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OSCILLATIONS OF A RELATIVISTIC ELECTRON PLASMA IN AN EXTERNAL MAGNETIC FIELD*

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A relativistic treatment of plasma oscillations is presented. The calculations are carried out for the so called low β plasma, where the pressure is negligible as compared to the magnetic energy density. The covariant dispersion relations are derived and their meaning is compared to that of the classical ones.

§ 1. Introduction

The aim of this work is to study the small amplitude waves of an electron plasma on the basis of relativistic dynamics, to deduce the covariant forms of the dispersion relations of the typical modes of oscillations.

In order to motivate the necessity of the relativistic treatment we mention only a theoretical remark. As is well known, the problem of plasma oscillations is studied in classical plasma physics by means of the Newtonian equations of motion and some restricted, approximative forms of Maxwell's equations. The results of this classical treatment are in general well compatible with the experimental data, a discrepancy arises in the fact, that Newtonian equations are Galilei invariant while Maxwell's equations are Lorentz invariant. So it can well happen that this half Galileian half Lorentzian treatment will lead to false results. Therefore it is desirable to investigate this problem in a fully covariant manner. It is generally cited that relativistic effects become important only at high temperatures, which statement is only the half of the truth. Relativistic effects will be important also in the case when the involved energies — e.g. electromagnetic energy, compression energy, etc. — are comparable with the rest energy of the system.

Here for the sake of simplicity we consider an ideal electron plasma and we shall investigate the possible modes in a covariant manner. It is supposed that the electron gas is a classical one, and it is neutralized by a homogeneous positive background which is immobile in the rest frame and does not take part

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in the oscillations. Restriction is made to the so called low β plasma case, i.e. it is supposed that the density of electromagnetic energy is much greater than the compression energy, so the latter is omitted for the moment. Since the pressure is neglected, there will be no information concerning the acoustic waves of the electron plasma in this model.

In the treatment we shall consider the oscillations of the electron plasma both without and with external magnetic fields and in the discussion of the results comparison will be made with the classical dispersion relations.

§ 2. Fundamental equations

When the electron gas is not too dense the electromagnetic field can be described by the field tensor

$$F_{ik}=\begin{Bmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{Bmatrix}, \tag{1}$$

which satisfies the source free group of Maxwell's equations:

$$\varepsilon_{iklm}\,\partial_k F_{lm}=0\,. \tag{2}$$

Here $\partial_k=\partial/\partial x_k$, $x_4=ict$ and ε_{iklm} is the completely antisymmetric Levi-Civita tensor with $\varepsilon_{1234}=+1$. The summation convention is understood for doubly occurring Latin indices.

The electromagnetic field F_{ik} is coupled to its sources by Maxwell's second equation

$$\partial_k F_{ik}=\frac{4\pi}{c}j_i, \tag{3}$$

where the current four vector

$$i_i=env_i\,, \tag{4}$$

e being the electronic charge, the invariant scalar n is the density of electrons (measured in the comoving frame of reference), and v_i is the four velocity of the electrons.

The equations of motion is

$$\partial_k(nv_i\,v_k)=\frac{en}{mc}F_{ik}\,v_k\,, \tag{5}$$

where the pressure term is omitted. This approximation is valid if

$$\beta = \left| \frac{p}{\frac{1}{8\pi} \vec{H}^2} \right| = \left| \frac{p}{\frac{1}{16\pi} F_{rs} F_{rs}} \right| \ll 1,$$

i.e. we consider a low β plasma. Here m is the rest mass of the electron. The velocity four vector must satisfy the equation

$$v_r v_r = - c^2 . \tag{6}$$

The charge conservation

$$\partial_i \partial_k F_{ik} = \frac{4\pi e}{c} \partial_i (n v_i) = 0$$

leads to

$$v_k \partial_k v_i = \frac{e}{mc} F_{ik} v_k \tag{7}$$

the final form of the equation of motion we use.

The extension of the electron plasma is supposed to be infinite, to avoid boundary value problems for the moment.

The system of fundamental equations of the low β plasma is then

$$\left.\begin{aligned} &\varepsilon_{iklm} \partial_k F_{lm} = 0 \\ &\partial_k F_{ik} = \frac{4\pi e}{c} n v_i \\ &\partial_k (n v_k) = 0 \\ &v_r v_r = - c^2 \\ &v_k \partial_k v_i = \frac{e}{mc} F_{ik} v_k \end{aligned}\right\} , \tag{8}$$

which is a coupled system of nonlinear partial differential equations.

In order to linearize (8) we suppose that the electron gas is in a nonperturbed equilibrium state. In this state its charge is completely neutralized by the immobile positive background. There is no charge, no current without external perturbation. In covariant way this reads

$$(Ne^+ + Ne^-) u_k = 0 ,$$

where u_k is the electron equilibrium four velocity, N the equilibrium particle density. If F_{ik} denotes the field in the equilibrium system, then in the unperturbed state

$$\dot{F}_{ik} u_k = 0 .$$

Therefore $(\dot{F}_{ik}, u_k, N) = (\text{const})$ is a solution to (8), if $u_k u_k = -c^2$ in the unperturbed equilibrium state.

Then we superimpose to this state some small amplitude disturbances $(\delta F_{ik}, \delta u_k, \delta n)$ and retain those terms where only one disturbance occurs. This procedure leads to

$$\left.\begin{aligned} &\varepsilon_{iklm}\, \partial_k\, \delta F_{lm} = 0 \\ &\partial_k \partial F_{ik} = \frac{4\pi e}{c}\,(N\delta u_i + u_i\,\delta n) \\ &N\, \partial_r\, \delta u_r + n_r\, \partial_r\, \delta n = 0 \\ &u_k \partial_k \delta u_i = \frac{e}{mc}\,(u_k\, \delta F_{ik} + \dot{F}_{ik}\, \delta u_k) \\ &u_r\, \delta u_r = 0 \end{aligned}\right\}, \tag{9}$$

which is a coupled system of linear partial differential equations for the disturbances. In order to solve these coupled first order homogeneous linear differential equations we use the trial functions

$$\begin{Bmatrix} \delta n\,(x_r) \\ \delta u_i\,(x_r) \\ \delta F_{ik}\,(x_r) \end{Bmatrix} = \begin{Bmatrix} \delta n \\ \delta u_i \\ \delta F_{ik} \end{Bmatrix} \exp\, ik_r\, x_r \tag{10}$$

stating the validity of the superposition principle. Here δn, δu_i and δF_{ik} will be appropriately chosen constants, the determination of which requires a non-trivial solution of the algebraic system of homogeneous equations

$$\left.\begin{aligned} &\varepsilon_{iklm}\, k_k\, \delta F_{lm} = 0 \\ &k_k\, \delta F_{ik} = \frac{4\pi e}{ic}\,(N\delta u_i + u_i\, \delta n) \\ &Nk_r\, \delta u_r + u_r\, k_r\, \delta n = 0 \\ &iu_r\, k_r\, \delta u_i = \frac{e}{mc}\, u_r\, \delta F_{ir} + \frac{e}{mc}\, \dot{F}_{ik}\, \delta u_k \\ &u_r\, \delta u_r = 0 \end{aligned}\right\} \tag{11}$$

Before looking for the solutions of (11), some transformations will be useful.

The decomposition of the first equation of (11) gives

$$k_k \delta F_{lm} + k_m \delta F_{kl} + k_l \delta F_{mk} = 0. \tag{12}$$

Then multiplication with u_m and summation over m provides an equation for the quantity

$$\delta\Phi_i = u_k \delta F_{ik}, \tag{13}$$

which gives the perturbation of the electric field in the comoving frame, namely

$$L \delta F_{kl} = k_l \delta\Phi_k - k_k \delta\Phi_l, \tag{14}$$

where

$$L = k_r u_r. \tag{15}$$

This combined with the second equation of (11) gives

$$k_r k_r \delta\Phi_i - k_i k_r \delta\Phi_r = \frac{4\pi e L}{ic} (N\delta u_i + u_i \delta n). \tag{16}$$

Finally we introduce the notation

$$\Omega_{ik} = \frac{e}{mc} \dot{F}_{ik}. \tag{17}$$

Ω_{ik} is an antisymmetric tensor, built up from the equilibrium values of the electromagnetic fields. Since in equilibrium there is no electric field at all, we have

$$\Omega_{ik} u_k = \frac{e}{mc} \dot{F}_{ik} u_k \equiv 0.$$

In other words Ω_{ik} contains only the components of the electron cyclotron frequency $\vec{\omega}_c = (e/mc)\vec{H}_0$, or the components of the external magnetic field.

With this we write the fourth equation of (11) in the form

$$iL \delta u_i = \frac{e}{mc} \delta\Phi_i + \Omega_{ik} \delta u_k. \tag{18}$$

§ 3. Oscillations of the relativistic electron plasma when no external magnetic field is present

If there is no external magnetic field, we have $\Omega_{ik} = 0$ and the fundamental algebraic equations are

$$\left.\begin{aligned} &(Q^2 \delta_{ir} - k_i k_r) \delta\Phi_r = \frac{4\pi e L}{ic} (N\delta u_i + u_i \delta n) \\ &L\delta n + N k_r \delta u_r = 0 \\ &iL \delta u_i = \frac{e}{mc} \delta\Phi_i \end{aligned}\right\}, \tag{19}$$

where

$$Q^2 = k_r k_r \,. \tag{20}$$

It is easy to see that (19) incorporates all the equations of (11), making use of (13) and (14).

The system (19) can be written in the form

$$\delta u_i = \frac{e}{imc\,L}\,\delta\Phi_i \,, \tag{21}$$

$$\delta n = -\frac{Ne}{imcL^2}\,k_r\,\delta\Phi_r \tag{22}$$

and

$$\left[Q^2\,\delta_{ir} - k_i\,k_r + \frac{\omega_\pi^2}{c^2}\,\delta_{ir} - \frac{\omega_\pi^2}{c^2}\,\frac{u_i\,k_r}{L}\right]\delta\Phi_r = 0 \,, \tag{23}$$

where

$$\omega_\pi^2 = \frac{4\pi e^2 N}{m} \tag{24}$$

is the electron plasma frequency, an invariant scalar. One has to solve (23) first and then use (21) and (22) to derive the accompanying perturbations of the density and the velocity.

Instead of attacking directly Equ. (23) we decompose it into transverse and longitudinal perturbations.

A transverse oscillation of $\delta\Phi_r$ satisfies

$$k_r\,\delta\Phi_r = 0 \,, \tag{25}$$

whereas the longitudinal one obeys

$$(k_r\,k_r\,\delta_{is} - k_i\,k_s)\,\delta\Phi_s = 0. \tag{26}$$

Therefore the *transverse amplitude* is chosen by the projection tensor

$$\Pi_{ij} = \left(\delta_{ij} - k_i\,k_j\,\frac{1}{k_p\,k_p}\right) . \tag{27}$$

Application of (27) to (23) gives

$$\left(Q^2 + \frac{\omega_\pi^2}{c^2}\right)\Pi_{ir}\,\delta\Phi_r = 0$$

and a nontrivial transverse oscillation can exist if and only if

$$c^2 k_r k_r + \omega_\pi^2 = 0. \tag{28}$$

This equation is the covariant dispersion relation of a transverse plasma oscillation without external magnetic field. In the rest frame

$$k_r = \left(k_1, k_2, k_3, \frac{i}{c}\Omega\right)$$

and (28) will give

$$\Omega^2 = \omega_\pi^2 + c^2 \vec{k}^2$$

in accordance with the classical results. This transverse perturbation of $\delta\Phi_r$ is accompanied neither by density fluctuation nor longitudinal velocity fluctuation as it is guaranteed by (21) and (22).

In the case of a longitudinal oscillation from (23) we obtain by means of multiplication with u_i, and some manipulation

$$(\omega_\pi^2 - L^2) k_r \delta\Phi_r = 0\,, \tag{29}$$

where $k_r \delta\Phi_r \neq 0$, therefore

$$(k_r u_r)^2 = \omega_\pi^2$$

is the covariant dispersion relation of a longitudinal plasma oscillation without external magnetic field. In the rest frame this reduces to the classical form:

$$\Omega^2 = \omega_\pi^2\,.$$

This longitudinal perturbation of $\delta\Phi_r$ is accompanied by a longitudinal velocity fluctuation and a density fluctuation, as it can be seen from (21) and (22), but there is no transverse perturbation.

In the absence of external fields, the longitudinal and transverse oscillations decouple, and the covariant dispersion relations reduce to the classical ones in the rest frame.

§ 4. Oscillations of the relativistic electron plasma in the presence of an external magnetic field

a) *Reduction of the equations*

Let us suppose that the electron plasma is subjected to an external magnetic field in the rest frame. Then the basic algebraic equations we have to solve are the following

$$L\delta n + Nk_r\,\delta u_r = 0\,, \tag{30}$$

$$iL\,\delta u_i = \frac{e}{mc}\,\delta\Phi_i + \Omega_{ik}\,\delta u_k\,, \tag{31}$$

$$(Q^2\,\delta_{ik} - k_i\,k_k)\;\delta\Phi_r = \frac{4\pi eL}{ic}\,(N\delta u_i + u_i\,\delta n)\,. \tag{32}$$

We use (31) to express δu_i in terms of $\delta\Phi_i$. Since

$$\frac{e}{mc}\,\delta\Phi_i = (iL\,\delta_{ik} - \Omega_{ik})\,\delta u_k$$

and

$$\mathrm{Det}\,\{iL\,\delta_{ik} - \Omega_{ik}\} = L^2\left(L^2 + \frac{1}{4}\,\Omega_{rs}\,\Omega_{sr}\right) \neq 0\,,$$

we have

$$\delta u_r = \frac{L^2\,\delta_{ir} - iL\,\Omega_{ri} - \Omega_{rs}\,\Omega_{si} + \dfrac{1}{4}\,\delta_{ri}\,\Omega_{kl}\,\Omega_{lk}}{iL\left(L^2 + \dfrac{1}{4}\,\Omega_{pq}\Omega_{qp}\right)}\;\frac{e}{mc}\,\delta\Phi_i\,. \tag{33}$$

Making use of (31)

$$\delta n = -\frac{N}{L}\,k_r\,\delta u_r$$

and (33) we arrive at

$$\left[\left(Q^2 + \frac{\omega_\pi^2}{c^2}\right)\delta_{ir} - k_i\,k_r - \frac{\omega_\pi^2}{c^2}\left\{\frac{1}{L}\,u_i\,k_r + \right.\right.$$

$$\left.\left. + \frac{iL\,\Omega_{ir} + \Omega_{it}\,\Omega_{tr}}{L^2 + \dfrac{1}{4}\,\Omega_{pq}\,\Omega_{qp}} - \frac{u_i\,k_m}{L}\,\frac{iL\,\Omega_{mr} + \Omega_{mt}\,\Omega_{tr}}{L^2 + \dfrac{1}{4}\,\Omega_{pq}\,\Omega_{qp}}\right\}\right]\delta\Phi_r = 0. \tag{34}$$

In this model all information concerning plasma oscillations in the presence of external magnetic field is contained in (34). Before discussing the special modes of oscillation, however, it will be advantageous to simplify (34) by means of the introduction of a convenient four vector in the place of Ω_{pq}, as we have done with the introduction of Φ_i.

Starting from $F_{ik} = -F_{ki}$, and following the idea of A. M. PRATELLI [1], we have defined

$$\Phi_i = F_{ik}\,u_k$$

which in the rest frame has the components

$$\Phi_i = (E_x, E_y, E_z, 0) .$$

If we define the dual tensor of F_{ik} by the relation

$$F^*_{ik} = \frac{1}{2} \varepsilon_{ikrs} F_{rs}$$

we have the opportunity to define a four vector

$$h_i = - \frac{1}{ic} F^*_{ik} u_k$$

which in the rest frame has the components

$$h_i = (H_x, H_y, H_z, 0) .$$

Then it is possible to write F_{ik} in terms of Φ_i and h_i. It is easy to verify

$$F_{mn} = \frac{i}{c} \varepsilon_{ilmn} h_i u_l + \frac{1}{c} (\Phi_n u_m - \Phi_m u_n) .$$

Applying this to the particular $\overset{\circ}{F}_{ik}$, we have

$$\Omega_{mn} = \frac{i}{c} \varepsilon_{ilmn} \omega_i u_l , \tag{35}$$

where in the rest frame

$$\omega_i = \left(\frac{eH_x}{mc}, \quad \frac{eH_y}{mc}, \quad \frac{eH_y}{mc}, 0 \right) \tag{36}$$

and may therefore be called the cyclotron frequency four vector. We need other relations, too, namely

$$- \frac{1}{4} \Omega_{rs} \Omega_{rs} = \omega_l \omega_l \tag{37}$$

and since $\omega_q u_q = 0$ in all inertial frames, we have

$$\Omega_{mt} \Omega_{tr} = \frac{1}{c^2} \left[\omega_p \omega_p (u_m u_r - c^2 \delta_{mr}) + c^2 \omega_r \omega_m \right] . \tag{38}$$

Making use of these notations (34) can be written in the form

$$\Bigg[\left(Q^2+\frac{\omega_\pi^2}{c^2}\right)\delta_{ir}-k_i k_r-\frac{\omega_\pi^2}{c^2}\Bigg\{\frac{u_i k_r}{L}+\frac{L}{c}\,\frac{1}{L^2-\omega_q\omega_q}\,\varepsilon_{plir}\,\omega_p\,u_l+$$
$$+\frac{1}{L^2-\omega_q\omega_q}\,(\omega_r\omega_i-\omega_p\omega_p\delta_{ri})-$$
$$-\frac{1}{c}\,u_i k_m\,\frac{1}{L^2-\omega_q\omega_q}\,\varepsilon_{plmr}\,\omega_p\,u_l-\tag{39}$$
$$-\frac{1}{L}\,u_i k_m\,\frac{1}{L^2-\omega_q\omega_q}\,(\omega_r\omega_m-\omega_p\omega_p\delta_{rm})\Bigg\}\Bigg]\delta\Phi_r=0\,.$$

b) *Discussion of the oscillations*

I. Transverse waves. For transverse oscillations the equation (39) reduces to

$$\Bigg[\left(Q^2+\frac{\omega_\pi^2}{c_2}+\frac{\omega_\pi^2}{c^2}\,\frac{1}{L^2-\omega_q\omega_q}\right)\delta_{ir}-$$
$$-\frac{\omega_\pi^2}{c^2}\Bigg\{\frac{L}{c}\,\frac{1}{L^2-\omega_q\omega_q}\,\varepsilon_{plir}\,\omega_p\,u_l+\frac{\omega_i\omega_r}{L^2-\omega_q\omega_q}-\tag{40}$$
$$-\frac{\varepsilon_{plmr}\,u_i k_m\,\omega_p\,u_l}{c(L^2-\omega_q\omega_q)}-\frac{u_i k_m\omega_m\omega_r}{L(L^2-\omega_q\omega_q}\Bigg\}\,\Pi_{rt}\,\delta\Phi_t=0$$

where $\Pi_{rt}\delta\Phi_t$ is the transverse part of $\delta\Phi_i$.

The transverse oscillations may be decomposed into two cases, polarized perpendicularly

$$\omega_r\,\delta\Phi_r=0\tag{41}$$

or parallel

$$\left(\delta_{ir}-(\omega_s\omega_s)^{-1}\omega_i\omega_r\right)\delta\Phi_r=0,\tag{42}$$

to the external field ω_s. In the case (41), Equ. (40) takes the simpler form

$$(T\delta_{ir}+R\varepsilon_{plir}\,\omega_p\,u_l+S\varepsilon_{plmr}\,\omega_p\,u_l\,k_m\,u_i)\,\Pi_{rt}\,\delta\Phi_t=0\,,\tag{43}$$

where

$$T=Q^2+\frac{\omega_\pi^2}{c^2}\left(1+\frac{\omega_p\omega_p}{L^2-\omega_q\omega_q}\right),\tag{44}$$

$$R=-\frac{\omega_\pi^2}{c^2}\,\frac{L}{c}\,\frac{1}{L^2-\omega_q\omega_q}\,,\tag{45}$$

$$S=\frac{\omega_\pi^2}{c^2}\,\frac{1}{L^2-\omega_q\omega_q}\,.\tag{46}$$

In this (41) case, we may consider *propagation along the external magnetic field*, $k_r = \alpha\omega_r$, with a convenient constant α. So (43) takes the form

$$(T\delta_{ir} + R\varepsilon_{plir}\,\omega_p\,u_l)\,\Pi_{rt}\,\delta\Phi_t = 0\,. \tag{47}$$

The existence of a nontrivial solution to Equ. (47) is expressed by the vanishing of its determinant:

$$\left(\frac{T}{R}\right)^2\left\{\left(\frac{T}{R}\right)^2 - c^2\,\omega_p\,\omega_p\right\} = 0. \tag{48}$$

From $T = 0$ we obtain

$$c^2\,k_r\,k_r\,\{(u_s\,k_s)^2 - \omega_s\,\omega_s\} + \omega_\pi^2\,(u_s\,k_s)^2 = 0. \tag{49}$$

This is the covariant dispersion relation of a transverse plasma oscillation, which propagates along the external magnetic field. In the rest frame (49) gives

$$\Omega^2 = c^2\,\vec{k}^2 + \omega_\pi^2\,\frac{\Omega^2}{\Omega^2 - \omega^2}\,. \tag{50}$$

If $V = \Omega/k$ means phase velocity of the wave (49) in the rest frame of the plasma, then according to the dispersion relation (49)

$$\left(\frac{V}{c}\right)^2 = \frac{1}{1 - \dfrac{\omega_\pi^2}{\Omega^2 - \omega_c^2}} \tag{51}$$

This mode of oscillation cannot propagate in the frequency interval

$$\sqrt{\omega_c^2 + \omega_\pi^2} < \Omega < \omega_c\,. \tag{52}$$

A prototype of this propagating wave in low frequency approximation ($\Omega/\omega_c \ll 1$) is the Alfvén wave, for which (50) yields

$$\Omega^2 = \frac{k^2\,c_A^2}{1 + \dfrac{c_A^2}{c^2}}\,, \tag{53}$$

where

$$c_A^2 = \frac{H_0^2}{4\pi\,mN}$$

is the square of the Alfvén velocity.

This filter type behaviour of the electron plasma for this frequency interval is in connection with the cyclotron resonance.

Always in the case (41) the vanishing of the second factor in (48) yields

$$k_r k_r + \frac{\omega_\pi^2}{c^2} \; \frac{(k_r u_r)^2 \pm \omega k_r u_r}{(k_p u_p)^2 - \omega^2} = 0, \tag{54}$$

where $\omega = \omega_c = (\omega_s \omega_s)^{1/2}$. In the rest frame we obtain

$$\Omega^2 = c^2 \vec{k}^2 + \omega_\pi^2 \; \frac{\Omega}{\Omega \pm \omega}, \tag{55}$$

the two signs corresponding to the two different circular polarizations. This type of oscillation will not propagate if

$$\omega < \Omega < \frac{1}{2} [- |\omega| + \sqrt{\omega^2 + 4\omega_\pi^2}]$$

for the upper sign and

$$\Omega > \frac{1}{2} [- |\omega| + \sqrt{\omega^2 + 4\omega_\pi^2}]$$

for the lower sign.

The other type of class transverse waves polarized perpendicularly to ω_r contains the ones which *propagate across the external magnetic field.* In this case $k_r \omega_r = 0$, and the relevant equation is:

$$0 = \left[\left(Q^2 + \frac{\omega_\pi^2}{c^2} \; \frac{L^2}{L^2 - \omega^2} \right) \delta_{ir} - \frac{\omega_\pi^2}{c^3} \; \frac{1}{L^2 - \omega^2} \{ L \varepsilon_{plir} \omega_p u_l - \right.$$
$$\left. - \varepsilon_{plmr} \omega_p u_l u_i k_m \} \right] \delta \Phi_{ir}. \tag{56}$$

Since in this case the covariant dispersion relation would involve the evaluation of the determinant

$$D = \det \{ a \, \delta_{ir} - P_{ir} + u_i R_r \}, \tag{57}$$

where

$$a = \frac{1}{L} \; \frac{Q^2 + \dfrac{\omega_\pi^2}{c^2} \; \dfrac{L^2}{L^2 - \omega^2}}{\dfrac{\omega_\pi^2}{c^3} \; \dfrac{1}{L^2 - \omega^2}},$$

$$P_i = \varepsilon_{plir} \omega_p u_l, \qquad R_r = \frac{1}{L} k_m P_{mr},$$

we write here only the classical limit of (57):

$$\Omega^2 = c^2 \vec{k}^2 + \omega_\pi^2 \left(\frac{\mu^2}{\Omega^2 - \omega^2} \pm \frac{\omega \Omega}{\Omega^2 - \omega^2} \right), \tag{58}$$

where $\omega = \omega_c = (\omega_s \omega_s)^{1/2}$. The expression (54) is the dispersion relation of the transverse oscillations propagating across and polarized perpendicularly to the external magnetic field.

II. Longitudinal waves. For longitudinal waves k_r and $\delta\Phi_r$ must be parallel, and k_r and ω_r may be either parallel or perpendicular. We can therefore speak of propagation across $(\omega_r k_r = 0)$ and along $(\omega_r || k_r)$ the external magnetic field.

In the case of propagation across the magnetic field we have the dispersion relation from (39) (by multiplication with u_i)

$$L^2 = \omega_\pi^2 + c^2 \frac{\omega^2}{L^2 - \omega^2} \tag{59}$$

with $\omega^2 = (\omega_q \omega_q)$. This in covariant form

$$(k_r u_r)^2 = \omega_\pi^2 + c^2 \frac{\omega_q \omega_q}{(k_r u_r)^2 - \omega_q \omega_q}$$

or in the rest frame

$$\Omega^2 (\Omega^2 - \omega^2 - \omega_\pi^2) = 0$$

contains only the classical result.

Finally in the case of propagation along the external magnetic field by multiplication with ω_i we obtain from (39) the dispersion relation

$$c^2 k_r k_r + \omega_\pi^2 = 0 \tag{60}$$

or in the rest frame

$$\Omega^2 = c^2 \vec{k}^2 + \omega_\pi^2$$

being also the classical result.

§ 5. Acknowledgements

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КОЛЕБАНИЯ РЕЛЯТИВИСТИЧЕСКОЙ ЭЛЕКТРОННОЙ ПЛАЗМЫ ВО ВНЕШНЕМ МАГНИТНОМ ПОЛЕ

Г. РОТ

Резюме

Дается релятивистическое рассмотрение колебаний плазмы. Вычисления проведены для так называемой низкой β-плазмы, в случае которой давлением можно пренебречь по сравнению с плотностью магнитной энергии Выводятся ковариантные дисперсионные соотношения и их смысл сравнивается со значением подобных классических выражений.