.

The aim of this paper is to give a statistically correct method, making use of the full information available, to detect or else to contest the existence of a periodicity with given length of period and strictly constant shape, as well as to determine the shape of such a periodic change.

§ 2. The Fourier method, i.e. that of expressing the shape of the periodicity to be investigated by means of a trigonometrical polynome, has almost exclusively been used to investigate periodicities in the intensity of cosmic rays with given lengths of period. This method has, however, two serious shortcomings:

2.1. If the question to be decided upon is whether a periodicity with given length of period does exist or not, it is not sufficient to content oneself with determining the first few Fourier coefficients or, as it is sometimes done, the amplitude of the first harmonic, rather should the maximum possible number of Fourier amplitudes be taken into consideration so as to make use

* Dedicated to Prof. P. GOMBÁS on his 60th birthday.

of the maximum amount of information available in the form of the measured data. To calculate the maximum possible number of Fourier coefficients requires, however, rather tedious calculations. It will be shown that exactly the same amount of information can be gained on the basis of the same measured data in a way much simpler than that of calculating Fourier coefficients.

2.2. If, in addition to proving the existence of the periodicity, the shape of the periodic function is also to be determined, the application of the Fourier method may lead to difficulties. The Fourier method is justified only in the case when the periodic function is really a trigonometrical polynome. If a function other than a trigonometrical polynome is approximated by trigonometric polynomes, the approximation obtained bears only a weak resemblance to the function to be determined and, in addition to this, the coefficients of the polynome approximating the unknown function are "void" in the sense that they do not have any direct physical meaning, when considered individually; right on the contrary, they may be misleading sometimes. Variations of the cosmic ray intensity may often be sinusoidal and the second harmonic may also have physical meaning in certain cases. Fourier coefficients of the higher order have, however, no direct physical meaning in cosmic ray variations; at least, as for the present there has been no reason to attribute them any.

§ 3. Both disadvantages mentioned in the preceding paragraph are get rid of when approximating the unknown periodic function by means of a simple step function instead of a trigonometrical polynome. Numerical calculation of the heights of the maximum possible number of steps is a task by far simpler than that of calculating the maximum possible number of Fourier coefficients. In addition to this, the heights of the individual steps have straightforward physical meanings, i.e. they are equal to the mean values of the intensity during the time intervals corresponding to the widths of the steps.

Although the special conditions encountered in cosmic ray investigations are born in mind throughout this paper, the methods outlined and the results obtained apply to a large variety of other problems as well.

II. Formulation of the problem. Basic notations

§ 4. Let us denote by $n_\nu(\nu = 1,2, \ldots, N)$ the rates of a certain kind of cosmic ray particles as observed during N consecutive unit time intervals. Let us assume that the measured $n_\nu(\nu = 1,2, \ldots, N)$ values are not affected by any *systematic* errors, i.e. that $\langle n_\nu \rangle$, the expected value of n_ν, is equal to the mean value of the intensity during the unit time interval.

Furthermore, let us assume that the stochastic variables $n_\nu(\nu = 1,2, \ldots N)$ are independent of each other and are distributed normally. These conditions are generally not met rigorously, they, however, can be regarded as sufficiently good approximations in many cases.

Let us temporarily assume that the variances of all the variables n_ν are equal to σ^2. In Section VI the more general case with variables n_ν having different variances σ_ν^2 will also be dealt with.

§ 5. We have to test the hypothesis that, apart from certain types of changes, the intensity be a periodic function of time with a given period length, q. Furthermore, if this hypothesis turns out to be true, the shape of the periodic function is to be determined.

The unit of time should be chosen in such a way that q should be an integer, and the total number of measurements, N, should be $N = pq$, where p denotes an integer number. The case with p being a non-integer value is dealt with in Section VI.

The measured data, n_ν ($\nu = 1,2, \ldots ,N$) should then be arranged to form a matrix $\mathbf{n}(p, q) = \mathbf{n}$ with p rows and q columns in such a way that the elements n_{ij} of the matrix should be equal to n_ν in the following order:

$$n_{ij} = n_\nu \qquad \text{if} \qquad \nu = (i-1)\,q+j$$
$$(i = 1, 2, \ldots, p \qquad j = 1, 2, \ldots, q \qquad \nu = 1, 2, \ldots, pq)$$

$p \geq 2$ and $q \geq 2$ will be assumed throughout this paper.

§ 6. A few more notations:

a) Matrices will always be denoted by bold characters. Upper indices written in brackets and applied to a matrix symbol denote the numbers of rows and columns, respectively, of the matrix. These indices will be dropped if no ambiguity arises by doing so.

b) A dot on the place of a running index denotes the arithmetic mean of the quantities involved, when the index replaced by the dot runs through its usual range.

E.g.

$$n_{\cdot j} = \frac{1}{p} \cdot \sum_{i=1}^{p} n_{ij},$$

$$n_{\cdot\cdot} = \frac{1}{pq} \sum_{i=1}^{p} \sum_{j=1}^{q} n_{ij}, \qquad \text{and so on.}$$

c) Two identical running indices within a single term denote summation extended over the usual range of the identical indices.

d) A bar over a symbol denotes the estimated (measured) value of an unknown parameter. Thus symbols with bars represent always stochastic variables.

e) The variance of a stochastic variable x will be denoted by σ_x^2. (The estimated value of σ_x^2 will be denoted by $\overline{\sigma_x^2}$). σ^2 and $\overline{\sigma^2}$ without any index denote the variance of the stochastic variable n_{ij} and its estimated value, respectively.

III. The simple step function method in the case of a "pure" periodicity

§ 7. Let us assume for the moment that the intensity has no changes except of the hypothetical periodicity with the period length of q. This case will be referred to as the case of a "pure" periodicity. The mean values of the intensity during the subsequent unit time intervals within the length of a period should be denoted by $a_j (j = 1,2, \ldots, q)$. The periodicity does exist, if at least two of the a_j $(j = 1,2, \ldots q)$ values are not equal to each other. Clearly

$$\langle n_{ij} \rangle = a_j \qquad (i = 1, 2, \ldots, p \quad j = 1, 2, \ldots, q) . \tag{1}$$

$\langle n_{ij} \rangle$ is independent of the row index, that means periodicity.

We are thus facing the following problems:

7.1 $\bar{a}_j (j = 1,2, \ldots, q)$ i.e. the estimated values of the parameters $a_j (j = 1,2, \ldots, q)$ are to be determined, together with the estimated values of their respective statistical errors, $\overline{\sigma^2}_{\bar{a}_j}$.

7.2 The probability ε that the deviations of all the $\bar{a}_j (j = 1,2, \ldots, q)$ values from each other are due only to statistical fluctuations is to be determined. If this probability turns out to be very small, the existence of the periodicity may be regarded as proved.

7.3. $\overline{a^2}$ i.e. the estimated value of

$$a^2 = \frac{1}{q} \sum_{j=1}^{q} (a_j - a.)^2$$

is to be determined, together with its statistical error, $\bar{a}^2$ is the estimated value of the mean square amplitude of the periodic step function.

§ 8. Problems 7.1 and 7.2 are basic problems of the analysis of variance. Their solutions, which can be found in text books, are as follows (see e.g. [1]):

8.1

$$\bar{a}_j = n_{\cdot j}. \tag{2}$$

Let us introduce the following notations:

$$\left.\begin{aligned} p' &= q(p-1) , \\ Q_1 &= \sum_{j=1}^{q} (\bar{a}_j - \bar{a}.)^2 , \\ Q_2 &= \sum_{i=1}^{p} \sum_{j=1}^{q} (n_{ij} - \bar{a}_j)^2 . \end{aligned}\right\} \tag{3}$$

Since

$$\sigma^2_{\bar{a}_j} = \frac{\sigma^2}{p} \tag{4a}$$

we have

$$\overline{\sigma^2_{\bar{a}_j}} = \frac{\bar{\sigma}^2}{p} = \frac{Q_2}{pp'} . \tag{4b}$$

8.2 Introducing the notation

$$x = \frac{1}{2} \ln \left(\frac{pp'}{q-1} \frac{Q_1}{Q_2} \right)$$

the probability, ε, that the deviations of the quantities $\bar{a}_j (j = 1,2, \ldots, q)$ from each other are due exclusively to statistical fluctuations is given by

$$\varepsilon = P_{q-1,p'} (\zeta > x),$$

where $P_{\nu,\mu}(\zeta > x)$ stands for FISHER's z distribution with μ and ν degrees of freedom.

8.3

$$\overline{a^2} = \frac{1}{q} \left[Q_1 - \frac{q-1}{pp'} Q_2 \right]. \tag{5}$$

The problem of determining $\overline{\sigma^2_{a^2}}$ is not treated in the literature. According to calculations outlined in Appendix II we have

$$\bar{\sigma}^2_{a^2} = \frac{2Q_2}{pp'q^2} \left[2\, Q_1 - \frac{(q-1)^2}{pp'(p'+2)} \left(\frac{p'}{q-1} - 1 \right) Q_2 \right]. \tag{6}$$

If $p \gg 2$, this turns into

$$\overline{\sigma^2_{a^2}} = \frac{2\,Q_2}{pp'\,q} \left(\frac{Q_1}{q} + \overline{a^2} \right). \tag{6*}$$

IV. Comparison of the simple step function method and the Fourier method in the case of pure periodicity

§ 9. The Fourier method is based on the hypothesis that the intensity $I(t)$, as a function of time, t, has the form of a trigonometric polynome.

$$I(t) = A_0 + \sum_{\mu=1}^{m} A^*_\mu \cos \left(\mu \frac{2\pi}{q} t \right) + \sum_{\mu=1}^{m'} B^*_\mu \sin \left(\mu \frac{2\pi}{q} t \right) . \tag{7}$$

The expected values of the measured data are equal to the mean values of $I(t)$ as taken during subsequent unit time intervals:

$$\langle n_{ij}\rangle = \int_{\nu-1}^{\nu} I(t)\,dt = A_0 + \sum_{\mu=1}^{m} A_\mu \cos\left[\mu\frac{2\pi}{q}\left(j-\frac{1}{2}\right)\right] + \sum_{\mu=1}^{m'} B_\mu \sin\left[\mu\frac{2\pi}{q}\left(j-\frac{1}{2}\right)\right] \tag{8}$$

$$(i=1,2,\ldots,p;\ j=1,2,\ldots,q;\ |m-m'|\le 1),$$

where ν stands for $(i-1)q+j$ and

$$\frac{A_\mu}{A_\mu^*} = \frac{B_\mu}{B_\mu^*} = \frac{\sin(\mu\pi/q)}{(\mu\pi/q)}\,.$$

The estimated values of the Fourier coefficients $A_\mu(\mu=1,2,\ldots,m)$ and $B_\mu(\mu=1,2,\ldots,m)$ have to be determined. The maximum possible number of the parameters which can be determined is equal to* q; in particular

$$m \le E\left[\frac{q-1}{2}\right] \quad\text{and}\quad m' \le E\left[\frac{q}{2}\right],$$

where $E[x]$ stands for the largest integer number not larger than x.

It is well known that the estimated values of the Fourier coefficients and their statistical errors are the following:

$$\bar{A}_0 = n_{..}\,, \qquad \overline{\sigma^2_{\bar{A}_0}} = \frac{\overline{\sigma^2}}{pq}$$

$$\bar{A}_\mu = \frac{2}{q}\sum_{j=1}^{q} n_{.j}\cos\left[\mu\frac{2\pi}{q}\left(j-\frac{1}{2}\right)\right], \qquad \overline{\sigma^2_{\bar{A}_\mu}} = \frac{2\overline{\sigma^2}}{pq}$$

$$\left(\mu = 1,2,\ldots,m \le E\left[\frac{q-1}{2}\right]\right)$$

$$\bar{B}_\mu = \frac{2}{q}\sum_{j=1}^{q} n_{.j}\sin\left[\mu\frac{2\pi}{q}\left(j-\frac{1}{2}\right)\right], \qquad \overline{\sigma^2_{\bar{B}_\mu}} = \frac{2\overline{\sigma^2}}{pq} \tag{9}$$

$$\left(\mu = 1,2,\ldots,m' \le E\left[\frac{q-1}{2}\right]\right)$$

If $q = 2m'$, then

$$\bar{B}_{m'} = \frac{1}{q}\sum_{j=1}^{q}(-1)^j n_{.j}\,, \qquad \overline{\sigma^2_{\bar{B}_{m'}}} = \frac{\overline{\sigma^2}}{pq}$$

* Compare, however, [2], where this fact is disregarded.

It should be pointed out that also the expression (8) has the form of a step function. The step function representation of $\langle n_{ij}\rangle$ is inevitable since the number of measurements during a period is by all means finite. Expression (8) is, however, considerably more complicated than the simple expression (1) on which the method outlined in Section III was based. The question arises, whether the much more complicated method based on expression (8) yields statistically more information as to the existence of the periodicity, than does the simple step function method based on (1). It will be shown, in what follows, that the answer is negative.

Let be

$$m = E\left[\frac{q-1}{2}\right] \quad \text{and} \quad m' = E\left[\frac{q}{2}\right], \tag{9*}$$

i.e.

$$m + m' + 1 = q.$$

It will be shown that, in this case, the Fourier procedure is statistically equivalent to the simple step function method based on (1).

§ 10. First it will be shown that the estimated value of the mean square amplitude as obtained by the Fourier method and the simple step function method are exactly the same, i.e. they are identical second order expressions of the quantities n_{ij}.

A simple calculation shows that the mean square amplitude of the function (8) is equal to

$$A^2 = \frac{1}{2}\left[\sum_{\mu=1}^{m} A_\mu^2 + \sum_{\mu=1}^{m'} B_\mu^2\right] + \varepsilon_q B_{m'}^2, \tag{10}$$

where

$$\varepsilon_q = E\left[\frac{q}{2}\right] - \frac{q-1}{2}.$$

In order to determine $\overline{A^2}$ it should be noted that

$$\overline{A_\mu^2} = \bar{A}_\mu^2 - \overline{\sigma_{A\mu}^2} \qquad (\mu = 1, 2, \ldots, m),$$

and

$$\overline{B_\mu^2} = \bar{B}_\mu^2 - \overline{\sigma_{B\mu}^2} \qquad (\mu = 1, 2, \ldots, m').$$

Furthermore, it can be shown by rather longish elementary calculations that

$$\frac{1}{2}\left[\sum_{\mu=1}^{m} \bar{A}_\mu^2 + \sum_{\mu=1}^{m'} \bar{B}_\mu^2\right] + \varepsilon_q \bar{B}_{m'}^2 = \frac{1}{q}\sum_{j=1}(n_{\cdot j} - n_{\cdot\cdot})^2$$

whence, on the basis of (10), (2), and (5), it can be seen that

$$\overline{a^2} = \overline{A^2},$$

i.e. the estimated values of the mean square amplitude of the periodic function as obtained on the basis of (1) or (8) are identical expressions of the n_{ij} data.

§ 11. We proceed with showing that the simple step function method based on (1), and the Fourier method based on (8) reduce the (empirical) mean square deviation of the n_{ij} values from their mean to exactly the same extent. The reduced mean square deviation divided by $p' = q(p-1)$ gives the value of $\overline{\sigma^2}$. Thus both procedures lead exactly the same value of $\overline{\sigma^2}$

$\overline{\sigma^2}$ as determined on the basis of (1) is given by

$$\overline{\sigma^2} = \frac{Q_2}{p'}. \tag{11}$$

(The notations are explained in Equs. (3)). In the case of the Fourier method we have, on the other hand,

$$\begin{aligned}\overline{\sigma^2} = \frac{1}{p'} \sum_{i=1}^{p} \sum_{j=1}^{q} \Bigg\{ n_{ij} - \bar{A}_0 - \sum_{\mu=1}^{m} \bar{A}_\mu \cos\left[\mu \frac{2\pi}{q}\left(j - \frac{1}{2}\right)\right] - \\ - \sum_{\mu=1}^{m'} \bar{B}_\mu \sin\left[\mu \frac{2\pi}{q}\left(j - \frac{1}{2}\right)\right]\Bigg\}^2.\end{aligned} \tag{12}$$

It can be shown by direct calculation that the right hand sides of Equs. (11) and (12) are identical second order expressions of the values n_{ij} if (9*) is true.

§ 12. It seems to be worth while to emphasize that the unbiased estimate of the mean square amplitude of the periodic function (8) is given by

$$\overline{A^2} = \frac{1}{2}\left[\sum_{\mu=1}^{m}{}' \overline{A_\mu^2} + \sum_{\mu=1}^{m'}{}' \overline{B_\mu^2}\right] + \varepsilon_q \overline{B_{m'}^2},$$

i.e. *not* by the expression

$$U^2 = \frac{1}{2}\left[\sum_{\mu=1}^{m}{}' \bar{A}_\mu^2 + \sum_{\mu=1}^{m'}{}' \bar{B}_\mu^2\right] + \varepsilon_q \bar{B}_{m'}^2,$$

which is used in many cases in spite of the fact that U^2 is *not* an *unbiased* estimate of A^2. It can be easily seen that in all cases (except that of n_{ij} = constant)

$$U^2 > \overline{A^2} \quad \text{and thus} \quad \langle U^2 \rangle > A^2.$$

The difference $U^2 - A^2$ is large just in the delicate cases when the statistical errors of the coefficients $\bar{A}_\mu$, B_μ are large and the existence of the periodicity is questionable. The incorrect value U^2 unjustly favours the hypothesis that the periodicity exists.

V. The simple step function method in the case of mixed periodicity

§ 13. The intensity of any component of the cosmic radiation shows, in all cases, also systematic variations others than the periodicity to be investigated (cases of "mixed" periodicity). There are systematic variations like the absorption effect or the decay effect, which depend on quantities, like the barometric pressure or the height of a certain isobaric level, which in principle, can be measured regularly and have thus known values β_{ijk} $(i = 1,2, \ldots, p;\ j = 1,2, \ldots q;\ k = 1,2, \ldots, r)$ during the time interval (i, j). (Let us take e.g. β_{ij1} to be the average barometric pressure during the time interval (i, j), β_{ij2} the average height of the 100 mb isobaric level during the name interval, and so on.) Supposing that the intensity be a linear function of the quantities $\beta_{ijk} (k = 1,2, \ldots, r)$ we have, in the case of a periodicity with the period length q

$$\begin{gathered}\langle n_{ij} \rangle = a_j + \beta_{ijk}\, b_k \\ (i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q;\ k = 1, 2, \ldots, r)\end{gathered} \tag{13}$$

where $b_k (k = 1,2, \ldots, r)$ are unknown parameters, e.g. b_1 the partial barometric coefficient, b_2 the decay coefficient, and so on.

It can always be assumed, without any restriction of generality, that the (sum of any type of the quantities β_{ijk} taken for the total time of measurement is zero i.e.

$$\beta_{\cdot\cdot k} = 0 \quad (k = 1, 2, \ldots, r).$$

Note, however, that generally

$$\beta_{\cdot jk} \neq 0\,, \qquad \beta_{i\cdot k} \neq 0.)$$

It would be *incorrect* to determine the values $\bar{b}_k (k = 1,2, \ldots, r)$ separately, on the basis of the equations

$$\langle n_{ij} \rangle = \beta_{ijk}\, b_k^*$$

and then to use the "corrected" values $n_{ij}^* = n_{ij} - \beta_{ijk}\, \bar{b}_k^*$ to determine $\bar{a}_j (j = 1,2, \ldots, q)$ on the basis of

$$\langle n_{ij}^* \rangle = a_j^*$$

since in general

$$b_k^* \neq b_k \quad \text{and} \quad a_j^* \neq a_j$$

and the values $b_k^* (k = 1,2, \ldots, r)$ and $a_j^* (j = 1,2, \ldots, q)$ have no physical meaning.

Sometimes it may be of advantage to use corrected values $n_{ij} - \beta_{ijk} \bar{b}'_k$ with predetermined $\bar{b}'_k$. This may be the case when the total time of measurement is short and there are previous measurements available with considerably longer duration thus yielding considerably more exact values $\bar{b}'_k$. It is, however, indispensable that these predetermined coefficients should have been calculated on the basis of equations allowing also for the periodicity in question, i.e.

$$\langle n'_{ij} \rangle = a'_j + \beta'_{ijk} b'_k$$

$$(i = 1, 2, \ldots, p'; \quad j = 1, 2, \ldots, q'; \quad k = 1, 2, \ldots, r'$$

with

$$p' \gg p, \quad q' = q, \quad r' = r),$$

where n'_{ij} are the results of the older measurements and β'_{ijk} are the meteorological factors pertaining to them.

§ 14. Problems of the type of Equ. (13) have also been dealt with in the literature (see e.g. [3]). The solutions found are, however, incomplete in the sense that they do not give answers to all questions raised in cosmic ray investigations.

A full treatment of the problem will be given in what follows. The method applied will be that of the maximum likelihood which allows a compact treatment of the problem on the one hand, and may be used also in cases, when the n_{ij} variables are not distributed normally, on the other.

§ 15. The problem must be formulated in a more general way than that indicated in Equ. (13) so as to allow for systematic variations which are not of the type $\beta_{ijk} b_k$. There may be e.g. periodic variations with a period length other than q or aperiodic changes, like a slow recovery after a Forbush effect. The general character of these systematic variations must be known a priori or, at least, a reasonable hypothesis must be made as to the general trend of these variations. New terms containing some unknown parameters must then be inserted into Equ. (13). The estimated values of these parameters can be determined by means of the maximum likelihood method as discussed below.

No special hypothesis has to be made as to the general character of the remaining systematic variations if these are "slow", i.e. the changes due to them during the time q are small as compared to $\sqrt{a^2}$ where a^2 stands for

$$\frac{1}{q} \sum_{j=1}^{q} (a_j - a.)^2$$

as before. "Slow" systematic variations can be approximated by a step function in which the width of the steps are equal to q. Denoting the heights of the steps by c_i Equ. (13) becomes

$$\langle n_{ij}\rangle = c_i + a_j + \beta_{ijk}\, b_k, \tag{14}$$
$$c_1 = 0 \ ^{*)}, \qquad \beta_{\cdot\cdot k} = 0,$$
$$(i = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, q; \quad k = 1, 2, \ldots, r)$$

or, in a more symmetric form:

$$\begin{gathered}\langle n_{ij}\rangle = A + c_i + a_j + \beta_{ijk}\, b_k, \\ c. = 0\ ^{*}, \qquad a. = 0\ ^{*}, \qquad \beta_{\cdot\cdot k} = 0, \\ (i = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, q; \quad k = 1, 2, \ldots, r).\end{gathered} \tag{15}$$

If the systematic variation characterized by the constants c_i is slow enough, then the width of the steps may be chosen to be larger than q, e.g. an arbitrary integer multiple of q. Alternatively, a "slow" variation can be approximated also by a polygonal line connecting equidistant points separated by the distance q or an arbitrary integer multiple of q. The most suitable hypothesis as to the form of the slow variation must be chosen individually in each case on the basis of careful consideration of the circumstances.

The problem as specified by Equ. (15) will be dealt with in what follows. It can be seen that

$$\langle n_{\cdot\cdot}\rangle = A,$$

i.e. A is the average intensity as taken during the total time of measurement.

Again the problem is a threefold one:

(I) Unbiased estimated values of the parameters A,

$$A, c_i\,(i = 1, 2, \ldots, p), \quad a_j\,(j = 1, 2, \ldots, q), \quad b_k\,(k = 1, 2, \ldots, r)$$

have to be determined together with their respective statistical errors and covariances.

(II) The probability has to be calculated, that the deviation from zero of the estimated values $\tilde{a}_j (j = 1, 2, \ldots, q)$ are due to statistical fluctuations alone.

(III) An unbiased estimated value of the quantity

$$a^2 = \frac{1}{q} \sum_{j=1}^{q} a_j^2,$$

i.e. the mean square amplitude of the periodicity in question, has to be determined together with its statistical error.

* Without this (these) assumption(s) the problem would become indeterminate.

§ 16. Let us introduce the following notations:

$$\delta_{mn} = 1 \quad \text{if} \quad m = n \quad \text{and} \quad \delta_{mn} = 0 \quad \text{if} \quad m \neq n .$$

The transpose of the matrix $\mathbf{M}^{(m,n)}$ should be denoted by $\widetilde{\mathbf{M}}^{(n,m)}$.

The inverse of the matrix $\mathbf{M}$ consisting of the elements M_{ij} should be denoted by $\mathbf{M}^{+}$, the elements of $\mathbf{M}^{+}$ by $\mathbf{M}^{+}_{ij}$.

$\mathbf{I}^{(m)}$ should denote a unit matrix with m rows and columns.

$\mathbf{J}^{(m,n)}$ should denote a matrix whose elements are all equal to 1, and $\mathbf{0}^{(m,n)}$ a matrix with elements all equal to 0.

$\mathbf{B} = \mathbf{B}^{(r,r)}$ should denote the matrix with elements $B_{kk'} = \beta_{ijk}\beta_{ijk'}$,

$\mathbf{B}^* = \mathbf{B}^{*(r,r}$ should denote the matrix with elements $B^*_{kk'} = B_{kk} - p\beta_{\cdot ijk}\beta_{\cdot jk'} - q\beta_{i\cdot k}\beta_{i\cdot k'}$,

$\mathbf{N} = \mathbf{N}^{(r,1)}$ should denote the matrix with elements $N_k = n_{ij}\beta_{ijk} - pn_{\cdot j}\beta_{\cdot jk} - qn_{i\cdot}\beta_{i\cdot k}$,

$\mathbf{B}_c = \mathbf{B}_c^{(p,r)}$ should denote the matrix with elements $B^c_{ik} = \beta_{i\cdot k}$,

$\mathbf{B}_a = \mathbf{B}_a^{(q,r)}$ should denote the matrix with elements $B^a_{jk} = \beta_{\cdot jk}$,

$\mathscr{B}_c = \mathscr{B}_c^{(p,r)}$ should denote the matrix with elements $\mathscr{B}^c_{ik} = \beta_{i\cdot k'} B^{*+}_{k'k}$,

$\mathscr{B}_a = \mathscr{B}_a^{(q,r)}$ should denote the matrix with elements $\mathscr{B}^a_{jk} = \beta_{\cdot jk'} B^{*+}_{kk'}$;

$\mathscr{B}_{cc} = \mathscr{B}_{cc}^{(p,p)}$ should denote the matrix with elements $\mathscr{B}^{cc}_{ii'} = \beta_{i\cdot k}\mathscr{B}^c_{i'k} + \dfrac{1}{q}\,\delta_{ii'} - \dfrac{1}{pq}$,

$\mathscr{B}_{ac} = \mathscr{B}_{ac}^{(p,q)}$ should denote the matrix with elements $\mathscr{B}^{ac}_{ij} = \beta_{i\cdot k}\mathscr{B}^a_{jk} = \beta_{\cdot jk}\mathscr{B}^c_{ik}$,

$\mathscr{B}_{aa} = \mathscr{B}_{aa}^{(q,q)}$ should denote the matrix with elements $\mathscr{B}^{aa}_{jj'} = \beta_{\cdot jk}\mathscr{B}^a_{j'k} + \dfrac{1}{p}\,\delta_{jj'} - \dfrac{1}{pq}$.

It can be seen easily that all quadratic matrices defined above are symmetrical.

§ 17. (Problem I of § 15).

According to the method of maximum likelihood, unbiased estimates of the parameters A, c_i, a_j, b_k can be obtained (see e.g. [4]) as the solutions of the simultaneous equations

$$\left.\begin{aligned}
&\frac{\partial P}{\partial A} = 0 \\
&\frac{\partial P}{\partial c_i} = 0 \quad (i = 1, 2, \ldots, p) \\
&\frac{\partial P}{\partial a_j} = 0 \quad (j = 1, 2, \ldots, q) \\
&\frac{\partial P}{\partial b_k} = 0 \quad (k = 1, 2, \ldots, r) \\
&\frac{pq}{\sigma^2} c. = 0 \\
&\frac{pq}{\sigma^2} a. = 0
\end{aligned}\right\}, \tag{16}$$

where

$$P = \ln P^* - \frac{pq}{\sigma^2} \lambda_c c. - \frac{pq}{\sigma^2} \lambda_a a. ,$$

with

$$\frac{pq}{\sigma^2} \lambda_c \quad \text{and} \quad \frac{pq}{\sigma^2} \lambda_a$$

denoting Lagrange multipliers, and

$$P^* = \prod_{i=1}^{p} \prod_{j=1}^{q} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\varkappa^2} (n_{ij} - \langle n_{ij} \rangle)^2\right],$$

i.e. P^* stands for the joint distribution of the stochastic variables $n_{ij}(i = 1,2, \ldots, p;\ j = 1,2, \ldots, q)$. If the variables n_{ij} are not independent and normal, P^* has another form which, at any rate, must be known. The explicit formulae given in what follows refer to the case when n_{ij} are independent and normal.

The solution of the simultaneous equations are the following:

$$\left.\begin{aligned}
&\bar{A} = n.. \\
&\bar{c}_i = n_i. - n.. - \beta_{i\cdot k} B^{*+}_{kk'} N_{k'} \\
&\qquad (i = 1, 2, \ldots, p) \\
&\bar{a}_j = n._j - n.. - \beta_{\cdot jk} B^{*+}_{kk'} N_{k'} \\
&\qquad (j = 1, 2, \ldots, q) \\
&\bar{b}_k = B^{*+}_{kk'} N_{k'} \\
&\qquad (k = 1, 2, \ldots, r) \\
&\lambda_a = \lambda_c = 0 .
\end{aligned}\right\}. \tag{17}$$

In order to obtain the statistical errors and covariances of the estimated values as indicated in Equs. (17), let us write down the matrix of the simultaneous equations (16) multiplied by -1. This is a matrix of $p+q+r+3$ rows and columns and is composed of submatrices in the following way:

$$\mathbf{M} = \frac{1}{\sigma^2} \left\| \begin{matrix} pq & q\mathbf{J}^{(1,p)} & p\mathbf{J}^{(1,q)} & \mathbf{0}^{(1,r)} & 0 & 0 \\ q\mathbf{J}^{(p,1)} & q\mathbf{I}^{(q)} & \mathbf{J}^{(p,q)} & q\mathbf{B}_c^{(p,r)} & q\mathbf{J}^{(p,1)} & \mathbf{0}^{(q,1)} \\ p\mathbf{J}^{(q,1)} & \mathbf{J}^{(q,p)} & p\mathbf{I}^{(q)} & p\mathbf{B}_a^{(q,r)} & \mathbf{0}^{(q,1)} & p\mathbf{J}^{(q,1)} \\ \mathbf{0}^{(r,1)} & q\tilde{\mathbf{B}}_c^{(r,p)} & p\tilde{\mathbf{B}}_a^{(r,q)} & \mathbf{B}^{(r,r)} & \mathbf{0}^{(r,1)} & \mathbf{0}^{(r,1)} \\ 0 & q\mathbf{J}^{(1,p)} & \mathbf{0}^{(1,q)} & \mathbf{0}^{(1,r)} & 0 & 0 \\ 0 & \mathbf{0}^{(1,p)} & p\mathbf{J}^{(1,q)} & \mathbf{0}^{(1,r)} & 0 & 0 \end{matrix} \right\| .$$

The inverse of $\mathbf{M}$ is

$$\mathbf{M}^{+} = \sigma^2 \left\| \begin{matrix} \frac{1}{pq} & \mathbf{0}^{(1,p)} & \mathbf{0}^{(1,q)} & \mathbf{0}^{(1,r)} & -\frac{1}{pq} & -\frac{1}{pq} \\ \mathbf{0}^{(p,1)} & \mathcal{B}_{cc}^{(p,p)} & \mathcal{B}_{ac}^{(p,q)} & -\mathcal{B}_{c}^{(p,r)} & \frac{1}{pq}\mathbf{J}^{(p,1)} & \mathbf{0}^{(p,1)} \\ \mathbf{0}^{(q,1)} & \mathcal{B}_{ac}^{(q,p)} & \mathcal{B}_{aa}^{(q,q)} & -\mathcal{B}_{a}^{(q,r)} & \mathbf{0}^{(q,1)} & \frac{1}{pq}\mathbf{J}^{(q,1)} \\ \mathbf{0}^{(r,1)} & \mathcal{B}_{c}^{(r,p)} & -\mathcal{B}_{a}^{(r,q)} & \mathbf{B}^{*+(r,r)} & \mathbf{0}^{(r,1)} & \mathbf{0}^{(r,1)} \\ -\frac{1}{pq} & \frac{1}{pq}\mathbf{J}^{(1,p)} & \mathbf{0}^{(1,q)} & \mathbf{0}^{(1,r)} & 0 & 0 \\ -\frac{1}{pq} & \mathbf{0}^{(1,p)} & \frac{1}{pq}\mathbf{J}^{(1,q)} & \mathbf{0}^{(1,r)} & 0 & 0 \end{matrix} \right\|$$

The meanings of the symbols can be found in § 16.

Let $\mathcal{M}^{+}$ be the matrix obtained when deleting the last two rows and columns in $\mathbf{M}^{+} \cdot \mathcal{M}^{+}$ is, of course, degenerate because of $c. = a. = 0$ and is of the rank $p+q+r-1$. Let s' and s be integer numbers such that $1 \leq s' \leq \leq p$ and $1 \leq s \leq q$. Let us omit the rows and columns of $\mathcal{M}^{+}$ standing on the places numbered $1+s'$ and $1+p+s$. The resulting matrix, $\mathcal{M}^{+}_{s's}$ is the covariance matrix of the variables $\bar{A}$, $\bar{c}_i(i = 1,2, \ldots, s'-1, s'+1, \ldots, p)$, $\bar{a}_j(j = 1,2, \ldots, s-1, s+1, \ldots, q)$, $\bar{b}_k(k = 1,2, \ldots, r)$.

It can thus be seen that $\bar{A}$ is independent of the measured values of all remaining parameters and has the estimated variance

$$\overline{\sigma^2_{\bar{A}}} = \frac{1}{pq}\overline{\sigma^2}.$$

The estimated values of the parameters $\bar{c}_i$, $\bar{a}_j$, $\bar{b}_k$ are not independent of each other. Their estimated variances are the diagonal elements of the matrices $\overline{\sigma^2}\,\mathscr{B}_{cc}$, $\overline{\sigma^2}\,\mathscr{B}_{aa}$ and $\overline{\sigma^2}\,\mathscr{B}^{*+}$ respectively; i.e.

$$\left.\begin{aligned} \overline{\sigma^2_{\bar{c}_i}} &= \mathscr{B}^{cc}_{ii'}\overline{\sigma^2} && (\text{with } i'=i) \\ \overline{\sigma^2_{\bar{a}_j}} &= \mathscr{B}^{aa}_{jj'}\overline{\sigma^2} && (\text{with } j'=j) \\ \overline{\sigma^2_{\bar{b}_k}} &= \mathscr{B}^{*+}_{kk'} && (\text{with } k'=k) \end{aligned}\right\}. \tag{18}$$

It can be easily seen that the estimated value of σ^2 is

$$\begin{aligned} \overline{\sigma^2} &= \frac{1}{q'}(n_{ij}-\bar{n}_{ij})(n_{ij}-\bar{n}_{ij}) = \\ &= \frac{1}{q'}\left[n_{ij}n_{ij} - pq\bar{A}^2 - qn_{i.}\bar{c}_i - pn_{.j}\bar{a}_j - n_{ij}\beta_{ijk}\bar{b}_k\right], \end{aligned} \tag{19}$$

where $q' = (p-1)(q-1) - r$ and

$$\bar{n}_{ij} = \bar{A} + \bar{c}_i + \bar{a}_j + \beta_{ijk}\bar{b}_k\,.$$

The quantity $q'\,\overline{\sigma^2}/\sigma^2$ is a stochastic variable with a χ^2 distribution of q' degrees of freedom.

On the basis of the identity

$$\sum_{i=1}^{p}\sum_{j=1}^{q}(n_{ij}-n_{..})^2 = \sum_{i=1}^{p}\sum_{j=1}^{q}(n_{ij}-\bar{n}_{ij})^2 + \sum_{i=1}^{p}\sum_{j=1}^{q}(\bar{c}_i+\bar{a}_j+\beta_{ijk}\bar{b}_k)^2$$

making use of the theorem of COCHRAN [5] and FISHER's lemma, it can be seen that all quantities $n_{ij}-\bar{n}_{ij}$ $(i=1,2,\ldots,p;\ j=1,2,\ldots,q)$ are independent of $\bar{A}$, $\bar{c}_i$ $(i=1,2,\ldots,p)$ $\bar{a}_j (j=1,2,\ldots,q)$ and $\bar{b}_k (k=1,2,\ldots,r)$ Thus, e.g., $(n_{ij}-\bar{n}_{ij})(n_{ij}-\bar{n}_{ij})$ is independent of $f(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_q)$, this latter being an arbitrary function of the variables $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_q$.

§ 18. (Problem II of § 15). In order to calculate the probability ε that the deviations from zero of the values $\bar{a}_j (j=1,2,\ldots,q)$ are due to statistical fluctuations alone, let us introduce the following notations:

$\bar{\mathbf{a}}_s$ should denote the matrix

$$\bar{\mathbf{a}}_s^{(1,q-1)} = \|\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{s-1}, \bar{a}_{s+1}, \ldots, \bar{a}_q\|$$

with s being an integer number chosen arbitrarily within the interval

$$1 \le s \le q.$$

$\mathscr{B}_s^{q-1,q-1}$ should be the matrix obtained when omitting the row and column in the matrix $\mathscr{B}_{aa}^{(q,q)}$ numbered s.

First, let us calculate the probability ε_s that the deviations from zero of the elements of $\bar{\mathbf{a}}_s$ are due only to statistical fluctuations.

The covariance matrix of the elements of $\bar{\mathbf{a}}_s$ is $\sigma^2\mathscr{B}_s$. Assuming $a_j = \langle\bar{a}_j\rangle = 0$ $(j = 1,2, \ldots, q)$ the joint distribution function of the elements of $\bar{\mathbf{a}}_s$ is given by

$$S(\bar{\mathbf{a}}_s) = \frac{\sqrt{D_s^+}}{(2\pi)^{q-1/2}} \exp\left[-\frac{1}{2\sigma^2}\bar{\mathbf{a}}_s\mathscr{B}_s^+\tilde{\bar{\mathbf{a}}}_s\right],$$

where D_s^+ stands for the determinant of the matrix

$$\frac{1}{\sigma^2}\mathscr{B}_s^+.$$

It is well known that the quantity

$$Q_s = \frac{1}{\sigma^2}\bar{\mathbf{a}}_s\mathscr{B}_s^+\tilde{\bar{\mathbf{a}}}$$

is distributed according to a χ^2 distribution with $q - 1$ degrees of freedom. As it was shown at the end of § 17 $(n_{ij} - \bar{n}_{ij})(n_{ij} - \bar{n}_{ij})$ is independent of any function of $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_q$ thus $\overline{\sigma^2}$ and Q_s are independent of each other. As a consequence, the quantity

$$y_s = \frac{1}{2}\ln\frac{1}{q-1}\frac{\bar{\mathbf{a}}_s\mathscr{B}_s^+\tilde{\bar{\mathbf{a}}}}{\overline{\sigma^2}}$$

is distributed according to FISHER's z distribution with $q - 1$ and q' degrees of freedom. Denoting this latter distribution by $P_{q-1,q'}(\zeta > z)$ the result is obtained that

$$\varepsilon_s = P_{q-1,q'}(\zeta > y_s). \tag{20}$$

Applying the theorem proved in Appendix III it can be seen that $\bar{\mathbf{a}}_s\mathscr{B}_s^+\tilde{\bar{\mathbf{a}}}$, is independent of s and, as a consequence, the same is true also for y_s and ε_s It is therefore justified to regard the value $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_q = \varepsilon$ as being the probability that the deviations from zero of the values $\bar{a}_j$ $(j = 1,2, \ldots, q)$ are due to statistical fluctuations alone. The value of $\varepsilon = \varepsilon_s$ is thus determined by Equs. (20).

§ 19. (Problem III of § 15). Since

$$\overline{a_j^2} = \overline{a_j^2} - \overline{\sigma_{\bar{a}j}^2} \qquad (j = 1, 2, \ldots, q),$$

therefore, taking into account Equ. (18), an unbiased estimate of $\mathbf{a}^2$ is

$$\overline{a^2} = \frac{1}{q} \left(\bar{a}_j \bar{a}_j - \mathscr{B}_{jj}^{aa} \overline{\sigma^2} \right). \tag{21}$$

Calculations outlined in Appendix II show that the estimated value of the variance of $\overline{a^2}$ is equal to

$$\overline{\sigma^2_{a^2}} = \frac{2 \overline{\sigma^2}}{q^2} \left[2 \bar{a}_j \mathscr{B}_{jj'}^{aa}, \bar{a}_{j'} - \frac{\overline{\sigma^2}}{q'+2} \left(q' \mathscr{B}_{jj'}^{aa} \mathscr{B}_{jj'}^{aa} - \mathscr{B}_{jj}^{aa} \mathscr{B}_{j'j'}^{aa} \right) \right] \tag{22}$$

the value of $\overline{\sigma^2}$ being given with Equ. (19).

VI. The case of the incomplete matrix n with elements of unequal variances*

§ 20. It may happen that the measurement of the intensity $\langle n_{ij} \rangle$ and/or some of the quantities β_{ijk} was interrupted during certain intervals (i, j), and thus the corresponding n_{ij} and/or β_{ijk} values are missing. It may also happen that the variances of the measured quantities n_{ij} are not equal to each other. The solutions of the three problems formulated in § 15 may then be found on the following lines.

Let us denote the variance of n_{ij} by

$$\sigma_{ij}^2 = \sigma^{2*} / w_{ij} ,$$

where the factors $1/w_{ij}$ must be known a priori and σ^{2*} is to be determined.

Furthermore, let be

$\Delta_{ij} = w_{ij}$ if the values $n_{ij}, \beta_{ij2}, \beta_{ij2}, \ldots, \beta_{ijr}$ are all available,

$\Delta_{ij} = 0$ if at least one of the values $n_{ij}, \beta_{ij1}, \beta_{ij2}, \ldots, \beta_{ijr}$ is missing.

If among the values $\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iq}$ there is not more than one different from zero, the ith rows of the matrices** $\mathbf{n}$ and $\boldsymbol{\beta}_k (k=1,2, \ldots, r)$ should be deleted and the numeration of the rows should be rearranged accordingly.

If not more than one among the quantities $\Delta_{1j}, \Delta_{2j}, \ldots, \Delta_{pj}$ is different from zero, the measurement must be continued until at least two of the Δ_{1j}, $\Delta_{2j}, \ldots, \Delta_{pj}, \Delta_{p+1,j}, \ldots, \Delta_{p+p*,j}$ values becomes different from zero, and the value $p + p^*$ should then be denoted by p.

Thus at least two elements in each row and column of the matrix $\Delta_{(p,q)}$ are different from zero.

* Identical indices do *not* mean summation in this chapter.

** $\boldsymbol{\beta}_k^{(p,q)}$ stands for the matrix with elements β_{ijk} $(i = 1,2, \ldots, p;\ j = 1,2, \ldots, q)$

The values $n_{ij}, \beta_{ijk} (k = 1,2, \ldots, r)$ will, in what follows, never occur without being multiplied by Δ_{ij}. Thus arbitrary values (e.g. 0) may be written in the place of n_{ij} and β_{ijk} $(k = 1,2, \ldots, r)$ if $\Delta_{ij} = 0$. This way the blank places in the matrices $\mathbf{n}$ and $\boldsymbol{\beta}_k$ $(k = 1,2, \ldots, r)$ can be filled in. This is also the case when the total time of measurement (N) is not an integer multiple of q; the last row can be completed arbitrarily only that the corresponding Δ_{pj} values must be put equal to 0. It will be assumed in what follows that there are no blank places in the matrices $\mathbf{n}$ and $\boldsymbol{\beta}_k$ $(k = 1,2, \ldots, r)$.

It can be assumed without restriction imposed upon the factors β_{ijk} that

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \Delta_{ij} \beta_{ijk} = 0 \qquad (k = 1, 2, \ldots, r).$$

The sum of the elements of the matrix Δ should be denoted by Δ and the number of elements in Δ which are different from 0 should be N_0. ($N_0 \leqslant pq$ is thus the total number of useful measurements.) The product of all non-zero values Δ_{ij} should be denoted by $\pi(\Delta)$.

§ 21. The basic assumption as to the form of the change of the intensity (Equs. (14)) may remain unchanged, neither does change the principle of deriving the equations for determining the estimated values of the parameters (Equs. (16)). The particular form of Equs. (16) will, however, be different from that in § 17. The matrix of the system of linear equations (multiplied by -1) will, in this case, be the following:

$$\mathbf{M} = \frac{1}{\sigma^{2*}} \left\| \begin{matrix} \Delta & \Delta_c^{(1,p)} & \Delta_a^{(1,q)} & \mathbf{O}^{(1,r)} & 0 & 0 \\ \Delta_a^{(p,1)} & \Delta_{cc}^{(p,p)} & \Delta^{(p,q)} & \mathbf{B}_c^{(p,r)} & q\mathbf{J}^{(p,1)} & \mathbf{O}^{(p,1)} \\ \tilde{\Delta}_a^{(q,1)} & \tilde{\Delta}^{(q,p)} & \Delta_{aa}^{(q,q)} & \mathbf{B}_a^{(q,r)} & \mathbf{O}^{(q,1)} & p\mathbf{J}^{(q,1)} \\ \mathbf{O}^{(r,1)} & \tilde{\mathbf{B}}_c^{(r,p)} & \tilde{\mathbf{B}}_a^{(r,q)} & \mathbf{B}^{(r,r)} & \mathbf{O}^{(r,1)} & \mathbf{O}^{(r,1)} \\ 0 & q\mathbf{J}^{(1,p)} & \mathbf{O}^{(1,q)} & \mathbf{O}^{(1,r)} & 0 & 0 \\ 0 & \mathbf{O}^{(1,p)} & p\mathbf{J}^{(1,q)} & \mathbf{O}^{(1,r)} & 0 & 0 \end{matrix} \right\|, \tag{23}$$

where

$\Delta_c^{(1,p)}$	stands for the matrix with elements	$\sum_{j=1}^{q} \Delta_{ij}$	$(i = 1, 2, \ldots, p)$
$\Delta_a^{(1,q)}$	stands for the matrix with elements	$\sum_{i=1}^{p} \Delta_{ij}$	$(j = 1, 2, \ldots, q)$
$\Delta_{cc}^{(p,p)}$	stands for the diagonal matrix with diagonal elements	$\sum_{j=1}^{q} \Delta_{ij}$	$(i = 1, 2, \ldots, p)$
$\Delta_{aa}^{(q,q)}$	stands for the diagonal matrix with diagonal elements	$\sum_{i=1}^{p} \Delta_{ij}$	$(j = 1, 2, \ldots, q)$

$\mathbf{B}_c^{(p,r)}$	stands for the matrix with elements	$\sum_{j=1}^{q} \Delta_{ij} \beta_{ijk}$	$(i = 1, 2, \ldots, p$; $k = 1, 2, \ldots, r)$
$\mathbf{B}_a^{(q,r)}$	stands for the matrix with elements	$\sum_{i=1}^{p} \Delta_{ij} \beta_{ijk}$	$(j = 1, 2, \ldots, q$; $k = 1, 2, \ldots, r)$
$\mathbf{B}^{(r,r)}$	stands for the matrix with elements	$\sum_{i=1}^{p} \sum_{j=1}^{q} \Delta_{ij} \beta_{ijk} \beta_{ijk'}$	$(k,k' = 1, 2, \ldots, r)$.

Δ is explained in § 20, $\mathbf{O}$ and $\mathbf{J}$ have the same meaning as in § 16.

In general, no simple expressions can be derived for the estimated values of the parameters which can, in principle, be calculated by inverting the matrix (23). If this has been done, every question can be answered exactly as it has been done in Section V. Note, however, that the estimated value of σ^2 will be

$$\overline{\sigma^{2*}} = \frac{1}{N_0 - p - q - r + 1} \sum_{i=1}^{p} \sum_{j=1}^{q} \Delta_{ij} (n_{ij} - \bar{n}_{ij})^2 .$$

Equations (20), (21), (22) remain valid if

$$q' = N_0 - p - q - r + 1 \text{ (instead of } (p-1)(q-1)-r),$$

and $\sigma^{2*} \mathscr{B}^{aa}_{jj'}$ denotes the covariance of $\bar{a}_j$ and $\bar{a}_{j'}$, and $\sigma^{2*} \mathscr{B}_s$ stands for the matrix obtained by omitting the row and column numbered s in the covariance matrix of the $\bar{\mathbf{a}}_s$ quantities.

§ 22. Sometimes it may be almost impossible to calculate the inverse of the matrix (23) even by means of electronic computers. This is the case if p is very large (e.g. ≥ 200), i.e. the matrix $\mathbf{n}$ consists of very many rows and thus there are lots of parameters c_i characterizing the slow change of the intensity. The inversion of the whole matrix (23) can, however, be avoided in the following way:

Let us introduce the following notations:

$$\left.\begin{aligned}
\Delta_i &= \sum_{j=1}^{q} \Delta_{ij} \\
n^i_{\cdot} &= \frac{1}{\Delta_i} \sum_{j=1}^{q} \Delta_{ij} n_{ij} \\
a^i_{\cdot} &= \frac{1}{\Delta_i} \sum_{j=1}^{q} \Delta_{ij} a_j \\
\beta^i_{\cdot k} &= \frac{1}{\Delta_i} \sum_{j=1}^{q} \Delta_{ij} \beta_{ijk} \\
\beta'_{ijk} &= \beta_{ijk} - \beta^i_{\cdot} \\
d_i &= A + c_i + a^i_{\cdot} + \beta^i_{\cdot k} b_k
\end{aligned}\right\} . \qquad (24)$$

Note that

$$\left.\begin{array}{lll} \sum_{j=1}^{q} \Delta_{ij}(n_{ij} - n_{\cdot}^{i}) = 0 & & (i = 1, 2, \ldots, p) \\ \sum_{j=1}^{q} \Delta_{ij}(a_j - a_{\cdot}^{i}) = 0 & & (i = 1, 2, \ldots, p) \\ \sum_{j=1}^{q} \Delta_{ij}\beta'_{ijk} = 0 & & \begin{array}{l}(i = 1, 2, \ldots, p \\ k = 1, 2, \ldots, r)\end{array} \end{array}\right\} . \tag{24a}$$

The basic assumption as to the change of the intensity can thus be written in the form

$$\langle n_{ij}\rangle = d_i + a_j - a_{\cdot}^{i} + \beta'_{ijk} b_k$$

$$(a_{\cdot} = 0\,, \quad 1 = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, q; \quad k = 1, 2, \ldots, r)$$

which is identical with the form as expressed by Equ. (14).

Regarding the quantities $d_i\,(i = 1,2, \ldots, p)$, $a_j\,(j = 1,2, \ldots, q)$ and $b_k(k = 1,2, \ldots, r)$ as unknown parameters the normal equations for determining $\bar{d}_i$, $\bar{a}_j$ and $\bar{b}_k$ will be the following:

$$\frac{\partial P}{\partial d_i} = \frac{1}{\sigma^{2*}} \sum_{j=1}^{q} \Delta_{ij}(n_{ij} - \bar{d}_i) = 0\,, \quad (i = 1, 2, \ldots, p) \tag{25}$$

$$\left.\begin{array}{l} \frac{\partial P}{\partial a_j} = \frac{1}{\sigma^{2*}} \sum_{j=1}^{p} \Delta_{ij}(n_{ij} - n^{i} - \bar{a}_j + \bar{a}_{\cdot}^{i} - \beta'_{ijk}\bar{b}_k) = 0 \\ \qquad\qquad (j = 1, 2, \ldots, q) \\ \frac{\partial P}{\partial b_k} = \frac{1}{\sigma^{2*}} \sum_{i=1}^{p}\sum_{j=1}^{q} \Delta_{ij}\beta'_{ijk}(n_{ij} - n_{\cdot}^{i} - \bar{a}_j + \bar{a}_{\cdot}^{i} - \beta'_{ijk'}\bar{b}_{k'}) = 0 \\ \sum_{j=1} \bar{a}_j = 0\,. \end{array}\right\} , \tag{26}$$

It can be seen that Equs. (25) and (26) do not have common variables and

$$\bar{d}_i = n_{\cdot}^{i} \quad (i = 1, 2, \ldots, p)\,.$$

The system (26) consists of only $q + r + 1$ equations the matrix of which can be inverted much easier than the matrix (23), and the covariance matrix of the quantities $\bar{a}_j\,(j = 1, 2, \ldots, q)$ and $\bar{b}_k\,(k = 1,2, \ldots, r)$ can thus be determined.

The estimated values of the quantities $A + c_i (i = 1, 2, \ldots, p)$ can be obtained on the basis of $\bar{a}_j$, $\bar{b}_k$ and Equ. (24). The variances involving $\bar{A} + \bar{c}_i$ $(i = 1,2, \ldots, p)$ can also be calculated on the basis of (24) and of the fact that the $\bar{d}_i\,(i = 1,2, \ldots, p)$ are independent of $\bar{a}_j\,(j = 1,2, \ldots, q)$ and $\bar{b}_k\,(k = 1,2, \ldots, r)$, E.g.

covariance of $(\bar{A} + \bar{c}_i, \bar{a}_j) = -$ covariance of $(\bar{a}_{\cdot}^{i}, \bar{a}_j) - \sum_{k=1}^{r}$ covariance of $(\beta_{\cdot k}^{i}\bar{b}_k, \bar{a}_j)$.

VII. Concluding remarks

§ 23. It has been shown that the representation

$$\langle n_{ij} \rangle = A + c_i + a_j + \beta_{ijk} b_k$$
$$(i = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, q; \quad k = 1, 2, \ldots, r) \tag{27}$$
$$a. = c. = 0$$

may be very often preferred to that of the Fourier type. Assuming n_{ij} to be independent Gaussian variables, explicite expressions have been given for the maximum likelihood estimations of the parameters A, c_i, a_j, b_k together with the variances and covariances of the estimates. The probability that the deviations of the values $\bar{a}_j$ $(j = 1,2, \ldots, q)$ from zero are due to statistical fluctuations only has been derived as well as an unbiased estimated value of the mean square amplitude

$$a^2 = \frac{1}{q} \sum_{j=1}^{q} a_j^2$$

and the variance of this estimate.

The advantage of the representation (27) is that

a) it does not involve the assumption that the periodic function be a trigonometric polynome, and

b) the numerical calculations required are by far simpler than those involved in a Fourier representation with the same number of parameters.

There are cases when the first and may be also the second Fourier components have direct physical meanings and should be estimated. In such cases the Fourier method and the simple step function method may be combined to yield the estimated values of the amplitudes of the first and second harmonics and the rest may be investigated by the step function representation.

§ 24. The existence of a slow variation of the intensity, i.e. the probability ε_c that the deviations from zero of the estimated values $\bar{c}_i$ $(i = 1,2, \ldots, p)$ are due to statistical fluctuations alone, can be determined just in the same way as it has been done in the case of the periodic part of the variation. All what we have to do is to change the notations referring to "a" into those referring to "c". We thus have

$$\varepsilon_c = P_{p-1,q'}(\zeta > y_c)$$

with

$$y_c = \frac{1}{2} \ln \frac{1}{p-1} \frac{\bar{\mathbf{c}}_s \mathscr{B}_{sc}^{+} \tilde{\bar{\mathbf{c}}}_s}{o^2},$$

where $\mathscr{B}_{sc}$ stands for the matrix obtained when omitting the row and column numbered s in the matrix $\mathscr{B}_{cc}$ (see § 18).

The unbiased estimated value of

$$c^2 = \frac{1}{p} \sum_{i=1}^{p} c_i^2$$

the mean square amplitude of the slow variation and the variance of this estimate can also be obtained by using the formulae referring to a^2 and changing the notations accordingly.

§ 26. Applications of some parts of the results reported on in this paper may be found in [6] and [7]. Full applications will be published later on.

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Appendix I

If the variables $x_i (i = 1,2, \ldots, n)$ are distributed normally according to the joint density function

$$\frac{\sqrt{\det(A_{ik})}}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2} A_{ik}(x_i - a_i)(x_k - a_k)\right],$$

$$(i, k = 1, 2, \ldots, n)$$

then the variable

$$u = \sum_{i=1}^{n} (x_i - x.)^2$$

has the expected value

$$\langle u \rangle = A_i A_i + A^*_{ii}, \tag{A1}$$

and the variance

$$\sigma_u^2 = 2 A^*_{ij}(A^*_{ij} + 2 A_i A_j), \tag{A2}$$

where

$$A_i = a_i - a$$

and

$$A^*_{ij} = A^+_{ij} - A^+_{i.} - A^+_{.j} + A^+_{..} \,. \tag{A3}$$

Proof. The quantities

$$x'_i = x_i - x. \qquad (i = 1, 2, \ldots, n)$$

have the expected values

$$\langle x_i' \rangle = A_i$$

and their covariance matrix is composed of the elements $A_{ij}^*(i,\ j = 1,2,\ldots, n)$, It can be shown by means of simple calculations that the quantity

$$u = x_i' x_i'$$

has the expected value and variance as given by Equs. ($A1$) ($A2$), respectively.

Appendix II

The variables x_i $(i = 1,2,\ldots, n)$ should be distributed normally with

$$\langle x_i \rangle = a_i \qquad (i = 1, 2, \ldots, n)$$

and

$$\langle (x_i - a_i)(x_j - a_j \rangle = \sigma^2 A_{ij}, \qquad (i, j = 1, 2, \ldots, n)\ ,$$

where the A_{ij} elements are known, the rank of the matrix $\| A_{ij} \|$ is at least $n - 1$, a_i and σ^2 are unknown, but σ^2 has a known estimate $\overline{\sigma^2}$, such that $p'\overline{\sigma^2}/\sigma^2$ is distributed according to χ^2 with p' degrees of freedom, and $\overline{\sigma^2}$ is independent of $\chi_i(i = 1,2,\ldots, n)$.

Clearly the value

$$\overline{a^2} = \frac{1}{n}\left[x_i x_i - A_{ii}\overline{\sigma^2}\right] \tag{A4}$$

is an unbiased estimate of

$$a^2 = \frac{1}{n} a_i a_i .$$

It will be shown that the estimated variance of $\overline{a}^2$ is

$$\overline{\sigma^2_{\overline{a}^2}} = \frac{2\overline{\sigma^2}}{n^2}\left[2 Q_3 - \frac{\overline{\sigma^2}}{p'+2}(p' A_{ij} A_{ij} - A_{ii} A_{jj})\right], \tag{A5}$$

where

$$Q_3 = x_i A_{ij} x_j .$$

According to Appendix I

$$\sigma^2_{x_i x_i} = 2\sigma^2 A_{ij}(\sigma^2 A_{ij} + 2 a_i a_j) .$$

Taking into account that

$$\overline{(\sigma^2)^2} = \frac{p'}{p'+2}(\overline{\sigma^2})^2$$

and

$$\overline{\sigma^2_{\overline{\sigma^2}}} = \frac{2}{p'}\,\overline{(\sigma^2)^2} = \frac{2}{p'+2}\,(\overline{\sigma^2})^2,$$

furthermore that $x_i x_i$ is independent of $\overline{\sigma^2}$, and that the estimated value of $a_i a_j \sigma^2$ is

$$x_i\,x_j\,\overline{\sigma^2} - \frac{p'}{p'+2}\,A_{ij}\,(\overline{\sigma^2})^2\,.$$

Equs. (A5) can be obtained without difficulty.

Appendix III

A one-row matrix $\mathbf{b}^{(1,n)}$ with elements $b_i (i = 1,2, \ldots, n)$ all different from zero should be given together with a one-column matrix $\mathbf{a}^{(n,1)}$ with elements $a_i (i = 1,2, \ldots, n)$, and a symmetric matrix $\mathbf{D}^{(n,n)}$ of the rank $n - 1$. Each diagonal submatrix of $\mathbf{D}$ with $n - 1$ rows and $n - 1$ columns should be of the rank $n - 1$. The elements of $\mathbf{a}$ and $\mathbf{D}$ should satisfy the equations

$$\mathbf{b}\mathbf{a} = \mathbf{O}^{(1,1)} = O \quad \text{i.e.} \quad b_i\,a_i = O \tag{A6}$$

and

$$\mathbf{D}\tilde{\mathbf{b}} = \mathbf{O}^{(n,1)}, \quad \text{i.e.} \quad D_{ij}\,b_j = O \quad (i = 1, 2, \ldots, n). \tag{A7}$$

Let us denote by $\mathbf{a}_r^{(n-1,1)}$ the one-column matrix with elements $a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n$ and by $\mathbf{D}_r^{(n-1,n-1)}$ the matrix with elements $D_{ij} (i,j = 1, \ldots, r - 1, r + 1, \ldots, n)$.

It will be shown that the quantity

$$Q_r = \tilde{\mathbf{a}}_r\,\mathbf{D}_r^{+}\,\mathbf{a}_r \tag{A8}$$

is independent of r. This can be done as follows:

If the one-column matrix $\mathbf{y}_0^{(n,1)}$ is a solution of the equation

$$\mathbf{D}\mathbf{y}_0 = \mathbf{a}\,, \tag{A9}$$

then all solutions of (A9) may be written in the form

$$\mathbf{y} = \mathbf{y}_0 + \lambda\tilde{\mathbf{b}}\,,$$

where λ denotes an arbitrary scalar quantity. Introducing the notation

$$\mathbf{z}_r^{(n-1,1)} = \mathbf{D}_r^{+}\,\mathbf{a}_r\,, \tag{A10}$$

it can be seen that

$$\mathbf{D}_r z_r = \mathbf{a}_r . \tag{A11}$$

Let us denote the elements of $\mathbf{z}_r$ by $z_1, z_2, \ldots, z_{r-1}, z_{r+1}, \ldots, z_n$ and let us denote by $\mathbf{z}_r'^{(n,1)}$ the matrix consisting of the elements $z_1, \ldots, z_{r-1}$, O, $z_{r+1}, \ldots, z_n$. It follows from (A6), (A7), and (A11) that $\mathbf{z}_r'$ is a solution of (A9), may thus be written in the form

$$\mathbf{z}_r' = \mathbf{y}_0 + \lambda_r \tilde{\mathbf{b}} . \tag{A12}$$

Making use of Q_r may be expressed by means of $\mathbf{z}'_r$ in the form

$$Q_r = \tilde{\mathbf{a}}\mathbf{z}_r' . \tag{A13}$$

Let us now investigate another quantity (A8), e.g.

$$Q_s = \tilde{\mathbf{a}}_s \mathbf{D}_s^+ \mathbf{a}_s .$$

Repeating the considerations above we arrive at

$$\mathbf{z}_s' = \mathbf{y}_0 + \lambda_s \tilde{\mathbf{b}} \tag{A14}$$

and

$$Q_s = \tilde{\mathbf{a}}\mathbf{z}_s' . \tag{A15}$$

On the basis of (A12) — (A15), and (A6) we have

$$Q_r - Q_s = (\lambda_r - \lambda_s)\, \widetilde{\mathbf{a}\mathbf{b}} = 0$$

what was to be shown.

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НЕКОТОРЫЕ ПРОБЛЕМЫ ИССЛЕДОВАННОЙ ПЕРИОДИЧНОСТИ ВРЕМЕННОЙ СЕРИИ КОСМИЧЕСКИХ ЛУЧЕЙ

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Резюме

Дается обобщение метода анализа расхождения с целью исследования существования и вида периодичности с данной длиной периода. Принимаются во внимание как переменные изменения интенсивности космических лучей, так и метеорологические эффекты. Далее, кроме точного исследования существования периодичности излагаются максимальные вероятные оценки как для постоянных, характеризующих вид периодичности, так и для главной квадратичной амплитуды периодической функции вместе с их относительной статистической ошибкой в случае произвольного числа метеорологических факторов, влияющих на интенсивность космического излучения.

Показываются невыгоды применения метода Фурье в исследовании периодичности с определенной длиной периода Наконец, показывается, что факт статистического определения метеорологических коэффициентов нельзя отделять от анализа периодичности.