Acta Physica Academiae Scientiarum Hungaricae, Tomus 27, pp. 269—276 (1969)

SMALL AMPLITUDE WAVES AND WEAK DISCONTINUITIES IN THE RELATIVISTIC HYDRODYNAMICS OF AN IDEAL FLUID*

By

I. Abonyi

INSTITUTE OF THEORETICAL PHYSICS, ROLAND EÖTVÖS UNIVERSITY, BUDAPEST

(Received 7. II. 1969)

Waves of small amplitude and the propagation of the surfaces of weak discontinuity (jumping derivatives) are studied in relativistic fluid dynamics. It is shown that in analogy with classical fluid dynamics, small amplitude waves and weak discontinuities have similar character. Both small amplitude waves, and surfaces of weak discontinuity are propagated with the velocity of sound, which, however, contains a relativistic correction.

1. Introduction — Basic assumptions

In this paper we consider a relativistic ideal fluid. The equation of state of a simple one component fluid can be written in the form

$$\mu^0 = \mu^0 (p, s), \tag{1}$$

where μ^0 is mass of the fluid in unit co-moving volume, p is the pressure, s is the entropy of the fluid in unit co-moving volume.

It is required that the equation of state should be valid following the motion of the volume element

$$u_k \partial_k \mu^0 = \left(\frac{\partial \mu^0}{\partial p}\right)_s u_k \partial_k p + \left(\frac{\partial u^0}{\partial s}\right)_p u_k \partial_k s . \tag{2}$$

Here u_k stands for the four vector velocity of the fluid and ∂_k denotes the four vector gradient. (Summation convention is understood for doubly occurring Latin indices, with $x_4 = ic$.)

As is well known, the energy-momentum tensor T_{ik} of an ideal fluid has the form

$$T_{ik} = \frac{1}{c^2} (\varepsilon + p)\, u_i\, u_k + p\delta_{ik}, \tag{3}$$

* Dedicated to Prof. P. Gombás on his 60th birthday.

where ε is the energy of the fluid in unit co-moving volume. We may define the mass density

$$\mu = \frac{1}{c^2}(\varepsilon + p) \tag{4}$$

the mass equivalent of all energies in a unit co-moving volume. This, clearly, differs from μ^0, since ε may be decomposed to give

$$\frac{\varepsilon}{c^2} = \mu^0 + \frac{1}{c^2}\mu^0 \varepsilon^0 , \tag{5}$$

where μ^0 stands for the rest mass density, while ε^0 is the specific internal energy of the fluid.

So we shall write

$$\mu = \mu^0 \left(1 + \frac{\varepsilon^0}{c^2} + \frac{p}{\mu^0 c^2}\right) = \mu^0 v = \mu^0 \left(1 + \frac{1}{c^2} w\right), \tag{6}$$

where the symbol v is the so called "index" of the fluid, and w stands for the specific enthalpy.

Then, the energy momentum tensor is

$$T_{ik} = \mu^0 v u_i u_k + \delta_{ik} p , \tag{7}$$

and the equations of motion of the fluid are

$$\partial_k T_{ik} = 0. \tag{8}$$

We have to assure the interpretation of u_i as a velocity four vector therefore

$$u_k u_k = - c^2 \tag{9}$$

and the conservation of the number of particles

$$\partial_k (\mu^0 \mu_k) = 0 . \tag{10}$$

From equations (8), (9) and (10) it is easy to deduce by means of the thermodynamic relation

$$dw - \frac{1}{\mu^0} dp = T ds , \tag{11}$$

that the motion of the fluid is isentropic, namely

$$u_k \partial_k s = 0 . \tag{12}$$

So the equation of state (2) takes the simpler *barotropic* form: $\mu^0 = \mu^0(p)$, i.e.:

$$u_k \partial_k \mu^0 - a^2 u_k \partial_k p = 0\,, \tag{13}$$

where

$$a^2 = \left(\frac{\partial u^0}{\partial p}\right).$$

Then the basic equations of the simple one-component relativistic fluid are:

$$\left.\begin{aligned} &\mu^0 u_k \partial_k (v u_i) + \partial_i p = 0 \\ &\partial_k (\mu^0 u_k) = 0 \\ &u_k u_k = -c^2 \\ &u_k \partial_k \mu^0 - a^2 u_k \partial_k p = 0 \end{aligned}\right\}. \tag{14}$$

There are seven equations for seven variables (u_k, p, μ^0 and v). We stress that p, μ^0 and v are invariant scalars with respect to Lorentz-transformations.

In this treatment the fluid is assumed to have an infinite extent to avoid, for the time being, boundary condition problems.

2. Small amplitude waves

It can be seen that the basic equations (14) are solved by the system of variables

$$\left\{\begin{matrix} u_k \\ p \\ \mu^0 \\ v \end{matrix}\right\} = \text{constants in space and time,}$$

if u_k is chosen so as to obey (9). We then superpose small perturbations of the form

$$\left\{\begin{matrix} \delta\bar{u}_k \\ \delta\bar{p} \\ \delta\bar{\mu}^0 \\ \delta\bar{v} \end{matrix}\right\} = \left\{\begin{matrix} \delta u_k \\ \delta p \\ \delta\mu^0 \\ \delta v \end{matrix}\right\} \exp\, i k_r x_r\,, \tag{15}$$

where the amplitudes (δu_k, δp, $\delta\mu^0$, δv) are small quantities of the first order, and any term containing at least two first order factors will be omitted. Then the equations of motion will be

$$\mu^0 u_k \partial_k (v\,\delta\bar{u}_k + u_i\,\delta\bar{v}) + \partial_i \delta\bar{p} = 0\,,$$

$$\partial_k (\mu^0 \delta\bar{u}_k + u_k \delta\,\bar{\mu}^0) = 0\,,$$

$$(u_k + \delta\bar{u}_k)(u_k + \delta\bar{u}_k) = -c^2\,,$$

$$u_k \partial_k \delta\bar{\mu}^0 - a^2 u_k \partial_k \delta\bar{p} = 0\,.$$

In such a way we arrive at a set of linear algebraic equations, namely:

$$\left\{\begin{array}{l} u_i \mu^0 L \delta v + \mu^0 v L \delta u_i + k_i \delta p = 0 \,, \\ L \delta \mu^0 + \mu^0 k_r \delta u_r = 0 \,, \\ u_r \delta u_r = 0, \\ L \delta \mu^0 - a^2 L \delta p = 0 \,. \end{array}\right\} \tag{16}$$

Here $L = u_r k_r$. The existence of a nontrivial solution for the amplitudes $(\delta u_k, \delta p, \delta\mu^0, \delta v)$ is guaranteed by the vanishing of the determinant

$$D = \begin{vmatrix} \mu^0 Lv & 0 & 0 & 0 & \mu^0 Lu_1 & 0 & k_1 \\ 0 & \mu^0 Lv & 0 & 0 & \mu^0 Lu_2 & 0 & k_2 \\ 0 & 0 & \mu^0 Lv & 0 & \mu^0 Lu_3 & 0 & k_3 \\ 0 & 0 & 0 & \mu^0 Lv & \mu^0 Lu_4 & 0 & k_4 \\ \mu^0 k_1 & \mu^0 k_2 & \mu^0 k_3 & \mu^0 k_4 & 0 & L & 0 \\ u_1 & u_2 & u_3 & u_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L & -a^2 L \end{vmatrix} = 0. \tag{17}$$

This requirement gives

$$D = \mu^{05} v^4 L^4 \{L^2 (a^2 c^2 v - 1) - c^2 k_r k_r\} = 0 \,, \tag{18}$$

which is a relation between k_r and the properties of the fluid. Therefore, it can be envisaged as the dispersion relation of small amplitude waves. The system (15) can be a wave solution to the linearized basic equations (14) if and only if k_r is chosen so as to satisfy (18).

There are two possibilities. The first where $L \neq 0$, is

$$L^2 = (1 - a^2 c^2 v)^{-1} c^2 k_r k_r \,,$$

i.e.

$$(u_r k_r)^2 (a^2 c^2 v - 1) - c^2 k_r k_r = 0 \tag{19}$$

which, for the co-moving system of reference, where

$$u_r = \{0, 0, 0, ic\} \,, \quad k_r = \left\{k_1, k_2, k_3, \frac{i}{c} \omega \right\}$$

gives

$$V^2 \equiv \left(\frac{\omega}{k}\right)^2 = a^{-2} \frac{1}{v} = a^{-2} \frac{\mu^0}{\mu} \tag{20}$$

for the square of the phase velocity V of the wave with respect to the fluid. Since a is the reciprocal of the classical velocity of sound c_s, we have

$$V^2 = c_s \left(1 + \frac{\varepsilon^0}{\mu^0 c^2} + \frac{p}{\mu^0 c^2}\right)^{-1}. \tag{21}$$

This corresponds to the sound wave in the classical limit.

The other possibility is that

$$L = 0$$

which means that in the co-moving frame $\omega = 0$, therefore

$$V^* = \frac{\omega}{k} = 0\,; \tag{22}$$

this perturbation does not propagate with respect to the fluid.

In the first case equations (16) yield

$$\left.\begin{aligned} &k_r\, \delta u_r = -\frac{1}{L}\,\delta\mu^0, \quad \delta u_i = -\frac{a^{-2}}{\mu^0 vL}\left\{k_i + \frac{L}{c^2}\,u_i\right\}\delta\mu^0,\\ &\delta v = (\mu^0 c^2 a^2)^{-1}\,\delta\mu^0, \quad \delta p = a^{-2}\,\delta\mu^0, \end{aligned}\right\} \tag{23a}$$

the amplitudes of the longitudinal velocity perturbation, the "index" and pressure perturbations can be given in terms of the density perturbation $\delta\mu^0$. Therefore $\delta\mu^0$ must be different from zero, otherwise this type of small amplitude wave cannot exist.

In the second case, because $L = 0$, after some manipulations we obtain from (16)

$$\left.\begin{aligned} &k_r\,\delta u_r = 0, \quad \delta u_i = 0\,,\\ &\delta p = 0,\\ &\delta v = 0\,,\\ &\delta\mu^0 = \text{arbitrary}. \end{aligned}\right\} \tag{23b}$$

Because of (22), we may realize this perturbation as a stratification in the fluid which is immobile with respect to the fluid, and since now $\delta\mu^0$ is the only variable of the entropy, the stratification causes an immobile variation of entropy which does not propagate with respect to the fluid (entropy wave). And since the fluid is an ideal one, this stratification of entropy can move only together with the fluid.

3. Weak discontinuities

When the hydrodynamic quantities (u^k, p, μ^0, v) themselves are continuous, but their derivatives have jumps along a surface, we speak of that surface as a surface of weak discontinuity.

Let us denote by

$$f(x_1, x_2, x_3, x_4) = 0$$

the equation of the surface, across which the derivatives of the hydrodynamic quantities are not continuous.

The unit normal to this hypersurface has the components

$$N_k = \frac{\partial_k f}{(\partial_r f \partial_r f)^{1/2}}$$

and following [1, 2] we use $V = cU$ as the velocity of the hypersurface along its normal, with the definition

$$1 - U^2 = \frac{\sum_1^4 g_{rs} \partial_r f \partial_s f}{\sum_1^3 a_{\varrho\sigma} \partial_\varrho f \partial_\sigma f},$$

where g_{rs} is the four dimensional metric tensor, while $a_{\sigma\tau}$ is the three dimensional one. So

$$N_4 = \frac{-iU}{[1 - U^2]^{1/2}}. \tag{24}$$

It is obvious tat einhther the phase velocity of small amplitude waves, nor the velocity of this surface is a covariant notion.

By definition, the discontinuities along the surface in question of the hydrodynamical quantity F can be written in the form

$$[\partial_S F] = N_S \delta F,$$

where the difference

$$[\partial_S F] = \lim_{\rightarrow} \partial_S F - \lim_{\leftarrow} \partial_S F$$

involves the limits taken on the different sides of the hypersurfaces, and δF represents the jump of $\partial_s F$.

Effecting the limiting processes we obtain from the basic equations:

$$\left.\begin{aligned}&\mu^0 u_i \Lambda\delta v + \mu^0 v \Lambda\delta u_i + N_i \delta p = 0\\&\mu^0 N_k \delta u_k + \Lambda\delta\mu^0 = 0\\&u_k \delta u_k = 0\\&\Lambda\delta\mu^0 - a^2 \Lambda \delta p = 0\end{aligned}\right\}, \tag{25}$$

where

$$\Lambda = u_k N_k .$$

The (25) is a homogeneous linear system of seven algebraic equations for seven unknowns, the condition of the existence of a nontrivial solution is that the determinant

$$\Delta = \begin{vmatrix} \mu^0 \Lambda v & 0 & 0 & 0 & \mu^0 \Lambda u_1 & 0 & N_1 \\ 0 & \mu^0 \Lambda v & 0 & 0 & \mu^0 \Lambda u_2 & 0 & N_2 \\ 0 & 0 & \mu^0 \Lambda v & 0 & \mu^0 \Lambda u_3 & 0 & N_3 \\ 0 & 0 & 0 & \mu^0 \Lambda v & \mu^0 \Lambda u_4 & 0 & N_4 \\ \mu^0 N_1 & \mu^0 N_2 & \mu^0 N_3 & \mu^0 N_4 & 0 & \Lambda & 0 \\ u_1 & u_2 & u_3 & u_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda & -a^2 \Lambda \end{vmatrix} = 0,$$

which has the same structure as (17). This requirement gives

$$\Delta = \mu^{05} v^4 \Lambda^4 \{\Lambda^2 (a^2 c^2 v - 1) - c^2 N_r N_r\} = 0 .$$

There are two possibilities again. First let us take $\Lambda \neq 0$, then

$$\Lambda^2 = c^2 (1 - a^2 c^2 v)$$

and we obtain in the co-moving system

$$U^2 = \frac{1}{a^2 v} = c_s^2 \frac{\mu^0}{\mu} = c_s^2 \left(1 + \frac{\varepsilon^0}{\mu^0 c^2} + \frac{p}{\mu^0 c^2}\right)^{-1},$$

stating that this type of the surface of weak discontinuity propagates with the velocity of sound. In this case the jumps are connected by the relations (23a).

In the second case $\Lambda = 0$, which in the co-moving frame means that the surface does not propagate with respect to the fluid, the jumps are then connected by the relations (23b).

4. Conclusions

We have shown that the velocity of propagation of small amplitude waves is modified in the relativistic case by the presence of a factor

$$\left(1+\frac{\varepsilon^0}{\mu^0 c^2}+\frac{p}{\mu^0 c^2}\right)^{-1/2}.$$

This means that if the specific internal energy and the pressure of the system is comparable to its rest energy density, the corrective factor may be important. The well known modification of sound velocity [3] because of the extreme relativistic equation of state

$$p=\frac{1}{3}e\,,$$

where e means the total energy density of the system makes its important contribution particularly to c_s, and not to the correcting factor.

Finally we have shown that, as in classical hydrodynamics, the surfaces of weak discontinuity propagate with the same velocity as do small amplitude waves, and can be classified in an analogous manner.

REFERENCES

1. A. Lichnerowicz, Théories relativistes de la gravitation et de l'électromagnétisme, Masson Paris 1955.
2. A. C. Giere, Journal of Math. Phys., 7, 1648, 1966.
3. L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Pergamon Press, 1963.

ВОЛНЫ НЕБОЛЬШИХ АМПЛИТУД И СЛАБЫЕ РАЗРЫВЫ В РЕЛЯТИВИСТИЧЕСКОЙ ГИДРОДИНАМИКЕ ИДЕАЛЬНОЙ ЖИДКОСТИ

И. АБОНИ

Резюме

Изучаются волны небольших амплитуд и распространение поверхностей слабого разрыва (скачок производных) в релятивистской динамике жидкости. Показывается, что в аналоге с классической динамикой жидкости волны небольшой амплитуды и слабые разрывы имеют подобный характер. Как волны небольшой амплитуды, так и поверхности слабого разрыва распространяются со скоростью звука, которая, однако, содержит релятивистическую поправку.