ON AN EQUIVALENCE THEOREM FOR INTEGRO-DIFFERENTIAL EQUATIONS OCCURRING IN FIELD THEORIES

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A direct proof is given for the equivalence of two types of integro-differential equations treated formerly by Królikowski and Rzewuski. The calculation shows a close connection between the equivalence theorem and the well-known initial value problem.

In the course of mathematical calculations in field theory we meet frequently integro-differential equations of the type

$$D\psi = \int K\psi \, dx \, ,$$

where D is some differential operator, K is the kernel of the euqation and ψ is the unknown function of any number of points of the four-dimensional world. The integral is taken (for any number of arguments) over a domain of space-time contained between two arbitrary space-like surfaces. This domain is most frequently the whole four-dimensional world. Equations of the above type occur e.g. in non-local field theories or in different approximation methods. Such equations are the Tamm-Dancoff or the Bethe-Salpeter equations.

It is of great importance for the better physical understanding of the problem (from the practical just as much as from the theoretical point of view) that this equation is equivalent to an equation of the type

$$D\psi = \int A\psi \, d\sigma$$

as it was proved [1]—[4] and applied [4]—[5] by Królikowski and Rzewuski in a series of papers. Here on the right side the integration is extended over one (or any number of) space-like surface, i.e. over a three-dimensional volume.

Here we give a direct proof for the equivalence which shows the connection between the equivalence theorem and the well-known initial value problem, thus it might serve for the better understanding of the former. In the course of the calculations, moreover we obtain the kernel A directly (naturally as a power series in the coupling constant only) without facing the necessity of determining it first from an extra integral equation as in [1]—[4].

For the sake of simplicity we confine ourselves to the treatment of the equation

$$(\gamma_\mu \partial_\mu + \varkappa)\, \psi(x) = g \int K(x, x')\, \psi(x')\, dx', \tag{1}$$

more complicated equations can be treated similarly. The integral equation corresponding to (1), satisfying the initial condition $\psi(x)$ equal a prescribed given $\psi(x)$ on an arbitrary space-like surface σ, is

$$\psi(x) = \psi_\sigma(x) + g \int N_\sigma(x, x')\, \psi(x')\, dx', \tag{2}$$

where

$$N_\sigma(x, x') = \{S_\sigma K\}(x, x') \tag{3}$$

with the denotation

$$\{AB..C\} = \int..\int A(x, \xi^1)\, B(\xi^1, \xi^2)..\, C(\xi^n, x')\, d\xi^1..\, d\xi^n.$$

S_σ is the Yang-Feldman's Green function

$$S_\sigma(x, x') = S^{(l)}(x - x') - \int_\sigma S(x - x'')\, \gamma_\nu\, S^{(l)}(x'' - x')\, d\sigma_\nu(x''), \tag{4}$$

where $S^{(l)}$ is any of the Green's function satisfying

$$(\gamma_\mu \partial_\mu + \varkappa)\, S^{(l)}(x) = \delta(x).$$

$$\psi_\sigma(x) = \int_\sigma S(x - x')\, \gamma_\nu\, \psi(x')\, d\sigma_\nu(x'), \tag{5}$$

thus

$$(\gamma_\mu \partial_\mu + \varkappa)\, \psi_\sigma(x) = 0,$$

the value of which is the prescribed $\psi(x)$ on σ. In this case the right side of (2) is indeed equal to the prescribed $\psi(x)$, if $x \in \sigma$, since from (4) $S_\sigma(x, x') = 0$ if $x \in \sigma$.

Supposing (2) has a unique solution (which naturally depends on the kernel K) which can be obtained by means of successive approximation, we have

$$\psi(x) = \psi_\sigma(x) + g \int R_\sigma(x, x')\, \psi_\sigma(x')\, dx', \tag{6}$$

where

$$R_\sigma(x, x') = \sum_{n=1}^{\infty} g^{n-1} \{N_\sigma^n\}(x, x'). \tag{7}$$

Substituting (5) into (6) we get

$$\psi(x) = \psi_\sigma(x) + g \int_\sigma L_\sigma(x, x') \gamma_\nu \psi(x') \, d\sigma_\nu(x') , \tag{8}$$

which is the solution of the initial value problem. Here

$$L_\sigma(x, x') = \{R_\sigma S\}(x, x') . \tag{9}$$

Applying the operator $(\gamma_\mu \partial_\mu + \varkappa)$ to (8) we obtain

$$(\gamma_\mu \partial_\mu + \varkappa) \psi(x) = g \int_\sigma A_\sigma(x, x') \gamma_\nu \psi(x') \, d\sigma_\nu(x') , \tag{10}$$

which is the equation we wanted to derive. In (10) using (3), (7) and (9)

$$\begin{aligned} A_\sigma(x, x') = (\gamma_\mu \partial_\mu + \varkappa) L_\sigma(x, x') = \sum_{n=1}^{\infty} g^{n-1} \{K N_\sigma^{n-1} S\}(x, x') = \\ = \{K S\}(x, x') + g \{K R_\sigma S\}(x, x'). \end{aligned} \tag{11}$$

The initial value problem of (10) can be solved by only one integration in contrast with eq. (1), because we obtained (10) from (8) by means of one differentiation. Indeed from (10) we get

$$\psi(x) = \psi_\sigma(x) + g \iint_\sigma S_\sigma(x, x') A_\sigma(x', x'') \gamma_\nu \psi(x'') \, d\sigma \ (x'') \, dx'. \tag{12}$$

Here on the right side only the arbitrary form of ψ on the surface σ occurs. It can be seen indeed from (9) and (11) that

$$L_\sigma(x, x') = \{S_\sigma A_\sigma\}(x, x') .$$

σ in the right side of (10) is arbitrary and the right side is independent of the special choice of σ. This can be seen explicitly, forming the functional derivative of (10). Using (10) and

$$A_\sigma(x, x') (\gamma_\mu \overleftarrow{\partial'_\mu} + \varkappa) = 0,$$

$$\frac{\delta A_\sigma(x, x')}{\delta\sigma(x'')} = - g A_\sigma(x, x'') A_\sigma(x'', x'),$$

which can be obtained from (11) the independence can be easily proved. (11) is the solution of the integral equation for A which can be found e.g. in [3] p. 199.

Because in (10) σ can be chosen arbitrarily it is advantageous to choose it as a plane going through the point x. In this case ψ contains in each side of the equation the same time arguments only. This makes possible the separation of the time variable, leaving thus only an eigenvalue problem with a non-local "potential". This is just the chief advantage of (10) in comparison to (1).

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О ТЕОРЕМЕ ЭКВИВАЛЕНТНОСТИ В СЛУЧАЕ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ, ИМЕЮЩИХ МЕСТО В ТЕОРИИ ПОЛЯ

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Резюме

Непосредственно доказывается эквивалентность интегро-дифференциальных уравнений двух видов, исследуемых ранее Круликовским и Жевуским. В вычислениях иллюстрируется связь теоремы эквивалентности с известной задачей начальных значений.