Acta Physica Academiae Scientiarum Hungaricae, Tomus 27, pp. 261—268 (1969)

IS POINCARÉ INVARIANCE COMPATIBLE WITH GENERAL RELATIVITY?*

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BUDAPEST

(Received 3. II. 1969)

We have shown previously that an invariance principle is defined in curved space by invariance under the Brandt groupoid consisting of elements given by parallel displacements along all possible curves in space-time. It is argued here that the Brandt groupoid might contain the Poincaré group as a local group in each tangent space but then space-time must have non-vanishing torsion. Such a conclusion might also be implied by recent measurements of SADEH et al. For an Einstein manifold, on the other hand, the Brandt groupoid contains only the homogeneous Lorentz group.

1. Introduction

SADEH, KNOWLES and YAPLEE [1,2] observed an anomalous decrease of frequency in the 21 cm absorbtion line from Taurus A. They also found the effect when a terrestrial source was used and they found the decrease roughly proportional to distance. This decrease cannot be accounted for by general relativity and in a recent paper SZEKERES [3] argues this indicates that space-time is not Einstein—Riemannian (a space with a symmetric metric and symmetric connection). He assumes a linear connection with non-vanishing torsion and calculates the contribution of the torsion part to the shift of frequency and finds it in agreement with the observations of SADEH et al. He therefore concludes that these measurements do suggest a physical space-time of non-vanishing torsion.

In this paper, while we do not argue with his physical interpretation of the torsion tensor, we want to point out that a similar conclusion is already implied by our previous work, though in an indirect way. More precisely, our previous results imply that either the Poincaré group is a good local symmetry group and then space-time has torsion or general relativity holds (no torsion) and then the local invariance group is the homogeneous Lorentz group without translations.

In our earlier papers we made an attempt to introduce an invariance principle in curved space-time. The first thing in such an attempt is to face

* Dedicated to Prof. P. GOMBÁS on his 60th birthday.

the fact that there are only local inertial systems, namely the geodesic systems at each point x. We assumed that these local inertial systems are still physically completely equivalent and then determined the invariance transformations connecting them. Let T_x and T_y denote ennuples of unit vectors in the local inertial systems, the geodesic systems, introduced at space-time points x and y, respectively. Then one of our main results is as follows [4]. The invariance transformations mapping T_x into T_y are defined by Levi-Civita parallel displacement along all possible curves connecting x and y. Different experimental consequences of this invariance have been worked out [4, 5, 6]. For example it has been shown that the three experimental tests of general relativity follow.

Consider now the set $B(x, y)$ of transformations defined by parallel displacement along all possible oriented curves connecting x and y and denote by B the set as x and y run through all points of space-time. There is nothing that would distinguish one space-time point among the others, therefore each element of the set B is an invariance transformation. Therefore the set B defines an invariance principle in curved space and an important problem is then what is the structure of B. Clearly it cannot be a group but one can show [7] that it is a Brandt groupoid.

The fundamental fact in our present argument is that the Lorentz group is contained in the groupoid B as a subset working in each tangent space T_x, it is indeed the holonomy group ψ_x, defined by parallel displacement along all possible closed curves through x of the underlying space-time manifold, discussed extensively [5, 10]. Indeed, the identity component* of the holonomy group (hg) is the six-dimensional homogeneous Lorentz group for a non-vacuum Einstein manifold. However, the Poincaré group P can never be realised as the hg of an Einstein manifold since the hg associated with a symmetric connection is always homogeneous.

Now it is known that the hg associated with a linear, non-symmetric, connection is inhomogeneous and therefore the Poincaré group could be interpreted as a local invariance group in such a space.

However, our invariance principle in curved space is defined by the Brandt groupoid B which is an object more general than a group. To see the intimate relationship between measurements of the red-shift type and local invariance groups, such as the Lorentz, or Poincaré, group we first give in Section 3 a remarkable decomposition theorem for the groupoid B.

* It can be shown [10] that the existence of inversions in ψ_x depends on the topological properties of space-time. Introduce topology by defining space-time to be a differentiable manifold M_n. Let π_1 be the first homotopy group of M_n and ψ_x^0 the identity component of ψ_x. Then one can prove [10] that the homomorphism $\pi_1 \to \psi_x/\psi_x^0$ exists and the problem of the existence of inversions is therefore reduced to the computation of π_1.

2. The Brandt groupoid

A Brandt groupoid [8] (also [9], p. 121) is a set G of elements in which the product exists only for certain pairs and which satisfies the following conditions.

I. If for three elements $a, b, c, \in G$ the relation

$$ab = c$$

holds, then each of them is uniquely determined by the other two.

II. If ab and bc exist then there exist also $(ab)c$ and $a(bc)$, if ab and $(ab)c$ exist, then there exist also bc and $a(bc)$, if bc and $a(bc)$ exist, then there exist also ab and $(ab)c$. In all three cases the equality

$$(ab)c = a(bc)$$

holds.

III. For every element $b \in G$ there exists a uniquely determined element $i(b)$, the right unit, a uniquely determined element $i'(b)$, the left unit, and a uniquely determined inverse element b^{-1} such that

$$bi(b) = i'(b)b = b,$$

$$b^{-1}b = i, \ bb^{-1} = i'.$$

IV. For any two units i and i' there exists at least one element $b \in G$ such that i is the right unit and i' is the left unit of b.

It is easy to see that our set B of invariance transformations defined by parallel displacements satisfies these axioms. Indeed, for any parametrised curve $x^\alpha = x^\alpha(t)$, the equations

$$\frac{du^\alpha}{dt} = \omega^\alpha_\beta (t)\, u^\beta, \tag{1}$$

where $\omega^\alpha_\beta(t) = - \left\{ {\alpha \atop \beta\gamma} \right\} dx^\gamma/dt$, have a unique set of solutions of the form [7, 11]

$$u^\alpha (t) = b^\alpha_\beta (t, t_0)\, u^\beta (t_0)\,, \tag{2}$$

where the matrices b^α_β are non-singular (at least under the conditions discussed below). This defines a linear homogeneous isometry $u(t_0) \to u(t)$ from the tangent space at $x(t_0)$ to that at $x(t)$. The matrices also satisfy [11]

$$\left. \begin{aligned} b^\alpha_\beta (t, t')\, b^\beta_\gamma (t', t'') &= b^\alpha_\gamma (t, t'') \\ b^\alpha_\beta (t, t')\, b^\beta_\gamma (t', t) &= \delta^\alpha_\gamma . \end{aligned} \right\} \tag{3}$$

Denote now by $b(t_y,t_x)$ any curve with parameter t connecting x and y and denote the matrix (2) defined by the curve also by the same symbol $b(t_y,t_x)$.

Elements of the groupoid B are of the form $b(t_y,t_x)$ (we use matrix notation) for the oriented curves from x to y, in this order, as x and y run through all points. Define multiplication in B as: Two elements $b(t_x, t_z)$ and $b(t_\omega, t_y)$ can be multiplied, in this order, if $z = \omega$ and only in this case and the product is then given by the appropriate form of Equs.(3). The set B with this multiplication is clearly a Brandt groupoid.

Indeed, consider first Condition IV. Our transformations (2) defined by Equs. (1) are determined by the Christoffel symbols, which in turn are functions of the metric tensor and its derivatives. IV thus implies that a continuous metric tensor must exist everywhere. It was pointed out [10] that topological properties must be introduced into the definition of space-time. One way to do this is to suppose [10] space-time to be a differentiable manifold M_n. Now a differentiable manifold always admits a positive definite metric tensor, but admits a continuous metric tensor of signature (3,1) if and only if the Euler—Poincaré characteristic $\chi = \sum_k (-1)^k v_k$, where v_k is the *kth* Betti number, vanishes ([12], p. 18). In this case then the matrix in Equ. (2) is non-singular.

Condition II is trivially satisfied and the non-singular nature of $[b(t_y,t_x)]$ ensures I. III is also satisfied: For any element $b(t_x,t_y)$ the left and right units $b(t_x,t_x)$ and $b(t_y,t_y)$, the (unit) matrices associated with the points x and y, respectively, and the inverse $b^{-1}(t_x,t_y) = b(t_y,t_x)$ associated with the inversely oriented curve, clearly exist. This proves the groupoid nature of B under the above restriction.

3. Direct product decomposition of the groupoid

We now construct two subsets of B and then show that B is the direct product of these.

It is clear from axioms I to IV that the condition for B to be a group is that it should contain a single unit element. Therefore B can never be group in curved space, nevertheless it contains groups.

Consider indeed the subset of all those elements b for which the left and right units coincide $i(b) = i'(b) = i_x$. It is clear from the axioms that this set is a group consisting of the elements defined by transformations along all closed curves through the point x. It is indeed the holonomy group $\psi_x(M_n)$ at x discussed extensively [5,10].

We now construct another subset of B as follows. Take the right and left unit elements $i(x)$ and $i'(y)$ at two arbitrary but fixed points x and y, in this order, and connect x and y by a single, but arbitrarily chosen, curve $b(y,x)$. Denote the transformation along the curve also by the same symbol $b(y,x) \in B$. When x and y run through all points of M_n we obtain a subset S_b of B which is also a groupoid. Since S_b has the same set of units as B it is a subgroupoid of B and it is also clear from the construction that this is the minimal sub-groupoid which has the same set of units as B.

If we connect x and y with any different but well defined curve $c(y,x)$ we get another subgroupoid S_c which is isomorphic with S_b since they have the same set of units.

Let us now fix S_b and let $x \in M_n$ be any arbitrary but fixed point. We want to prove that $B = S_b \otimes \psi_x(M_n)$ where $\otimes$ is direct product.

To this end we first remark that any arbitrary element $t(z, y)$ of B can be written as the product $b(z,x)a(x)b(x,y)$, where $a(x) \in \psi_x$ and $b(z,x)$, $b(x,y) \in S_b$, in a unique way. To see that this is so one has only to choose $a(x) = b^{-1}(z,x)t(z,y)b^{-1}(x,y)$ which is clearly a transformation defined by the closed loop $b^{-1}(z,x)t(z,y)b^{-1}(x,y)$ through x.

Let now $t(v,\omega) = b(v,x)a'(x)b(x,\omega)$, where $a'(x) \in \psi_x$ and $b(v,x)$, $b(x,\omega) \in S_b$, be the above product decomposition of any other arbitrary element $t(v, \omega)$ of B. Then the product $t(v, \omega)t(z,y)$ exists only if $\omega = z$ and it is in this case $t(v,y) = b(v,x)\ a'(x)a(x)\ b(x,y)$.

This shows that in the product of arbitrary elements $t(v,\omega)$ and $t(z,y)$ of B, elements of S_b and elements of ψ_x are multiplied separately. In other words we have

$$B = S_b \otimes \psi_x(M_n)\,. \tag{4}$$

4. Local groups

We are now able to discuss the problem put forward in the Introduction. In our effort to introduce an invariance principle in curved space the starting point was [4] the problem of how to compare physical quantities in the (inertial) tangent spaces at different space-time points. We have seen that we can compare physical quantities by means of the transformations contained in the groupoid B.

Obviously, the decomposition (4) classifies physical measurements into two classes:

a) local measurements, in which quantities in the same tangent space T_x are compared, can be evaluated by the local invariance group ψ_x,

b) measurements, in which quantities in the tangent spaces at different space-time points are compared, can be evaluated by elements of the minimal sub-groupoid S_b.

Supposing now general relativity, the three crucial tests fall into class b). Red shift between any x and y has been shown [4,6] to follow from invarianc under parallel displacement along any curve connecting x and y, i.e. from invariance under an element of S_b. Also, if one chooses geodesics for elements of S_b then the geodesic axioms follow.

As to class a) it is well known that the holonomy group ψ_x is subgroup of the homogeneous Lorentz group for an Einstein manifold and the Lie-algebra of ψ_x, which defines local invariance in the tangent space T_x, is spanned [13] by the $^{\varkappa}_{\lambda}$-domains of the curvature tensor and its covariant derivatives

$$p^\alpha q^\beta R^{\varkappa}_{\alpha\beta\lambda}, q^\omega \nabla_\omega p^\alpha q^\beta R^{\varkappa}_{\alpha\beta\lambda}, \ldots, (q^\omega \nabla_\omega)^k p^\alpha q^\beta R^{\varkappa}_{\alpha\beta\lambda}, \tag{5}$$

where the arbitrary vectors p,q and the curvature tensor R and its covariant derivatives are to be understood at x.

It is seen from expression (5) that ψ_x is reduced to the identity for a flat manifold. We have on the other hand the important theorem of BEIGLBÖCH [14], which says that the Lie-algebra of ψ_x is always six-dimensional for a non-vacuum Einstein manifold. Therefore the restricted Lorentz group $L^\uparrow_+$ (for inversions see [10] and also the footnote on p. 262) can be interpreted as a local property of a non-vacuum Einstein manifold. However, the local invariance group $\psi_x(M_n)$ is always homogeneous for an Einstein manifold. This follows from the fact that the hg associated with a symmetric connection is homogeneous. This is an unpleasant feature of local invariance since translation invariance has deep physical consequences and there is therefore interest in more general spaces for which the hg is non-homogeneous.

Maybe the simplest such generalization is in which the Chistoffel symbols are replaced by a non-symmetric connection $\Gamma^{\varrho}_{\mu\nu}$. It is indeed well known that the hg associated with such a connection is non-homogeneous and the infinitesimal translations at x are generated by expressions of the form ([13] p. 362)

$$-T^k_{\mu\nu}\, df^{\mu\nu}, \tag{6}$$

where $df^{\mu\nu}$ is an infinitesimal facet at x and $T^k_{\mu\nu} = \Gamma^k_{\mu\nu} - \left\{ {k \atop \mu\nu} \right\}$ is the torsion tensor.

Consider now the set B' (Section 2) of invariance transformations defined by parallel displacement associated with this new connection. B' is again a Brandt groupoid and the decomposition of Section 3 also holds. In this way the Poincaré group might be obtained, just as $L^\uparrow_+$ has been in the case of an Einstein manifold, as the hg of this generalized space.

Suppose now that the Poincaré group is a local invariance group. Then if our invariance principle, i.e. invariance under the Brandt groupoid, is valid, then invariance, in measurements of class b), under S'_b must also hold as can be seen from the decomposition (4). But this is just the interpretation of SZEKERES of the anomalous frequency shift found by SADEH et al.

Clearly, the measurements of SADEH et al. belong to class b) and what SZEKERES calculates is just the contribution to parallel displacement of the non-symmetric part of the connection when S_b' is constructed from geodesics.

We do not want here to argue about the physical interpretation of the torsion tensor, only want to point out that, if the Poincaré group is to be interpreted as a local invariance group in a curved space, then a non-vanishing torsion tensor must be involved.

5. Discussion

Recent cosmological observations seem to confirm that physical space-time is curved. The Lorentz, or the Poincaré group cannot then be interpreted as the motion group of that space. In a curved space we have only local inertial systems and in this case the considerations of this, and previous papers (see the Introduction) are relevant. We want here to emphasise that our basic assumption is that these local systems are still physically equivalent. At the basis of this assumption is really the Eötvös experiment. The choice of the connection, which defines the invariance transformations connecting these systems, is a matter of experiment.

Once, however, a particular connection is selected then the structure of, for example, the local invariance group is determined. In particular the argument presented here suggests that either general relativity holds and then the local invariance group is only the homogeneous Lorentz group in each tangent space, or the Poincaré group is good and then the underlying space-time manifold has non-vanishing torsion.

In conclusion it must be emphasized that our full invariance principle is defined not by a group but a Brandt groupoid which is a more general object.

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ИНВАРИАНТНОСТЬ ПУАНКАРЕ СОВМЕСТИМА С ОБЩЕЙ ТЕОРИЕЙ ОТНОСИТЕЛЬНОСТИ?

М. ШЮВЕГЕШ

Резюме

Предварительно показали, что определен инвариантный принцип в искривленном пространстве инвариантностью по отношению группоиде Брандта, состоящего из элементов, данных параллельными смещениями по всем возможным кривым в пространстве времени. Доказывается, что группоид Брандта может содержать группу Пуанкаре как локальную группу в каждом тангенциальном пространстве, но в этом случае пространство-время должно иметь неисчезающую крутизну Такое условие может быть применено и современными измерениями Саде и др. С другой стороны, в случае одного множества Эйнштейна группоид Брандта содержит только однородную группу Лоренца.