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A SIMPLIFIED METHOD FOR CALCULATING THE INTERNAL STRUCTURE OF NEUTRON STARS*

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A very economical method is proposed for solution of the hydrostatic equilibrium equations of neutron stars. As a simple example the model of homogeneous internal distribution is constructed. Utilizing BRUECKNER's equation of state the importance of the effect of correlation can be demonstrated.

I. Introduction

It was suggested by ZWICKY [1] that the suddenness and magnitude of energy generation in supernovae might be explained as a partial gravitational collapse of a star (or a gas cloud) into a neutron core. There has recently been renewed interest in this possibility because observations indicate that some of the remnants of supernovae are sources of X-rays, and it can be stated with reasonable assurance that a neutron star is responsible directly or indirectly for X-radiation [2].

In this paper we shall not deal with the problem of observing such neutron stars. Instead we confine ourselves to the internal structure of cold, degenerate neutron stars. To investigate the static structure of such a star, one needs an equation of state, i.e. an energy—density relation for neutron matter. In consideration of this, there are two great uncertainties. First, the behaviour of nuclear forces in the high-energy region is not well known. In principle, the neutron—neutron forces can be determined from proton—proton phase shift analysis, assuming charge independence and correcting for Coulomb effects. Various models of nuclear forces have been constructed by different authors which fit the phase shifts up to 300 MeV bombarding energy. But the results in the many-body problem (in the case of high density) for these nuclear force models may be seriously different, i.e. the equation of state is sensitive to the choice of nuclear force model.

The second uncertainty arises from the fact that certain terms must be neglected in any calculation method for the many-body problem. In most of

* Dedicated to Prof. P. GOMBÁS on his 60th birthday.

the various equations of state which have been proposed for neutron matter, very crude methods are used for solving the many-body problem. OPPENHEIMER and VOLKOFF [3] assume an equation of state of an ideal Fermi gas, i.e. the neutron—neutron forces are completely neglected. The semi-empirical equation of state due to SKYRME [4] is expected to describe average nuclear properties. SALPETER [5] worked out a more complete equation of state applying the scattering limit theory and used an effective range approximation. To interpret the effect of a hard core of neutrons WHEELER et al. [6] studied the incompressible fluid model of neutron matter. Several very simple forms of equation of state were derived taking into account the nuclear forces by the Hartree—Fock method [7]. In this paper we shall utilize the neutron matter calculation of BRUECKNER et al. [8] for GAMMEL—THALER potential with repulsive core. BRUECKNER's t-matrix approach to the many-body problem takes into account the two particle correlations exactly, and this is the most accurate neutron matter calculation which has been performed up to now.

The various equations of state may be used to construct neutron star models by solving the equations of hydrostatic equilibrium. Such calculations have been carried out by several authors. The OPPENHEIMER—VOLKOFF [3] star model consists of non-interacting neutrons, CAMERON's [9] model is based upon SKYRME's [4] equation of state. The ideal and real gas models constructed by AMBARTSUMYAN and SAAKYAN [10] take into account strange particles in the stellar matter. More recently two neutron star model calculations have been made, using Hartree—Fock equations of state. TSURUTA and CAMERON [11] (henceforth cited as "TC") assumed an equation of state as suggested by LEVINGER and SIMMONS [7], GOMBÁS and KISDI [7] an equation of state based upon GOMBÁS's semi-empirical nuclear force model (which does not fit the two-body data, but the average nuclear binding energies).

The integration of the hydrostatic equations, in principle, can be carried out without any difficulty. Because of the sufficient increase of pressure with increasing density the Schwarzschild singularity does not occur in neutron-star models, and any conventional method for numerical integration can be applied. Nevertheless, the construction of a star model is not a simple problem for it needs a lot of numerical calculations. In Section II of this paper we propose a new numerical method for solving the hydrostatic equations.* This may be more economical than the usual step-by-step methods and makes it possible to discuss the neutron star models without using digital computers. In this work we apply this method in the simplest form which results in a star model of homogeneous internal distribution (Section III). Inspecting the numerical solution of the hydrostatic equations of TC, we can conclude that

* Our method is closely connected with the method of momenta in differential equation theory. See, e.g. L. COLLATZ, Numerische und graphische Methoden. (Handbuch der Physik, Vol. II. Springer, Berlin, 1955.)

this homogeneous model is a reasonable approximation if the density is in the region of 10^{14}—10^{15} gcm^{-3}, which is the region of the heaviest neutron stars. In Section IV, we calculate the gravitational and proper masses and the (constant) density as functions of the radius, assuming the equation of state of BRUECKNER et al. [8]. The results are compared with those of some previous works.

II. The equations of hydrostatic equilibrium

A spherical neutron star in hydrostatic equilibrium is described by the following structure equations:

a) Pressure equation [12]

$$\frac{dP}{dr} = -\frac{G}{c^4} \cdot \frac{(\varepsilon + P)(mc^2 + 4\pi r^3 P)}{r^2\left(1 - \frac{2\,Gm}{c^2\,r}\right)}, \tag{1}$$

where P and ε are the local pressure and energy density at radius r, m is the mass inside the radius r, G and c are the gravity constant and the velocity of light, respectively.

b) Conservation of mass:

$$\frac{dm}{dr} = \frac{4\pi}{c^2} r^2 \varepsilon . \tag{2}$$

c) Equation of state:

$$\varepsilon = \varepsilon(\varrho) = \varrho \cdot \big(\mu c^2 + E_b(\varrho)\big), \tag{3}$$

where ϱ is the neutron number density, μ is the neutron mass and E_b is the energy per particle in a neutron gas of density ϱ. The relation between E_b and ϱ must be calculated from a many-body theory. In this Section we do not fix the special form of this E_b — ϱ relation.

The pressure is given by

$$P(\varrho) = \varrho \frac{\partial \varepsilon}{\partial \varrho} - \varepsilon = \varrho^2 \frac{\partial E_b}{\partial \varrho} . \tag{4}$$

d) The boundary conditions are

$$m = 0 \quad \text{at} \quad r = 0, \tag{5}$$

$$\varrho = 0 \quad \text{at} \quad r = R, \tag{6}$$

where R is the radius of the neutron star; the outer edge of the star is assumed to have zero density. From Eqs. (3) and (4) it then follows that both ε and P are also zero at the surface.

It is clear that Eqs. (1)—(6) uniquely determine the internal distribution of the quantities ϱ, ε, P and m. The star model has only one free characteristic quantity, the radius R. After solving the hydrostatic equations (1)—(6), the gravitational and proper masses are obtained from

$$M_g = m(R)\,, \tag{7}$$

$$M_p = \mu \int_0^R \frac{\varrho(r)\, 4\pi r^2\, dr}{\sqrt{1 - \frac{2\,Gm(r)}{c^2\, r}}}\,, \tag{8}$$

respectively. The gravitational mass of the star is the mass as perceived by a distant observer and the proper mass is the neutron number of the star, multiplied by the neutron mass μ.

We define the moment of order $\varkappa$, of the neutron distribution to be

$$\int_0^R \varrho(r)\, r^\varkappa\, dr\,, \tag{9}$$

where $\varkappa \geqslant 0$. Integrating by parts and taking into account the boundary condition $\varrho(R) = 0$, we get:

$$\begin{aligned}\int_0^R \varrho r^\varkappa\, dr &= -\frac{1}{\varkappa+1}\int_0^R \frac{d\varrho}{dr}\, r^{\varkappa+1}\, dr = \\ &= \frac{G}{(\varkappa+1)\, c^4}\int_0^R \frac{\varepsilon + P}{P'} \cdot \frac{mc^2 + 4\pi r^3 P}{r - \frac{2\,G}{c^2}\, m}\, r^\varkappa\, dr\,,\end{aligned} \tag{10}$$

where

$$P' = \frac{dP}{d\varrho} = \varrho\, \frac{\partial^2 \varepsilon}{\partial \varrho^2}\,.$$

This means that for every order $\varkappa$, the moment of ϱ is identical with

$$\frac{G}{(\varkappa+1)\, c^4}$$

times the moment of

$$\frac{\varepsilon + P}{P'} \cdot \frac{mc^2 + 4\pi r^3 P}{r - \frac{2G}{c^2}\, m}\,.$$

Now we can introduce a very economical approximation procedure for solving Eqs. (1)—(6). We take a trial function for ϱ which, besides r, depends on a set of parameters $\beta_1, \beta_2, \ldots, \beta_s$.

$$\varrho = \varrho(r; \beta_1, \beta_2, \ldots, \beta_s). \tag{11}$$

We suppose that the boundary condition (6) is satisfied for every value of the β-s. The energy density and the pressure are determined from ϱ by the equation of state (3) and (4):

$$\varepsilon = \varepsilon(r; \beta_1, \beta_2, \ldots, \beta_s), \tag{12}$$

$$P = P(r; \beta_1, \beta_2, \ldots, \beta_s), \tag{13}$$

and m is calculated from (2) and (5):

$$m = \frac{4\pi}{c^2}\int_0^r \varepsilon r^2\, dr = m(r; \beta_1, \beta_2, \ldots, \beta_s)\,. \tag{14}$$

Instead of (1) we require that the moment equation (10) must be satisfied for S different $\varkappa$ values. This determines the parameters $\beta_1, \beta_2, \ldots, \beta_s$. Putting these β-s into the expressions (11)—(14) we get the solution of our problem.

Naturally, we have drawn our procedure as general as possible. *How* to introduce the parameters $\beta_1, \beta_2, \ldots, \beta_s$, and for *what values of* $\varkappa$ the moment equation is satisfied, is, in principle, arbitrary. But one can imagine that the accuracy of our method depends on the proper choice of the trial function and the values of the $\varkappa$-s.

III. Star model of homogeneous internal distribution

As a simple example of our method we treat a neutron star model in which the internal density distribution is homogeneous.

$$\begin{aligned} \varrho(r) &= \varrho_0, \quad &\text{if} \quad r<R, \\ \varrho(r) &= 0, \quad &\text{if} \quad r>R. \end{aligned} \tag{15}$$

Here R is the radius of the star and ϱ_0 is the only parameter of the model which is to be determined from a moment equation.

The equation of state then gives the homogeneous distribution for the energy density and pressure:

$$\begin{aligned} \varepsilon(r) = \varepsilon_0 \quad &\text{and} \quad P(r) = P_0, \quad \text{if} \quad r<R, \\ \varepsilon(r) = 0 \quad &\text{and} \quad P(r) = 0, \quad \text{if} \quad r>R, \end{aligned} \tag{16}$$

where $\varepsilon_0 = \varepsilon(\varrho_0)$ and $P_0 = P(\varrho_0)$. The quantity m is determined by (2) and (5):

$$m(r) = \frac{4\pi r^3 \varepsilon_0}{3 c^2}, \quad \text{if} \quad r < R,$$
$$m(r) = \frac{4\pi R^3 \varepsilon_0}{3 c^2}, \quad \text{if} \quad r > R. \tag{17}$$

Putting the expressions (15), (16) and (17) into the moment equation (10) we obtain a transcendental equation for ϱ_0.

$$\varrho_0 = \frac{4\pi R^2 G}{3 c^4} \cdot \frac{(\varepsilon_0 + P_0)(\varepsilon_0 + 3P_0)}{P_0'} \int_0^1 \frac{x^{\varkappa+2}}{1 - \frac{8\pi R^2 G\varepsilon_0}{3 c^4} x^2} dx, \tag{18}$$

where $x = r/R$ and the abbreviation P_0' stands for

$$P'(\varrho_0) = \left(\frac{\partial P}{\partial \varrho}\right)_{\varrho=\varrho_0}$$

Introducing the dimensionless quantities

$$\xi = \frac{\varrho_0 \varepsilon_0 P_0'}{(\varepsilon_0 + P_0)(\varepsilon_0 + 3P_0)} \quad \text{and} \quad \eta = \frac{8\pi R^2 G\varepsilon_0}{3 c^4}, \tag{19}$$

Eq. (18) takes the very simple form

$$\xi = \frac{1}{2}\eta \int_0^1 \frac{x^{\varkappa+2}}{1 - \eta x^2} dx. \tag{20}$$

As our simple model has only one parameter, ϱ_0, the moment equation (18) or (20) can be satisfied only for one value of the order $\varkappa$. We choose the value $\varkappa = 2$, because then (10) guarantees a good approximation for the integral

$$\int_0^R \varrho \, 4\pi r^2 \, dr$$

which is closely connected with the mass of the star. For $\varkappa = 2$ Eq. (20) gives

$$\xi = \frac{1}{2}\left[\frac{1}{2\sqrt{\eta^3}} \ln\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\right) - \frac{1}{\eta} - \frac{1}{3}\right]. \tag{21}$$

Summarizing, the construction of the homogeneous model consists of the following steps:

a) Choose a value for ϱ_0. This can be regarded as the free characteristic quantity of the star model.

b) Determine ε_0, P_0 and P_0' from the equation of state and ξ according to (19).

c) Calculate η from (21). For this we need the inverse of the function which stands on the right-hand side of (21). This can be determined numerically.

d) Finally, the radius R can be obtained from (19):

$$R = \left(\frac{3\, c^4}{8\pi G} \cdot \frac{\eta}{\varepsilon_0} \right)^{1/2} . \tag{22}$$

In this way we get a radius — neutron density relation, which is one of the fundamental characteristic relations of the neutron star model.

If we have determined R by the above procedure, then we get the gravitational and proper masses from (7) and (8), respectively. The gravitational mass is

$$M_g = \frac{4\pi R^3\, \varepsilon_0}{3\, c^2} = \frac{c^2}{2G}\, \eta R\, , \tag{23}$$

and for the proper mass we have

$$M_p = 4\pi R^3\, \mu \varrho_0 \int_0^1 \frac{x^2\, dx}{\sqrt{1-\eta x^2}} = \frac{4\pi R^3\, \mu \varrho_0}{3} \cdot \frac{3}{2\eta} \left(\frac{\arcsin \sqrt{\eta}}{\sqrt{\eta}} - \sqrt{1-\eta} \right) . \tag{24}$$

From these equations we get M_g and M_p as functions of ϱ_0 (or R). These mass — density and mass — radius relations are also fundamental characteristics of the model.

IV. Results for Brueckner's equation of state

Brueckner et al. [8] have calculated the energy per particle, E_b, in neutron matter as a function of neutron density ϱ, solving the integral equations of the Brueckner's many-body theory exactly. We reproduce their result in Fig. 1. We have found that a convenient and very accurate analytical fit for Brueckner's $E_b = E_b(\varrho)$ is given by the following polytropic expression:

$$E_b = a \varrho^{\lambda}\, , \tag{25}$$

where $\lambda = 0.576$ and $a = 57.9$, if E_b is measured in MeV and ϱ in 10^{39} cm^{-3} units. Assuming (25), the equation of state (3) and (4) is

$$\begin{aligned} \varepsilon &= \varrho(\mu c^2 + a \varrho^{\lambda})\, , \\ P &= \lambda a \varrho^{\lambda+1} . \end{aligned} \tag{26}$$

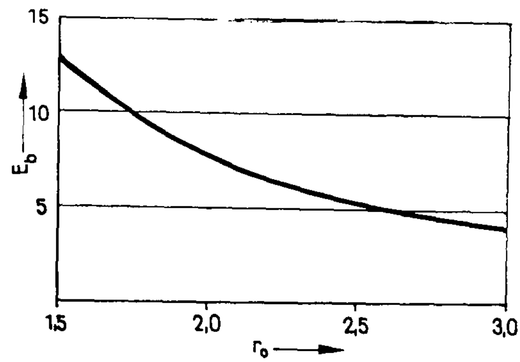

Fig. 1. Total energy per particle in a neutron gas. The distance parameter $r_0 = \left(\frac{3}{4\pi\varrho_0}\right)^{1/3}$ is given in 10^{-13} cm, the energy E_b in MeV units. (From BRUECKNER et al. [8])

In this case the quantity ξ, defined by (19), is the following function of ϱ_0

$$\xi = \frac{\lambda(\lambda+1)\,a\varrho_0^\lambda\,[\mu c^2 + a\varrho_0^\lambda]}{[\mu c^2 + (\lambda+1)\,a\varrho_0^\lambda][\mu c^2 + (3\lambda+1)\,a\varrho_0^\lambda]}\,. \tag{27}$$

Utilizing this, the radius R, the gravitational mass M_g and the proper mass M_p can be determined as has been described in Section III. The results are given in Table I.

In Fig. 2 the relation between the energy density ε_0 and the gravitational mass M_g is shown. For comparison the central energy density — gravitational mass relations of TC and of the OPPENHEIMER—VOLKOFF model are also plotted.

We believe the reality of our model lies only in the region II of Fig. 2. In the low density region (marked I) the neutrons are unstable against the decay $n \to p + e + \bar{\nu}$. The models of TC have taken into account this possibility and in this region, are composed of neutrons, electrons and heavy ions. Our simplified model works with stable "neutrons" and therefore cannot be compared, in this region, with the results of TC. On the other hand, the OPPENHEIMER—VOLKOFF model is also composed of stable Fermi particles. The big difference between our and their results shows the importance of the effect of nuclear forces. In the region of superdense neutron stars (marked III in Fig. 2) our homogeneous model is not reliable. According to TC, in this region the matter of the star accumulates near the centre and the deviation from constant density distribution becomes serious. To deal with this effect, our simple trial function must be replaced by a more flexible one.

The remnants of the heaviest supernovae lie around the mass peak in the intermediate density region (marked II in Fig. 2). The drastic deviation among the various results in this region give an indication of the poorness of

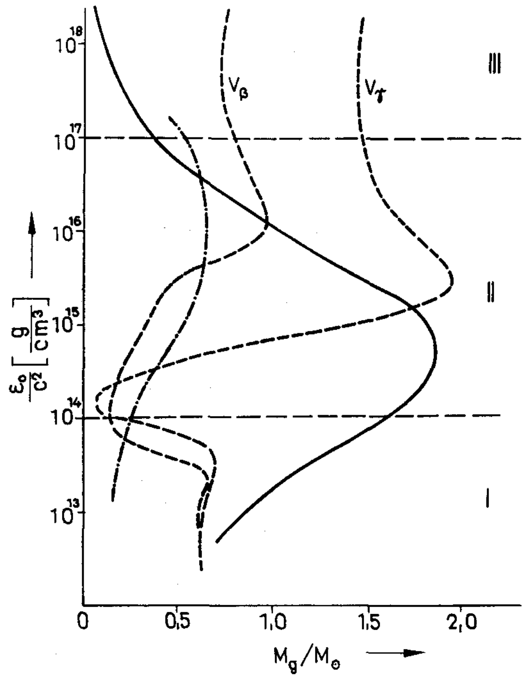

Fig. 2. Gravitational mass — energy density relation: ——— our result, —·—·— result of OPPENHEIMER and VOLKOFF [3] for an ideal Fermi gas, -------- results of TC for their nuclear forces V_β and V_γ, respectively

Table I

Characteristics of the homogeneous neutron star model with BRUECKNER's equation of state. ϱ_0 in $g\ cm^{-3}$, ε and P_0 in erg cm^{-3}, R in km and, M_g, M_p in $M_\odot$ units

ϱ_0	ε_0	P_0	ξ	η	R	M_g	M_p
7.19×10^{12}	6.48×10^{33}	9.94×10^{30}	0.00657	0.0626	37.4	0.792	0.805
1.44×10^{13}	1.30×10^{34}	2.96×10^{31}	0.00976	0.0908	31.8	0.976	0.999
2.88×10^{13}	2.60×10^{34}	8.83×10^{31}	0.0144	0.130	26.8	1.175	1.210
7.19×10^{13}	6.53×10^{34}	3.74×10^{32}	0.0242	0.205	21.3	1.479	1.568
1.44×10^{14}	1.31×10^{35}	1.12×10^{33}	0.0354	0.282	17.6	1.680	1.823
2.88×10^{14}	2.64×10^{35}	3.33×10^{33}	0.0522	0.377	14.4	1.832	2.059
7.19×10^{14}	6.71×10^{35}	1.41×10^{34}	0.0834	0.514	10.5	1.828	2.150
1.44×10^{15}	1.37×10^{36}	4.20×10^{34}	0.118	0.635	8.20	1.761	2.178
2.88×10^{15}	2.80×10^{36}	1.25×10^{35}	0.163	0.720	6.09	1.484	1.893
7.19×10^{15}	7.39×10^{36}	5.30×10^{35}	0.238	0.823	4.01	1.117	1.482
1.44×10^{16}	1.57×10^{37}	1.58×10^{36}	0.303	0.877	2.84	0.843	1.122
2.88×10^{16}	3.40×10^{37}	4.71×10^{36}	0.371	0.913	1.97	0.608	0.785
7.19×10^{16}	9.93×10^{37}	2.00×10^{37}	0.450	0.941	1.17	0.373	0.435
5.39×10^{17}	1.31×10^{39}	4.77×10^{38}	0.201	0.780	0.293	0.077	0.041

neutron star theories. The difference between the two curves of TC shows what uncertainty can arise from the extrapolation of nuclear potentials originally constructed for two body problems. One should not be surprised if our result deviates appreciably from the TC curves, since our model is based on a third nuclear potential (introduced by GAMMEL and THALER).

Furthermore, BRUECKNER's equation of state, which we have used in our model, takes into account the two particle correlations in neutron matter. The correlation has been neglected in previous neutron star models and the deviation of our model from others may be due partially to this. The effect of correlation seems to be very important and requires further investigations.

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УПРОЩЕННЫЙ МЕТОД ДЛЯ ОПРЕДЕЛЕНИЯ ВНУТРЕННЕЙ СТРУКТУРЫ НЕЙТРОННЫХ ЗВЕЗД

Д. КИШДИ

Резюме

Для определения гидростатического равновесия нейтронных звезд предлагается действительно эффективный метод. В качестве простого примера подробно рассматривается модель с постоянной плотностью. Применением уравнения состояния Брюкнера для нейтронной материи сможем демонтировать важность корреляционных эффектов.