Acta Physica Academiae Scientiarum Hungaricae, Tomus 27, pp. 435—439 (1969)

INTEGRATION OF THE DYNAMICAL SYMMETRY GROUPS FOR THE $-1/r$ POTENTIAL*

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(Received 20. II. 1969)

Finite dynamical symmetry transformations of the Kepler motion are given in closed analytic form.

In recent years there has been considerable interest in the problem of dynamical groups in Classical and Quantum Mechanics. For the classical one particle problem in a $-1/r$ potential, characterized by the energy function (Hamiltonian)

$$E = \frac{p^2}{2m} - \frac{g}{r}, \tag{1}$$

two well-known vectorial constants of motion exist (see e.g., [1]): the angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{2}$$

and the Laplace—Lenz—Runge vector

$$\mathbf{A} = \frac{mg}{p_0} \left(\frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times \mathbf{L}}{mg} \right), \tag{3}$$

where

$$p_0 = \sqrt{2m|E|}. \tag{4}$$

The Poisson bracket relations of **L** and **A** coincide with the defining relations of the SO(4) and SO(3,1) Lie algebras for negative and positive energies, respectively. Higher Lie algebras can also be constructed out of the primitive dynamical variables. The pioneering work in this direction has been done by BACRY [2]. One finds (see also [3]) that the quantities

$$\mathbf{B} = r\mathbf{p} \begin{Bmatrix} \cos \\ \mathrm{ch} \end{Bmatrix} \frac{p_0(\mathbf{pr})}{mg} + \frac{mg}{p_0} \left[\frac{\mathbf{r}}{r} - \frac{(\mathbf{rp})\,\mathbf{p}}{mg} \right] \begin{Bmatrix} \sin \\ \mathrm{sh} \end{Bmatrix} \frac{p_0(\mathbf{pr})}{mg}, \tag{5a}$$

$$S = \frac{mg}{p_0} \left(1 - \frac{rp^2}{mg} \right) \begin{Bmatrix} \cos \\ \mathrm{ch} \end{Bmatrix} \frac{p_0(\mathbf{pr})}{mg} \mp \mathbf{rp} \begin{Bmatrix} \sin \\ \mathrm{sh} \end{Bmatrix} \frac{p_0(\mathbf{pr})}{mg}, \tag{5b}$$

* Dedicated to Prof. P. GOMBÁS on his 60th birthday.

as well as **L** and **A**, obey the bracket relations of the SO(4,1) and SO(3,2) Lie algebras for negative and positive energies, respectively. Further,

$$\mathbf{C} = \pm \frac{mg}{p_0}\left[\frac{\mathbf{r}}{r} - \frac{(\mathbf{rp})\,\mathbf{p}}{mg}\right]\begin{Bmatrix}\cos\\ \mathrm{ch}\end{Bmatrix}\frac{p_0(\mathbf{pr})}{mg} - r\mathbf{p}\begin{Bmatrix}\sin\\ \mathrm{sh}\end{Bmatrix}\frac{p_0(\mathbf{pr})}{mg}, \tag{6a}$$

$$T = -\mathbf{rp}\begin{Bmatrix}\cos\\ \mathrm{ch}\end{Bmatrix}\frac{p_0(\mathbf{pr})}{mg} - \frac{mg}{p_0}\left(1 - \frac{rp^2}{mg}\right)\begin{Bmatrix}\sin\\ \mathrm{sh}\end{Bmatrix}\frac{p_0(\mathbf{pr})}{mg}, \tag{6b}$$

together with **L** and **A**, obey the bracket relations of the SO(4,1) algebra for both negative and positive energies. In (5) and (6) the trigonometric functions should be taken for bound states, and the hyperbolic functions for positive energy. (The upper and lower signs refer to negative and positive energies, respectively.) Finally, one finds that **L**, **A**, **B**, S **C** and T, taken together with the new quantity

$$M = \pm\sqrt{\mp\frac{mg^2}{2E}}, \tag{7}$$

satisfy the Poisson bracket relations of the Lie algebra of the SO(4,2) group, again for both positive and negative energies [3].

The three-dimensional finite rotations generated by the angular momentum vector **L** can be immediately obtained by integration. As to the finite transformations of the dynamical variables **r** and **p** generated by the Laplace—Lenz—Runge vector **A**, it has been pointed out in [4] that these transformations are non-linear and cannot be given in explicit analytic form. In [5] it has been proved that the transformations generated by the angular momentum and the Laplace—Lenz—Runge vector **A** form, indeed, a group of canonical transformations. The finite group elements have been obtained by integration, and the global structure of this dynamical (invariance) symmetry group has been clarified; the group is isomorphic to the four-dimensional rotation group SO(4).

In what follows the dynamical invariance and non-invariance symmetry transformations will be treated in a unified and general manner. The use of the Bacry variables [3] as the basic quantities makes it possible to obtain explicit transformation formulae. Introduce the four-dimensional notation

$$B_\alpha\,[\mathbf{B},(\pm)^{-1/2}S],\quad C_\alpha\,[\mathbf{C},(\pm)^{-1/2}T]$$

for the quantities (5) and (6). The Bacry variables may be defined as

$$b_\alpha = \pm\, mgM^{-2}\,B_\alpha, \qquad c_\alpha = \pm\,(mg)^{-1}\,MC_\alpha; \tag{8}$$

they can be used equivalently instead of the original **r**, **p** position and momentum variables. Thus, the transformation formulae for these basic variables b_α, c_α are determined by those valid for B_α, C_α and M.

The three-dimensional rotations generated by the angular momentum vector will not be considered. For an infinitesimal canonical transformation generated by the Laplace—Lenz—Runge vector one has

$$\left.\begin{aligned} \delta\mathbf{B} &= (\mathbf{B}, \mathbf{A}\delta\alpha) = - S\delta\alpha\,, \\ \delta S &= (S, \mathbf{A}\delta\alpha) = \pm\, \mathbf{B}\delta\alpha\,, \\ \delta\mathbf{C} &= (\mathbf{C}, \mathbf{A}\delta\alpha) = - T\delta\alpha\,, \\ \delta T &= (T, \mathbf{A}\delta\alpha) = \pm\, \mathbf{C}\delta\alpha\,, \\ \delta M &= (M, \mathbf{A}\delta\alpha) = 0\,. \end{aligned}\right\} \tag{9}$$

Further, for the infinitesimal canonical transformations generated by the quantities **B**, S, **C** and T the following formulae hold:

$$\left.\begin{aligned} \delta\mathbf{L} &= (\mathbf{L}, \mathbf{B}\delta\beta) = -\mathbf{B}\times\delta\beta\,, \\ \delta\mathbf{A} &= (\mathbf{A}, \mathbf{B}\delta\beta) = S\delta\beta\,, \\ \delta\mathbf{B} &= (\mathbf{B}, \mathbf{B}\delta\beta) = \mathbf{L}\times\delta\beta\,, \\ \delta S &= (S, \mathbf{B}\delta\beta) = \mathbf{A}\delta\beta\,, \\ \delta\mathbf{C} &= (\mathbf{C}, \mathbf{B}\delta\beta) = M\delta\beta\,, \\ \delta T &= (T, \mathbf{B}\delta\beta) = 0\,, \\ \delta M &= (M, \mathbf{B}\delta\beta) = \mathbf{C}\delta\beta\,; \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} \delta\mathbf{L} &= (\mathbf{L}, S\delta\sigma) = 0\,, \\ \delta\mathbf{A} &= (\mathbf{A}, S\delta\sigma) = \mp\, \mathbf{B}\delta\sigma\,, \\ \delta\mathbf{B} &= (\mathbf{B}, S\delta\sigma) = - \mathbf{A}\delta\sigma\,, \\ \delta S &= (S, S\delta\sigma) = 0\,, \\ \delta\mathbf{C} &= (\mathbf{C}, S\delta\sigma) = 0\,, \\ \delta T &= (T, S\delta\sigma) = \pm\, M\delta\sigma\,, \\ \delta M &= (M, T\delta\sigma) = T\delta\sigma\,; \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} \delta\mathbf{L} &= (\mathbf{L}, \mathbf{C}\delta\gamma) = -\mathbf{C}\times\delta\gamma\,, \\ \delta\mathbf{A} &= (\mathbf{A}. \mathbf{C}\delta\gamma) = T\delta\gamma\,, \\ \delta\mathbf{B} &= (\mathbf{B}, \mathbf{C}\delta\gamma) = - M\delta\gamma\,, \\ \delta S &= (S, \mathbf{C}\delta\gamma) = 0\,, \\ \delta\mathbf{C} &= (\mathbf{C}, \mathbf{C}\delta\gamma) = \pm\, \mathbf{L}\times\delta\gamma\,, \\ \delta T &= (T, \mathbf{C}\delta\gamma) = \pm\, \mathbf{A}\delta\gamma\,, \\ \delta M &= (M, \mathbf{C}\delta\gamma) = \mp\mathbf{B}\delta\gamma\,; \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} \delta\mathbf{L} &= (\mathbf{L}, T\delta\tau) = 0\,, \\ \delta\mathbf{A} &= (\mathbf{A}, T\delta\tau) = \mp\, \mathbf{C}\delta\tau\,, \\ \delta\mathbf{B} &= (\mathbf{B}, T\delta\tau) = 0\,, \\ \delta S &= (S, T\delta\tau) = \mp\, M\delta\tau\,, \\ \delta\mathbf{C} &= (\mathbf{C}, T\delta\tau) = \mp\, \mathbf{A}\delta\tau\,, \\ \delta T &= (T, T\delta\tau) = 0\,, \\ \delta M &= (M, T\delta\tau) = \mp\, S\delta\tau\,. \end{aligned}\right\} \tag{13}$$

The integration can be easily carried out. From (9) one obtains

$$\left.\begin{aligned} \mathbf{B}' &= \mathbf{B} - (\mathbf{Bn})\,\mathbf{n} + (\mathbf{Bn})\,\mathbf{n} \begin{Bmatrix}\cos\\ \mathrm{ch}\end{Bmatrix} \alpha - S\mathbf{n} \begin{Bmatrix}\sin\\ \mathrm{sh}\end{Bmatrix} \alpha, \\ S' &= S \begin{Bmatrix}\cos\\ \mathrm{ch}\end{Bmatrix} \alpha \pm \mathbf{Bn} \begin{Bmatrix}\sin\\ \mathrm{sh}\end{Bmatrix} \alpha, \\ \mathbf{C}' &= \mathbf{C} - (\mathbf{Cn})\,\mathbf{n} + (\mathbf{Cn})\,\mathbf{n} \begin{Bmatrix}\cos\\ \mathrm{ch}\end{Bmatrix} \alpha - T\mathbf{n} \begin{Bmatrix}\sin\\ \mathrm{sh}\end{Bmatrix} \alpha, \\ T' &= T \begin{Bmatrix}\cos\\ \mathrm{ch}\end{Bmatrix} \alpha \pm \mathbf{Cn} \begin{Bmatrix}\sin\\ \mathrm{sh}\end{Bmatrix} \alpha, \\ M' &= M \end{aligned}\right\} \tag{14}$$

($\alpha = |\boldsymbol{\alpha}|$, $\mathbf{n} = \boldsymbol{\alpha}/\alpha$); from (10) one has

$$\left.\begin{aligned} \mathbf{L}' &= (\mathbf{Ln})\,\mathbf{n} + [\mathbf{L} - (\mathbf{Ln})\,\mathbf{n}]\,\mathrm{ch}\,\beta - \mathbf{B}\times\mathbf{n}\,\mathrm{sh}\,\mathbf{B}\,, \\ \mathbf{B}' &= (\mathbf{Bn})\,\mathbf{n} + [\mathbf{B} - (\mathbf{Bn})\,\mathbf{n}]\,\mathrm{ch}\,\beta + \mathbf{L}\times\mathbf{n}\,\mathrm{sh}\,\beta\,, \\ \mathbf{A}' &= \mathbf{A} - (\mathbf{An})\,\mathbf{n} + (\mathbf{An})\mathbf{n}\,\mathrm{ch}\,\beta + S\mathbf{n}\,\mathrm{sh}\,\beta\,, \\ S' &= S\,\mathrm{ch}\,\beta + \mathbf{An}\,\mathrm{sh}\,\beta\,, \\ \mathbf{C}' &= \mathbf{C} - (\mathbf{Cn})\,\mathbf{n} + (\mathbf{Cn})\,\mathbf{n}\,\mathrm{ch}\,\beta + M\mathbf{n}\,\mathrm{sh}\,\beta\,, \\ T' &= T, \\ M' &= M\,\mathrm{ch}\,\beta + \mathbf{Cn}\,\mathrm{sh}\,\beta \end{aligned}\right\} \tag{15}$$

($\beta = |\boldsymbol{\beta}|$, $\mathbf{n} = \boldsymbol{\beta}/\beta$); from (11) one gets

$$\left.\begin{aligned} \mathbf{L}' &= \mathbf{L}\,, \\ \mathbf{A}' &= \mathbf{A} \begin{Bmatrix}\mathrm{ch}\\ \cos\end{Bmatrix} \sigma \mp \mathbf{B} \begin{Bmatrix}\mathrm{sh}\\ \sin\end{Bmatrix} \sigma\,, \\ \mathbf{B}' &= \mathbf{B} \begin{Bmatrix}\mathrm{ch}\\ \cos\end{Bmatrix} \sigma - \mathbf{A} \begin{Bmatrix}\mathrm{sh}\\ \sin\end{Bmatrix} \sigma\,, \\ S' &= S\,, \\ \mathbf{C}' &= \mathbf{C}\,, \\ T' &= T \begin{Bmatrix}\mathrm{ch}\\ \cos\end{Bmatrix} \sigma + M \begin{Bmatrix}\mathrm{sh}\\ \sin\end{Bmatrix} \sigma\,, \\ M' &= M \begin{Bmatrix}\mathrm{ch}\\ \cos\end{Bmatrix} \sigma \pm T \begin{Bmatrix}\mathrm{sh}\\ \sin\end{Bmatrix} \sigma\,; \end{aligned}\right\} \tag{16}$$

from (12) one has

$$\left.\begin{aligned}
\mathbf{L}' &= (\mathbf{L}\mathbf{n})\,\mathbf{n} + [\mathbf{L} - (\mathbf{L}\mathbf{n})\,\mathbf{n}]\begin{Bmatrix}\mathrm{ch}\\ \cos\end{Bmatrix}\gamma - \mathbf{C}\times\mathbf{n}\begin{Bmatrix}\mathrm{sh}\\ \sin\end{Bmatrix}\gamma\,,\\
\mathbf{A}' &= \mathbf{A} - (\mathbf{A}\mathbf{n})\,\mathbf{n} + (\mathbf{A}\mathbf{n})\,\mathbf{n}\begin{Bmatrix}\mathrm{ch}\\ \cos\end{Bmatrix}\gamma + T\begin{Bmatrix}\mathrm{sh}\\ \sin\end{Bmatrix}\gamma\,,\\
\mathbf{B}' &= \mathbf{B} - (\mathbf{B}\mathbf{n})\,\mathbf{n} + (\mathbf{B}\mathbf{n})\,\mathbf{n}\begin{Bmatrix}\mathrm{ch}\\ \cos\end{Bmatrix}\gamma - M\mathbf{n}\begin{Bmatrix}\mathrm{sh}\\ \sin\end{Bmatrix}\gamma\,,\\
S' &= S\,,\\
\mathbf{C}' &= (\mathbf{C}\mathbf{n})\,\mathbf{n} + [\mathbf{C} - (\mathbf{C}\mathbf{n})\,\mathbf{n}]\begin{Bmatrix}\mathrm{ch}\\ \cos\end{Bmatrix}\gamma \pm \mathbf{L}\times\mathbf{n}\begin{Bmatrix}\mathrm{sh}\\ \sin\end{Bmatrix}\gamma\,,\\
T' &= T\begin{Bmatrix}\mathrm{ch}\\ \cos\end{Bmatrix}\gamma \pm \mathbf{A}\mathbf{n}\begin{Bmatrix}\mathrm{sh}\\ \sin\end{Bmatrix}\gamma\,,\\
M' &= M\begin{Bmatrix}\mathrm{ch}\\ \cos\end{Bmatrix}\gamma \mp \mathbf{B}\mathbf{n}\begin{Bmatrix}\mathrm{sh}\\ \sin\end{Bmatrix}\gamma
\end{aligned}\right\} \tag{17}$$

$(\gamma = |\boldsymbol{\gamma}|,\ \mathbf{n} = \boldsymbol{\gamma}/\gamma)$; finally, from (13) it follows that

$$\left.\begin{aligned}
\mathbf{L}' &= \mathbf{L}\,,\\
\mathbf{A}' &= \mathbf{A}\,\mathrm{ch}\,\tau \mp \mathbf{C}\,\mathrm{sh}\,\tau\,,\\
\mathbf{B}' &= \mathbf{B}\,,\\
S' &= S\,\mathrm{ch}\,\tau \mp M\,\mathrm{sh}\,\tau\,,\\
\mathbf{C}' &= \mathbf{C}\,\mathrm{ch}\,\tau \mp \mathbf{A}\,\mathrm{sh}\,\tau\,,\\
T' &= T\\
M' &= M\,\mathrm{ch}\,\tau \mp S\,\mathrm{sh}\,\tau\,.
\end{aligned}\right\} \tag{18}$$

This completes the integration of the SO(4,2) full dynamical symmetry group for the classical Kepler problem. The transformation formulae (14)—(18) determine the analytic form of the transformation laws for the primitive dynamical variables b_α, c_α or $\mathbf{r}$, $\mathbf{p}$; though, the formulae cannot be solved explicitly for $\mathbf{r}'$ and $\mathbf{p}'$.

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ИНТЕГРИРОВАНИЕ ДИНАМИЧЕСКИХ ГРУПП СИММЕТРИИ ДЛЯ ПОТЕНЦИАЛА — 1/r

Г. ДЬЁРДИ

Резюме

В закрытой аналитической форме даются конечные динамические преобразования симметрии движения Кеплера.