A PROOF OF THE COMPLETENESS OF THE SOLUTIONS OF THE SCHRÖDINGER EQUATION IN THE λ-PLANE

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It is proved that the system of solutions of the Schrödinger equation in the complex λ-plane is complete.

Of great mathematical interest is the problem of determining the potential $V(r)$ from a given distribution of Regge poles, that is to say the establishment of equations in the λ-plane of the same type as those of Gel'fand and Levitan [1] or Marchenko [2], which are valid in the k-plane.

The first step to be overcome is the establishment of a completeness relation for the solutions of the Schrödinger equation in the complex angular momentum plane, and this is the aim of this paper.

We use the same method as Jost and Kohn [3] have used to prove the completeness relation in the k-plane.

Let us consider the integral:

$$I(k, r) = -\int_{\Gamma} \lambda d\,\lambda \int_{0}^{\infty} \frac{h(r')}{rr'}\, G(\lambda, k, r, r')\, dr' , \qquad (1)$$

where $h(r')$ is an arbitrary function which does not make the integral diverge,

$G(\lambda, h, r, r')$ is the Green's function which allows us to express a solution $\psi(\lambda, \mathrm{k}, r)$ of the Schrödinger equation in terms of $\psi_0(\lambda, k, r)$, the solution of the free-Schrödinger equation (i.e. with zero potential) following the relation

$$\psi(\lambda, k, r) = \psi_0(\lambda, k, r) + \int_{0}^{\infty} G(\lambda, k, r, r')\; V(r')\; \psi_0(\lambda, k, r^2)\, dr' . \qquad (2)$$

It can be shown that $G(\lambda, k, r, r')$ can be expressed in terms of a physical solution φ which is defined in the origin by:

$$\lim_{r \to 0}\; \varphi(\lambda, k, r) \cdot r^{-\lambda - \frac{1}{2}} = 1 \qquad (3)$$

and by a JOST solution defined at infinity by:

$$\lim_{r\to\infty} f(\lambda, -k, r) \cdot e^{-ikr} = 1\,, \tag{4}$$

so that G can be written as

$$G(\lambda, k, r, r') = -\frac{\varphi(\lambda, k, r_{>'}) f(\lambda, -k, r_{>})}{f(\lambda, -k)}\,, \tag{5}$$

where $f(\lambda - k)$ is the JOST function defined by

$$f(\lambda, -k) = W[f(\lambda, -k, r), \varphi(\lambda, k, r)]. \tag{6}$$

The contour Γ is the imaginary axis of the λ-plane completed by a semicircle at infinity in the $\mathrm{Re}\lambda \geqslant 0$ half-plane. The potential is assumed to be regular, i.e. its first and second absolute moments exist, which implies (4) that the functions $\varphi(\lambda, k, r)$ and $f(\lambda, -k, k)$ are holomorphic in the half-plane $\mathrm{Re}\lambda \geqslant 0$, such that the only poles are the zeros a_i of the JOST function $f(\lambda - k)$.

Under these assumptions we can evaluate the integral by the residue method

$$I(k, r) = 2\pi i \sum_j \int_0^\infty \frac{h(r')}{rr'} \cdot \frac{\varphi(a_j, k, r_<) f(a_j, -k, r_>)}{\left.\frac{\partial f(\lambda, -k)}{\partial \lambda}\right|_{\lambda = dj}} dr'\,. \tag{7}$$

Taking into account

$$\left.\frac{\partial f(\lambda, -k)}{\partial \lambda}\right|_{\lambda = d_j} = -i\frac{aj}{k} f(a_j, k) M^2(a_j, k), \tag{8}$$

where

$$M^2(a_j, k) = \int_0^\infty \frac{f^2(a_j, -k, r)}{r^2} dr \tag{9}$$

and

$$\varphi(\lambda, k, r) = \frac{1}{2ik}[f(\lambda, k) f(\lambda, -k, r) - f(\lambda, -k) f(\lambda, k, r)] \tag{10}$$

which for $\lambda = a_i$ reduces to

$$\varphi(a_j, k, r) = \frac{1}{2ik} f(a_j, k) f(a_j, -k, r)\,. \tag{11}$$

We obtain

$$I(k, r) = i\pi \sum_j \int_0^\infty \frac{h(r')}{rr'} \cdot \frac{f(a_j, -k, r) f(a_j - k, r')}{M^2(a_j, k)} dr'\,. \tag{12}$$

On the other hand we can evaluate the integral along the contour, along the imaginary λ axis, taking into account that

$$f(\lambda, -k, r) = f(-\lambda, -k, r) . \tag{13}$$

We can show that

$$\int_{i\infty}^{-i\infty} \ldots = \frac{1}{2} \int_{i\infty}^{-i\infty} \lambda d\lambda \int_0^\infty \frac{h(r')}{rr'} dr' \Bigg\{ f(\lambda, -k, r)\, \Theta(r-r') \left[\frac{\varphi(\lambda, k, r')}{f(\lambda, -k)} - \frac{\varphi(-\lambda, k, r')}{f(-\lambda, -k)} \right] + f(\lambda, -k, r')\, \Theta(r'-r) \left[\frac{\varphi(\lambda, k, r)}{f(\lambda, -k)} - \frac{\varphi(-\lambda, k, r)}{f(-\lambda, -k)} \right] \Bigg\} . \tag{14}$$

If now we use the relation between the JOST and the φ solutions

$$f(\lambda, -k, r) = \frac{1}{2\lambda} \left[f(\lambda, -k)\, \varphi(-\lambda, k, r) - f(-\lambda, -k)\, \varphi(\lambda, k, r) \right] \tag{15}$$

we have

$$\int_{i\infty}^{-i\infty} \ldots = -2 \int_0^{i\infty} \lambda^2\, d\lambda \int_0^\infty \frac{h(r')}{rr'} \cdot \frac{f(\lambda, -k, r) f(\lambda, -k, r')}{f(\lambda, -k) f(-\lambda, -k)} dr' . \tag{16}$$

To evaluate the integral along the curved portion we need the asymptotic forms of the solutions of the SCHRÖDINGER equation and of the JOST function for $|\lambda| \to \infty$.

We know (5) that for regular potentials the functions $\varphi(\lambda, k, r)$ and $f(\lambda, -k)$ tend to the correspondent quantities of the free SCHRÖDINGER equation as $|\lambda| \to \infty$. Hence we have

$$\lim_{|\lambda| \to \infty} \varphi(\lambda, k, r) = r^{\lambda + \frac{1}{2}} ,$$

$$\lim_{|\lambda| \to \infty} \varphi(-\lambda, k, r) = r^{-\lambda + \frac{1}{2}} , \tag{17}$$

$$\lim_{|\lambda| \to \infty} f(\lambda, -k) = 2 \cdot e^{-i\frac{\pi}{4}} \sqrt{\lambda k} \left(\frac{2\lambda}{k} \right)^{\lambda} e^{-\lambda\left(1 - i\frac{\pi}{2}\right)}$$

valid for $\qquad |\arg \lambda| < \pi - \varepsilon .$

Taking into account (15) and (17) we deduce $\lim_{|\lambda|\to\infty} f(\lambda, -k, r)$, and hence we can write the asymptotic form of the integral as

$$\int_c \ldots = \frac{1}{2}\int_0^\infty \frac{h(r')}{\sqrt{rr'}}\, dr' \left[-i \int_c \left(\frac{ek\sqrt{rr'}}{2\lambda} \right)^{2\lambda} d\lambda - \right.$$
$$\left. - \int_c \left[\left(\frac{r'}{r} \right)^\lambda \Theta(r - r') + \left(\frac{r}{r'} \right)^\lambda \Theta(r' - r) \right] d\lambda \right]. \tag{18}$$

The first integral in the bracket can be neglected and we are left with

$$\int_c \ldots = i \int_0 \frac{h(r')}{\sqrt{rr'}}\, dr' \cdot \lim_{\eta\to\infty} \frac{\sin \eta R}{R} \quad \text{with} \quad R = \log \frac{r'}{r} \tag{19}$$
$$= i\pi h(r)$$

in virtue of the properties of the δ-function.

We have finally

$$I(k, r) = i\pi h(r) - 2\int_0^{i\infty} \lambda^2\, d\lambda \int_0^\infty \frac{h(r')}{rr'} \cdot \frac{f(\lambda, -k, r) f(\lambda, -k, r')}{f(\lambda, -k) f(-\lambda, -k)}\, dr' . \tag{20}$$

If we compare the result (20) with that obtained by the residue method (16), we see that

$$h(r) = \int_0^\infty h(r')\, dr' \left[\sum_d M^{-2}(a_j, k) \frac{f(a_j, -k, r)}{r} \cdot \frac{f(a_j, -k, r')}{r'} + \right.$$
$$\left. + \frac{2i}{\pi} \int_0^{i\infty} \frac{\lambda^2}{f(\lambda, -k) f(-\lambda, -k)} \cdot \frac{f(\lambda, -k, r)}{r} \cdot \frac{f(\lambda, -k, r')}{r'}\, dh' \right]. \tag{21}$$

This relation (21) can only be satisfied for all and every square integrable $h(r)$ if

$$\sum_d M^{-2}(a_j, k) \frac{f(a_j, -k, r)}{r} \cdot \frac{f(a_j, -k, r')}{r'} +$$
$$+ \frac{2i}{\pi} \int_0^{i\infty} \frac{\lambda^2\, d\lambda}{f(\lambda, -k) f(-\lambda, -k)} \cdot \frac{f(\lambda, -k, r)}{r} \cdot \frac{f(\lambda, -k, r')}{r'} = \delta(r - r') . \tag{22}$$

This relation (22) can be written as a STIELTJES integral if we define a spectral function $p_k'(\lambda)$ in the following sense

$$\frac{dp_k(\lambda)}{d\lambda} = \begin{cases} \dfrac{2i}{\pi} \dfrac{\lambda^2}{f(\lambda, -k) f(-\lambda, -k)} & \text{for } \lambda \varepsilon [0, i\infty] \\ \sum\limits_{d} \dfrac{\delta(\lambda - a_j)}{M^2(a_j, k)} & \text{for the zeros of } f(\lambda - k) \end{cases} \tag{23}$$

and we obtain

$$\int \frac{f(\lambda, -k, r)}{r} \cdot \frac{f(\lambda, -k, r')}{r'} \, dp_k(\lambda) = \delta(r - r') \,. \tag{24}$$

(22) or (24) prove the completeness of the set of the JOST solution of the SCHRÖDINGER equation modified by the factor r^{-1}.

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ДОКАЗАТЕЛЬСТВО ПОЛНОТЫ РЕШЕНИЙ УРАВНЕНИЯ ШРЕДИНГЕРА В λ — ПЛОСКОСТИ

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Резюме

Доказано, что система решений уравнения Шредингера является полной в комплексной λ — плоскости.