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THE HYDRODYNAMICAL MODEL OF WAVE MECHANICS*

THE MANY BODY PROBLEM

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The problem of how to extend the hydrodynamical model — described in previous publications — to many body problems is discussed and it is shown that the difficulties which arise in such an attempt are not of mathematical nature, but they reflect upon a physical problem which in our opinion is unsolved so far. It is suggested that in the correct treatment of a many body problem one should try to select between the possible wave functions such functions which are very likely in a statistical sense. This proposed selection resembles the selection of the very likely configurations in statistical mechanics.

§ 1. We have shown in a number of publications [1—5] how the wave equation describing a one-body problem can be transformed into a mathematical equivalent form, so that in the new form the variables have a good meaning in the classical sense.

It is generally believed to be impossible to carry out a similar transformation of the wave equation describing the motion of a system consisting of several particles.

We analyse the latter problem and show that there are difficulties indeed to express the many-body problem in terms of hydrodynamical variables, but these difficulties are not of mathematical nature but are connected with a physical problem which does not seem to be solved so far.

I

§ 2. The wave equation describing the motion of N particles under the influence of a potential V can be written

$$\sum_{\nu=1}^{N}\left(-\frac{\hbar^2}{2m_\nu}\nabla_\nu^2\psi + V\psi\right) = i\hbar\dot{\psi}\,, \tag{1}$$

* This article is dedicated to the 60th birthday of my good friend, Academician P. GOMBÁS, whom I sincerely wish many more years of undiminished activity. I always followed with great interest his work, in particular on the statistical methods of obtaining the stationary states of many body systems. Since many years I am wondering, whether a dynamical version of this theory could be found? It may be that the ideas developed in this article have some connection with the latter problem.

where

$$\psi = \psi_\nu(\mathfrak{r}, t), \qquad \mathfrak{r} = \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots . \boldsymbol{r}_N \tag{2}$$

and ∇_ν is the nabla operator acting upon the components of $\boldsymbol{r}_\nu$.

V may be supposed to have the form

$$V(\mathfrak{r}) = \sum_\nu V_\nu(\boldsymbol{r}_\nu) + \sum_{\nu \neq \mu} V_{\nu\mu}(\boldsymbol{r}_\nu - \boldsymbol{r}_\mu). \tag{3}$$

A type of equation which somewhat resembles to the hydrodynamical relations can be obtained from (1) by writing

$$\psi = Re^{iS}, \tag{4}$$

where R and S depend on $\mathfrak{r}$ and also on the time t.

Making use of the relation (4) one finds

$$\frac{\nabla_\nu^2 \psi}{\psi} = \frac{\nabla_\nu^2 R}{R} - (\mathrm{grad}_\nu S)^2 + i\left(\frac{2\,\mathrm{grad}_\nu R\,\mathrm{grad}_\nu S}{R} + \nabla_\nu^2 S\right). \tag{5}$$

Introducing (5) into (1) one finds separating real and imaginary parts

$$\sum_\nu \mathrm{div}_\nu \varrho \mathfrak{v}_\nu + \frac{\partial \varrho}{\partial t} = 0 \tag{6}$$

with

$$\varrho = R^2 \qquad \mathfrak{v}_\nu = \frac{\hbar}{m_\nu} \mathrm{grad}_\nu S. \tag{7}$$

(We use gothic $\mathfrak{v}$ for the velocity distribution in $3N$ dimensions).

From the real part of (5) we find

$$-\mathrm{grad}_\mu \left[\sum_\nu \left(\frac{\hbar^2}{2m_\nu} \frac{\nabla_\nu^2 \varrho^{1/2}}{\varrho^{1/2}} + \frac{1}{2} m_\nu \mathfrak{v}_\nu^2\right) + V\right] = m_\mu \mathfrak{v}_\mu. \tag{8}$$

§ 3. Equations (6), (7) and (8) give a set of differential equations which can be solved for given initial conditions. These equations can be regarded as "hydrodynamical equations in a $3N$-dimensional space". Relations (5), (6) and (7) do not seem, however, to have particular physical significance as they express the motion of a physical system in a $3N$-dimensional configuration space. Such a description reveals just as little the physical significance of the process involved as the original wave equation.

II

§ 4. We obtain sets of equations referring to three-dimensional distributions by forming suitable averages. We may introduce three-dimensional densities by writing

$$\varrho_\nu(\boldsymbol{r}) = \underbrace{\int \dots \int}_{(N-1)} \varrho(\boldsymbol{r}_1 \dots \boldsymbol{r}_{\nu-1}, \boldsymbol{r}, \boldsymbol{r}_{\nu+1} \dots \boldsymbol{r}_N)\, d^3\boldsymbol{r}_1 \dots d^3\boldsymbol{r}_{\nu-1}\, d^3\boldsymbol{r}_{\nu+1} \dots d^3\boldsymbol{r}_N . \quad (9)$$

In place of (9) we can also write shorter

$$\varrho_\nu(\boldsymbol{r}) = \int\limits_{(\nu)} \varrho d^{3(N-1)}\boldsymbol{r} , \quad (10)$$

where the symbol on the right hand side of (10) is supposed to stand for the right hand expression of (9).

Similarly we may introduce

$$\varrho_\nu(\boldsymbol{r})\, \boldsymbol{v}_\nu(\boldsymbol{r}) = \int\limits_{(N-1)} \dots \int \varrho \mathfrak{v}(\boldsymbol{r}_1 \dots \boldsymbol{r}_{\nu-1}, \boldsymbol{r}, \boldsymbol{r}_{\nu+1} \dots \boldsymbol{r}_N)\, d\boldsymbol{r}_1 \dots d\boldsymbol{r}_{\nu-1}\, d\boldsymbol{r}_{\nu+1} \dots d\boldsymbol{r}_N . \quad (11)$$

Or writing short

$$\boldsymbol{v}_\nu(\boldsymbol{r}) = \int\limits_{(\nu)} \varrho \mathfrak{v}_\nu\, d^{3(N-1)}\boldsymbol{r} . \quad (12)$$

§ 5. Integrating the continuity relation (6) into (N—1) coordinate vectors $\boldsymbol{r}_\nu$ $\nu = 1, 2, \dots \mu$—1, $\mu + 1 \dots N$, i.e. integrating over all coordinate vectors with the exception of $\boldsymbol{r}_\mu$ we find that N—1 of the terms under the sum vanish and we obtain

$$\operatorname{div}(\varrho_\mu \boldsymbol{v}_\mu) + \frac{\partial \varrho_\mu}{\partial t} = 0 \qquad \mu = 1, 2, \dots, N . \quad (13)$$

We see thus that the three-dimensional distributions $\varrho_\mu(\boldsymbol{r})$ and $\boldsymbol{v}_\mu(\boldsymbol{r})$ $\mu = 1, 2, \dots, N$ represent flows each satisfying a continuity relation in three dimensions.

III

§ 6. So as to show that the N flows so obtained have a good physical significance, we note that we may introduce current and charge densities

$$\boldsymbol{i}_\nu = e_\nu \boldsymbol{v}_\nu \varrho_\nu / c , \qquad \varrho_{\mathrm{el}_\nu} = e_\nu \varrho_\nu , \qquad \nu = 1, 2, \dots, N , \quad (14)$$

where e_ν is the electric charge carried by the ν-th particle. Forming retarded potentials

$$\boldsymbol{A}_\nu = \int \frac{[\boldsymbol{i}_\nu]}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d^3\boldsymbol{r}', \quad \Phi_\nu = \int \frac{[\varrho e]_\nu}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d^3 r' \tag{15}$$

in the usual way, we can suppose that the electromagnetic field of the system can be derived from potentials

$$\boldsymbol{A} = \Sigma \boldsymbol{A}_\nu, \qquad \Phi = \Sigma \Phi_\nu. \tag{16}$$

Indeed the field obtained from $\boldsymbol{A}$ and Φ as given by (15) and (16) is exactly equal to the expectation values of the corresponding quantities obtained in terms of the usual operator formalism.

§ 7. Similarly we can introduce quantities like momentum and angular momentum writing

$$\boldsymbol{p}_\nu = \int \varrho_\nu \boldsymbol{v}_\nu\, d^3\boldsymbol{r}, \qquad \boldsymbol{M}_\nu = \int (\boldsymbol{r}\times\varrho_\nu \boldsymbol{v}_\nu)\, d^3\boldsymbol{r}, \tag{17}$$

and it can be verified easily that the system as a whole behaves as a system with total momentum respectively angular momentum given by

$$\boldsymbol{p} = \Sigma \boldsymbol{p}_\nu, \qquad \boldsymbol{M} = \Sigma \boldsymbol{M}_\nu.$$

We see therefore that the behaviour of the N particle system can be characterized indeed with the help of the densities $\varrho_\nu \boldsymbol{v}_\nu$ and ϱ_ν. Taking the spin of the particles into consideration we have to introduce further variables, e.g. $\boldsymbol{T}_\nu$ where $\boldsymbol{T}_\nu = \boldsymbol{T}_\nu(\boldsymbol{r}, t)$ are unit vectors characterizing the directions of spins of the various particles.

§ 8. Multiplying both sides of (8) by ϱ and integrating over N—1 coordinate vectors $\boldsymbol{r}_\nu \neq \boldsymbol{r}_\mu$ we obtain three-dimensional equations of motions of the form

$$\varrho_\mu \dot{\boldsymbol{v}}_\mu = \boldsymbol{F}_\mu \qquad \mu = 1, 2, \ldots, N, \tag{18}$$

where

$$\boldsymbol{F}_\mu = -\operatorname{grad} \int_{(\mu)} \varrho(\mathfrak{Q} + \mathfrak{R} + V)\, d^{(n-1)}\boldsymbol{r},$$

$$\mathfrak{Q} = \sum_\nu \frac{\hbar^2}{m_\nu} \frac{\nabla_\nu^2 \varrho^{1/2}}{\varrho^{1/2}}, \qquad \mathfrak{R} = \frac{1}{2} \sum_\nu m_\nu \mathfrak{v}_\nu^2. \tag{19}$$

IV

§ 9. The relations (18) and (19) have the form of three-dimensional equations of motions which describe the motion of N media simultaneously. However, there is an essential difference between the one-body problem with $N = 1$

and the more-body problems $N > 1$. Indeed, in the case $N = 1$ the wave function ψ can be unambiguously determined from the distributions $\boldsymbol{v}$ and ϱ. Therefore in the one-body problem the right hand expression (18) can be expressed in terms of $\boldsymbol{v}$ and ϱ and thus (18) together with the continuity relation (13) gives a set of differential equations which for a given initial condition can be integrated. Therefore if we describe a one-particle system by giving

$$\boldsymbol{v}(\boldsymbol{r}, 0) = \boldsymbol{v}^{(0)}(\boldsymbol{r}) \qquad \text{and} \qquad \varrho(\boldsymbol{r}, 0) = \varrho^{(0)}(\boldsymbol{r}),$$

then we can determine the motion of the system unambiguously.

§ 10. In the case of several particles the position is different. Let us consider $N = 2$.

Giving initial conditions

$$\left.\begin{aligned} \mathfrak{v}_\nu(\boldsymbol{r}_1, \boldsymbol{r}_2, 0) &= \mathfrak{v}_\nu^{(0)}(\boldsymbol{r}_1, \boldsymbol{r}_2) \qquad \nu = 1, 2 \\ \varrho(\boldsymbol{r}_1, \boldsymbol{r}_2, 0) &= \varrho^{(0)}(\boldsymbol{r}_1, \boldsymbol{r}_2) \end{aligned}\right\}. \tag{20}$$

We can determine the motion of the system in an unambiguous manner. However, if we give only

$$\left.\begin{aligned} \varrho_1^{(0)}\boldsymbol{v}_1^{(0)}(\boldsymbol{r}) &= \int \varrho \mathfrak{v}_1(\boldsymbol{r}, \boldsymbol{r}', 0)\, d^3\boldsymbol{r}, \quad \varrho_1^{(0)}(\boldsymbol{r}) = \int \varrho(\boldsymbol{r}, \boldsymbol{r}', 0)\, d^3\boldsymbol{r}' \\ \varrho_2^{(0)}\boldsymbol{v}_2^{(0)}(\boldsymbol{r}) &= \int \varrho \mathfrak{v}_2(\boldsymbol{r}', \boldsymbol{r}, 0)\, d^3\boldsymbol{r}', \quad \varrho_2^{(0)}(\boldsymbol{r}) = \int \varrho(\boldsymbol{r}', \boldsymbol{r}, 0)\, d^3\boldsymbol{r}' \end{aligned}\right\}, \tag{21}$$

then we cannot determine $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2, 0)$ from the initial condition; expressing $\mathfrak{v}_\nu$ respectively ϱ in terms of $\psi = Re^{iS}$ making use of (8) we find that there exist a very large number of functions ψ satisfying (21).

§ 11. In fact the densities $\varrho_1\boldsymbol{v}_1$ and $\varrho_2\boldsymbol{v}_2$ can be taken as two three-dimensional moments of a six-dimensional distribution. A number of three-dimensional moments restrict a six-dimensional distribution only to a very slight extent. Indeed, we may approximate ψ by a step function dividing each of the coordinate axes into n sections and giving the average value of ψ in any of the $m = n^{3N}$ $3N$-dimensional cubes thus obtained. The number of conditions which can be imposed by giving the three-dimensional moments corresponding to the distribution (21) is

$$M = 8\,Nn^3 \ll m\,.$$

If we were to give some more three-dimensional fields we could increase the number of conditions but still we could not determine ψ from a number of such moments.

V

§ 12. We note that the fact that ψ depends on $3N$ space variables introduces a large ambiguity concerning the variation of the three-dimensional densities. Indeed, if we give not only $\boldsymbol{v}_\mu$ and ϱ_μ for $t = 0$ but also the time

derivatives $\dot{\boldsymbol{v}}_\mu$, then we still can construct a great manifold of ψ-functions which reproduce this extended initial condition.

Differentiating the equations of motion (18) and (19) into the time, we can eliminate with the help of the wave equations (1) the time derivatives of ψ which appear in the right hand side of the differentiated expressions. We obtain thus relations of the type

$$m_\mu \frac{d^l \boldsymbol{v}_\mu}{dt^l} = \text{function of } \psi \text{ and space derivatives integrated over the } \boldsymbol{r}_\nu, \nu \neq \mu. \tag{21a}$$

However, the left hand expressions depend for a fixed μ on the coordinate $\boldsymbol{r}_\mu$ only, while the right hand expression contains the ψ and its derivatives suitably averaged over the variables $\boldsymbol{r}_\nu \neq \boldsymbol{r}_\mu$. Therefore the right hand expressions can be taken to be 3-dimensional moments of the $3N$-dimensional distribution ψ. One can therefore find an infinite number of distributions ψ which satisfy the conditions (21a) at $t = 0$ for arbitrarily given values of the $\boldsymbol{v}_\mu$ and their time derivatives.

The fact that we can impose arbitrary initial conditions for the $\boldsymbol{v}_\mu$'s and their time derivatives means that we can prescribe the change of time of the $\boldsymbol{v}_\mu$'s themselves. I.e. *we can give $\boldsymbol{v}_\mu(\boldsymbol{r}, t)$ for an extended interval of time and find a ψ-function which corresponds to a motion in which $\boldsymbol{v}_\mu(\boldsymbol{r}, t)$ takes up values in this arbitrary prescribed manner.*

The above result shows that something is missing in the theory. The task of a theory is to determine the motion of a system from suitable initial conditions. Here we meet a state of affairs such that we can prescribe arbitrarily the motion of a system and the theory yields inner parameters, i.e. the wave function ψ, which leads to the motion we prescribed.

The unsatisfactory feature of the theory for $N > 1$ could be avoided in a formal way if we were to admit that the initial condition of a physical system has to be given by a $3N$-dimensional wave function or by $3N$-dimensional density distributions rather than by the averages referring to the three-dimensional space.

§ 13. So as to see the physical contents of the difficulty more clearly we give the following analogy: consider opaque bodies moving before a screen and let us observe the two-dimensional shadows which the three-dimensional bodies throw on the screen.

Studying the motion of the shadows we find that their motions cannot be determined from initial conditions. Indeed, bodies of quite different shapes may produce in one instant similar shaped shadows, but the shadows of the differently shaped bodies will change in different manner according to the shape of the moving three-dimensional body.

§ 14. To return to our original problem we can take the densities $\varrho_\mu(\boldsymbol{r})$ and velocities $\boldsymbol{v}_\mu(\boldsymbol{r})$ to represent a kind of three-dimensional projections of the $3N$ dimensional distributions $\varrho(\mathfrak{r})$ and $\mathfrak{v}_\nu(\mathfrak{r})$. If the state of the system is in fact determined by the $3N$-dimensional distribution, then the motion of the "shadows" described by the three-dimensional distributions cannot be determined from their three-dimensional initial conditions. Indeed, considering physical systems described by two wave functions $\psi_1(\mathfrak{r}, t)$ and $\psi_2(\mathfrak{r}, t)$ which lead, say at $t = 0$, to the same three-dimensional distributions. The systems starting from identical three-dimensional initial conditions will show entirely different motions for $t > 0$.

§ 15. The initial conditions giving the three-dimensional distributions at $t = 0$ do not determine the motion of a system if the equations of motions can be expressed in the form of wave equation

$$H\psi = i\hbar\dot{\psi}, \qquad \psi = \psi(\mathfrak{r}, t). \tag{22}$$

The physical problem of describing the motion of atoms seems, however, to require predictions of the motion of atoms described by three-dimensional distributions as we show presently.

Indeed, if we observe atoms we can observe their electromagnetic fields, i.e. we can observe electromagnetic fields emitted in form of radiation; we can also observe electric- or magnetic dipole moments arising if the atoms are polarized. We can further observe energy and momentum of atoms in particular if they collide with a macroscopic body. We can observe angular momentum and other similar parameters of an atom. All these quantities enumerated above can be observed more or less directly and they can be expressed using three-dimensional distributions only. Thus the features of an atom which are observed in usual experiments seem to give information at most upon the three-dimensional distributions.

§ 16. When we make the above statement we restrict ourselves to quantities which can be measured indeed by real experiments. We disregard quantities which are supposed to be "measurable" in terms of an abstract theory. To illustrate our point of view let us consider a He-atom and the supposed measurement of the positions of both of its electrons.

The wave function $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$ describing a two-body configuration is supposed to give

$$P(\boldsymbol{r}_1, \boldsymbol{r}_2)\, d^3\boldsymbol{r}_1\, d^3\boldsymbol{r}_2 = |\psi|^2\, d^3\boldsymbol{r}_1\, d^3\boldsymbol{r}_2\,, \tag{23}$$

which quantity is supposed to give the probability to find the first particles of the system inside a volume element $d^3\boldsymbol{r}_1$, the second inside $d^3\boldsymbol{r}_2$. Taking the system, say, to consist of the two electrons of a He-atom, (23) gives the probability density of finding the electrons very near to points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ inside the atom.

Considering real experimental conditions, we feel that it is impossible to locate by real measurements the position of even one electron inside an atom and it appears absolutely phantastic to try to make statistics of "where we find the second electron once we have found the first electron inside an element $d^3\boldsymbol{r}_1$".

§ 17. The proposition that by a series of hypothetical measurement of coordinates we can determine experimentally $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)$ seems to be absurd. Nevertheless, the wave function ψ has a good physical significance. Any real measurement will lead to the determination of moments

$$M_k = \int \psi^*(\mathfrak{r})\, \mathfrak{M}_k(\mathfrak{r})\, \psi(\mathfrak{r})\, d^{3N}\mathfrak{r}\,, \tag{24}$$

where $\mathfrak{M}_k(\mathfrak{r})$ is a function of $\mathfrak{r}$ or some operator acting on the wave function $\psi(\mathfrak{r})$. However, determining moments of the type (24) we obtain only very weak restrictions as to the distribution of $\psi(\mathfrak{r})$ in a $3N$-dimensional space.

We may determine the field of an atom which determination amounts to determining some moment M_k as function of the coordinate vector $\boldsymbol{r}$. In this way we obtain relations which can be written symbolically

$$M_k(\boldsymbol{r}) = \int \psi^*(\mathfrak{r})\, \mathfrak{M}_k(\mathfrak{r})\, \psi(\mathfrak{r})\, d^{3N-3}\mathfrak{r}\,. \tag{25}$$

The relation (25) — if $M_k(\boldsymbol{r})$ is obtained by measurement for all values of $\boldsymbol{r}$ — gives merely a three-dimensional restriction upon $\psi(\mathfrak{r})$. As explained in § 11 even a number of such restrictions are utterly insufficient for the determination of the distribution $\psi(\mathfrak{r})$ itself.

We see therefore that the empirical information we can obtain as to physical state of an atom is restricted to three-dimensional moments of ψ and therefore a useful theory must attempt to make conclusions about the motion of atoms based on information consisting of such three-dimensional moments only.

VI

§ 18. Logically there seem to be two possibilities to avoid the ambiguity involved in the more-body problems. One might suppose that the wave equation (22) gives only a *necessary condition* for the motion of $\psi(\mathfrak{r}, t)$. One might suppose that the full equations of motion have the form

$$\left.\begin{aligned} H\psi &= i\hbar\dot{\psi}\,, \qquad (a)\\ A\psi &= 0\,, \qquad (b)\end{aligned}\right\} \tag{26}$$

where the second condition is supposed to be an auxiliary condition compatible with the wave equation. If the condition (26b) is a sufficiently strong condition the solutions of (26a), (26b) reduce to a manifold of the order of that of

three-dimensional distributions. If such a condition (26b) existed, then the relations (10) and (12) together with (26b) would be sufficient to determine $\psi(\mathfrak{r})$ uniquely and thus hydrodynamical equations of motion could be obtained in the case $N > 1$ also.

§ 19. A condition of the form (26b) is obtained e.g. by requiring the wave function to be antisymmetric in certain variables. Such a condition is compatible with (26a) but it gives a very weak restriction only on the wave function. The condition that the wave function should be antisymmetric does not permit to reduce the manifold of solutions of (26a) sufficiently so as to make the solutions to correspond to three-dimensional distributions.

The ground state of an atom is uniquely determined by a variational condition. I.e. the ground state is described by the wave function for which

$$\delta \int \psi^* H\psi \, d^{3N}\mathfrak{r} = 0 . \tag{27}$$

One might be tempted to think that the auxiliary condition could be taken to be a variational condition of the type (27). The condition (26b) might require to look for the minimum of the energy among those functions ψ which lead to a given three-dimensional current charge distribution

$$\boldsymbol{v}_\mu(\boldsymbol{r}) \quad \text{and} \quad \varrho_\mu(\boldsymbol{r}) . \tag{28}$$

Thus one might try to postulate that a system which exhibits a (three-dimensional) current charge distribution (28) is to be described by that wave function ψ which leads in terms of (10), (11) to the distribution (28) and which possesses the minimum energy which is compatible with the configuration (28).

§ 20. The solutions obeying (27) together with the condition (28) obey certain differential equations which are somewhat similar to the time-independent Schrödinger equation. However, one finds that inserting as initial condition into (22) a wave function which possesses a relative minimum thus described — one finds from (22) that in the course of time the function *loses* the minimum property. Thus the conditions (27) and (28) are in general *incompatible* with the wave equation (22).

§ 21. We think it likely that conditions of the form of (26b) do not exist which are: 1) compatible with the wave equation (26) and 2) reduce the set of solutions to a sufficient degree. Whether or not this supposition is correct could be investigated in the following way.

We may suppose that the ground state of an atom — which is described at, say, $t = 0$ by a wave function $\psi_0(\mathfrak{r})$ — is a possible state. Let us consider another state which at some fixed time $t = t_1 > 0$ is described by an arbitrarily given wave function $\psi_1(\mathfrak{r})$.

The question arises whether we can find some outside perturbations which e.g. can be described by a potential

$$V(\mathfrak{r}, t) = \sum_{\nu=1}^{N} V_\nu(\boldsymbol{r}_\mu, t)\,, \tag{29}$$

such that in the interval $t = 0$ to $t = t_1$ the action of the potential makes the wave function to transform from

$$\psi = \psi_0 \quad \text{to} \quad \psi = \psi_1\,.$$

Suppose e.g. a wave equation of the form:

$$\left\{\sum_{\nu=1}^{N}\left(-\frac{\hbar^2}{2m_\nu}\nabla_\nu^2 + V_\nu(r_\nu, t)\right) + V_0\right\}\psi = i\hbar\dot{\psi}\,, \tag{30}$$

where V_0 represents the interaction between the N particles. The question arises whether we can find potentials $V_\nu(r_\nu, t)$ such that starting from an initial condition $\psi = \psi_0$ the solution of (3) will be equal to $\psi = \psi_1$ at $t = t_1$.

§ 22. The answer to the above question is not trivial for the following reason. The change $\delta\psi$ of ψ in an infinitesimal interval δt can be written

$$\delta\psi = \dot{\psi}\delta t = -\frac{i}{\hbar}H\psi\,, \tag{31}$$

we can choose the potentials V_ν at a given time t arbitrarily. Nevertheless because H contains the sum of potentials V_ν each depending on one variable only, (31) cannot be satisfied for an arbitrarily given $\delta\psi$.

In a formal way we can suppose ψ and $\psi + \delta\psi$ to represent two points in Hilbert space. We find that using potentials of the form (29) we can obtain *direct transitions* from a point ψ of Hilbert space only to exceptional points in its vicinity if we restrict ourselves to potentials of the form (29). Thus most of the points in the vicinity of a point ψ are *inaccessible* by *direct transition.*

From this it does not follow that these points cannot be reached on a round about way. In fact we think that most of the points of Hilbert space can be reached from an initial point ψ_0 as the effect of a sufficiently well chosen — rather complicated — perturbation of the form (29).

We hope to comeback to the mathematical discussionof this question in a later paper.

§ 23. The second logical possibility is to suppose that there exist no strong restrictions as to the ψ-functions which are accessible — but like in statistical mechanics the states described by ψ have very different probabilities to become realized. If there exist ψ-functions which give very likely configurations and others which give very unlikely ones, then we can postulate that a given distribution (28) is represented in most cases by a "very likely" ψ. Therefore

the motion of the system will in most cases take place in a very good approximation, in a manner to be expected in terms of a "very likely" ψ-function.

The analogy of the concept and that of statistical mechanics is a very deep one. Indeed, in statistical mechanics the motion of say a gas can be calculated exactly if the initial conditions are given, containing the coordinates and velocities of all the N atoms of the gas at $t = 0$.

From the mechanical equations of motion, the motion of the gas as a whole can be determined. However, if we give only the macroscopic density distribution and the distribution of velocity of flow, then these distributions do not determine the initial conditions for the motion of the individual atoms. Constructing sufficiently artificial initial conditions for the atoms, we come to configurations which starting from a given macroscopic initial condition behave in a very abnormal manner. With sufficient skill we can find initial conditions for the atoms which lead to an almost arbitrary prescribed macroscopic motion of the gas.

The equations of motion which give the correct macroscopic motions of a gas are obtained if we choose among the possible microscopic configurations the very likely ones. Deviations from the macroscopic equations of motions arise because of fluctuations, i.e. because occasionally there occur distributions which deviate from those which are the most likely ones.

§ 24. In the case of the ψ-function one imagines that "smoothed" functions are more likely than those which show a great deal of irregular fluctuations. It seems to be an unsolved problem which ψ's are the very likely ones.

Although we can give no method of actually selecting the "very likely ψ-functions" it seems plausible that these functions are somewhere near those possessing minimum properties of the type described in § 18. Indeed, if an atomic system is in a state possessing a great deal of energy, then it is likely to emit electromagnetic radiation and thus to reduce its energy. We expect therefore a tendency of atoms to approach states with low energy.

§ 25. We close these qualitative considerations with another remark. From the solutions of the wave equations valid for many-body systems we find — in excellent agreement with observation — the spectra of many electron systems, e.g. we find the spectrum of the He atom. A radiating He atom is a moving system and therefore we are in fact in a position to determine correctly the motion of the charge of the electrons inside a He atom.

It must be emphasized, however, that taking the superpositions of an arbitrarily large number of stationary solutions of the wave equation — we can find a combination which produces for a limited period an oscillation with an arbitrarily given frequency ν — this frequency need not be anywhere near to the optical frequencies of the He atom.

That as the result of the beat of the natural frequencies of He we can obtain a state in which an arbitrary frequency ν arises, is analogous to the

problem of the Fourier analysis. Using a large number of Fourier components we can obtain (inside a limited interval) a harmonic function with an arbitrary period ν which may differ from the periods of the terms of the series.

The state in which a He atom oscillates for a time with an unusual frequency ν contains obviously components of rather higher energy — and after a certain time the accidental phase relations between the natural frequencies which produce the beat frequency ν will disappear.

When we state that according to the theory a He atom generally emits well-defined combination frequencies only, then we make already a strong restriction on the wave functions ψ which we think likely to occur — we suppose that in the observed states the wave function of the discrete states are predominant.

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ГИДРОДИНАМИЧЕСКАЯ МОДЕЛЬ ВОЛНОВОЙ МЕХАНИКИ

ЗАДАЧА МНОГИХ ТЕЛ

Л. ЯНОШИ

Р е з ю м е

Дискутируется проблема расширения гидродинамической модели—описанной в предыдущих работах — к задаче многих тел. Показывается, что появляющиеся в приложении трудности не математического характера, они отображают физическую проблему, не решенную по нашему мнению до настоящего времени. Оказывается возможным, что в точном решении проблемы многих тел удается выбирать из возможных волновых функций те, которые очень подобны в статистическом смысле. Предполагаемый отбор напоминает отбор очень подобных конфигураций в статистической механике.