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# ON THE QUANTUM THEORY OF PARAMETRIC FREQUENCY CONVERTER*

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A quantum mechanical treatment is given of the acoustooptic parametric conversion processes in dielectric crystals when the signal acoustical and idle light waves are transformed into each other in the presence of intensive optical pumping. The approximate Heisenberg equations of motion are found and solved for the creation and annihilation operators of signal and idle modes with due regard to the interaction of these modes with other light and vibratory modes of the crystal ("the thermostat"). It is shown that the thermostat influence results in noise and attenuation effects. These persistent noises are also converted from one mode into another and vice versa. Threshold conditions and asymptotic levels of noise are discussed.

One of the most interesting fields of quantum optics is the investigation of parametric processes of generation, amplification and frequency conversion. In the present paper we shall consider the interaction of three boson modes in dielectric crystals supposing that their frequencies and wave vectors fulfil the phase matching conditions $\omega_1 = \omega_2 + \omega_3$ and $\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$. Periodic boundary conditions on the crystal surface are assumed, thus the normal coordinates correspond to running waves. Let mode 3 be intensively excited by an external source. In this case mode 1 will be parametrically converted to mode 2 and vice versa. Such processes play an important role in physical phenomena as interaction between light and ultrasound, coupling of laser and infrared radiation. In the last case the parametric converter can be used for the detection of infrared radiation (up-conversion).

In [1] a quantum mechanical model of parametric conversion was given. In this model the interaction of the signal and idle modes with the remaining electromagnetic and vibratory modes of the crystal was not taken into account (the set of these modes we shall call "thermostat" in the present paper), consequently this model did not deal with noise and attenuation. However, if the parametric frequency converter is used for information processing the attenuation of the signal and the signal-noise ratio are important characteristics of the device. The parametric converter is an example of coupled vibrations. Investigation of noise and damping in such systems is of both theoretical and practical interest.

* Dedicated to Prof. I. Tarján on his 70th birthday.

In this paper we shall consider the quantum theory of the parametric frequency converter taking into account the interaction of signal and idle modes with the thermostat in a microscopic way. Without pumping this interaction causes attenuation and noise, excluding the coherent signal, the average number of quanta in a mode is determined by the temperature of the crystal according to Planck's formula. As soon as the pumping starts the modes get coupled. As a result, both signals and noises are transferred from one mode into another. A quantitative description of noise transformation in the process of parametric frequency conversion is the main result of this paper. Another consequence of mode coupling is the change in the character of attenuation.

The interaction of the signal and idle modes with the modes of the thermostat depends on the type of the modes. To be more specific we shall consider acousto-optical processes in dielectric crystals when the pumping and idle modes are light waves and the signal is a quasi-transverse acoustical wave.

We shall take into account the interaction between these modes proceeding from the Hamiltonian $H=H_0+V_1+V_2$, written in the representation of secondary quantization where $H_0$ describes non-interacting light and vibratory modes, $V_1$ is the cubic anharmonicity term and $V_2$ corresponds to the photon-phonon interaction. We shall find solutions of the approximate equations of motion for time dependent creation and annihilation operators which allow us to study the role of attenuation and noise in the considered process and investigate the statistical properties of the transformed signals.

## 1. Equations of motion and their approximate solution

Let us consider the interaction between a quasitransverse acoustical and two electromagnetic modes (signal, idle and pumping waves, respectively) described by a coupling constant $p$ in $V_2$. The crystal temperature is assumed to be so low that the Landau—Rumer criterion is fulfilled for the ultrasonic wave.

For the intensive pumping light we shall use the parametric approximation [1—3], i.e. we shall neglect its damping and substitute $C$-numbers $\alpha_3^*$ and $\alpha_3$ in place of its creation and annihilation operators $a_3^+$ and $a_3$. Terms in $H$, which describe the interaction of the optical idle (1) and acoustical signal (2) modes with the thermostat can be written as

$$\lambda_i \hbar (a_i Q_i^+ + a_i^+ Q_i),$$

where $i=1,2$ and the operators $Q_1$, $Q_2$ are given:

$$\lambda_1 Q_1 = \hbar^{1/2} \sum_{\mathbf{l},i,j} \varkappa_{\mathbf{k},i\mathbf{l},j(\mathbf{k}_1-\mathbf{l})} a_{i\mathbf{l}} [b_{j(\mathbf{k}_2-\mathbf{l})} + b^+_{j(\mathbf{l}-\mathbf{k}_1)}], \tag{1}$$

$$\lambda_2 Q_2 = \hbar^{1/2} \sum_{\mathbf{k},i,j} U_{-\mathbf{k}_2,i\mathbf{k},j(\mathbf{k}_2-\mathbf{k})} [b_{i\mathbf{k}} + b^+_{j(-\mathbf{k})}] [b_{j(\mathbf{k}_2-\mathbf{k})} b^+_{j(\mathbf{k}-\mathbf{k}_2)}] . \tag{2}$$

$a_{il}$ and $b_{j\mathbf{k}}$ are the annihilation operators of the thermostat, photons and phonons, respectively, $\varkappa$ and $U$ are photoelectric and anharmonic coefficients, $\mathbf{l}$ and $\mathbf{k}$ are wave vectors. The small parameters $\lambda_1$ and $\lambda_2$ are introduced to take into account explicitly the weakness of anharmonicity and of the interaction between light and sound.

Let us assume that the pumping starts at $t=0$. For $t \geqq 0$ from the Hamiltonian $H$ we obtain the Heisenberg equations of motion for the creation operator of a photon of the idle mode $a_1^+(t)$ and that of a phonon of the signal mode $a_2^+(t)$

$$
\begin{aligned}
\frac{\mathrm{d}a_1^+(t)}{\mathrm{d}t} &= i\omega_1 a_1^+(t) + ig^* e^{i\omega_3 t} a_2^+(t) + i\lambda_1 Q_1^+(t)\,, \\
\frac{\mathrm{d}a_2(t)}{\mathrm{d}t} &= i\omega_2 a_2^+(t) + ig e^{-i\omega_3 t} a_1^+(t) + i\lambda_2 Q_2^+(t)\,.
\end{aligned} \tag{3}
$$

Here $g = -\hbar^{-1}\alpha^* p$ and $p = \hbar^{3/2}\varkappa_{\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_2}$. Eq. (3) can be supplemented with equations of motion for $Q_i(t)$.

Knowing the solution of Eq. (3) the normally ordered characteristic functions $\chi_i(t,\eta) = \mathrm{Tr}\,\{\rho e^{\eta a_i^+(t)} e^{-\eta^* a_i(t)}\}$ can be obtained [3] where $\rho$ is the density operator of the whole system in the Heisenberg representation. These characteristic functions make it possible to evaluate the mean value of any operator of $[a_i^+(t)]^m [a_i(t)]^n$ type and they contain information about the statistical properties of the converted signals.

Solving Eq. (3) exactly with respect to $g$ we shall use perturbation theory regarding the small parameters $\lambda_1$ and $\lambda_2$.

Let us set some initial conditions at $t=0$. For large $t$ we retain only those terms of the perturbative solution of Eq. (3), which correspond to secular terms in the expansion of $\chi_{1,2}(t,\eta)$. To be more detailed: we sum up those terms in the characteristic function which are of $(\lambda_1^2 t)^m\,(\lambda_2^2 t)^n$ type $(m, n = 1, 2, \ldots)$ and neglect those of $\lambda_i^2\,(\lambda_1^2 t)^{n-1}\,(\lambda_2^2 t)^m$ type. Taking into account the fact that the macroscopic system of the thermostat is only weakly perturbed by the modes under consideration and neglecting the intrinsic anharmonicity of the thermostat we obtain from (3):

$$
\begin{aligned}
\frac{\mathrm{d}a_1^+(t)}{\mathrm{d}t} &= i(\omega_1 + \Delta\omega_1 + i\Gamma_1)a_1^+(t) + ig^* e^{i\omega_3 t} a_2^+(t) + i\lambda_1 Q_1^{+(0)}(t)\,, \\
\frac{\mathrm{d}a_2^+(t)}{\mathrm{d}t} &= i(\omega_2 + \Delta\omega_2 + i\Gamma_2)a_2^+(t) + ig e^{-i\omega_3 t} a_1^+(t) + i\lambda_2 Q_2^{+(0)}(t)\,,
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\Gamma_2 = \frac{\hbar\pi}{2} \sum_{i,j,\mathbf{k}} |U_{-\mathbf{k}_2, j\mathbf{k}, i(\mathbf{k}_2-\mathbf{k})}|^2 \{[1 + \hat{n}_{j\mathbf{k}} + \hat{n}_{i(\mathbf{k}_2-\mathbf{k})}] \\
\delta(\omega_2 - \omega_{j\mathbf{k}} - \omega_{i(\mathbf{k}_2-\mathbf{k})}) + \\
+ 2[\hat{n}_{j\mathbf{k}} - \hat{n}_{i(\mathbf{k}_2-\mathbf{k})}]\delta(\omega_{i(\mathbf{k}_2-\mathbf{k})} - \omega_{j\mathbf{k}} - \omega_2)\}\,, \\
\hat{n} = b^+ b\,.
\end{aligned} \tag{5}
$$

The expression for $\Delta\omega_2$ can be obtained from (5) by substituting $P\left(\frac{1}{x}\right)$ for $\delta(x)$ where $P(x)$ symbolizes the Cauchy principal value. Analogous expressions can be given for $\Gamma_1$ and $\Delta\omega_1$. In Eq. (4) and (5) $\Gamma$, $\Delta\omega$ and $Q^{(0)}(t)$ are operators acting in the Fock space of states. Each state in this space is characterized by fixed numbers of photons and phonons of the thermostat modes. The above operators shall be evaluated neglecting the interaction of the thermostat with the signal and idle modes.

The solution of the system of operator equations (4) is found to be of the form

$$a_2^+(t)=a_2^+(0)C(t)+a_1^+(0)S(t)+f(t)\,, \tag{6}$$

where

$$C(t) = \exp\left[\left(i\omega_2+i\Delta\omega_2-\frac{\Gamma_1+\Gamma_2}{2}\right)t\right]\left(\cos rt+\frac{\Gamma_1-\Gamma_2}{2r}\sin rt\right), \tag{7}$$

$$S(t) = \frac{ig}{r}\exp\left[\left(i\omega_2+i\Delta\omega_2-\frac{\Gamma_1+\Gamma_2}{2}\right)t\right]\sin rt\,, \tag{8}$$

$$f(t) = i\int_0^t dt'\,[C(t-t')\lambda_2 Q_2^{+(0)}(t')+S(t-t')\lambda_1 Q_1^{+(0)}(t')e^{-i\omega_3 t'}]\,, \tag{9}$$

$$r = \left[|g|^2-\frac{(\Gamma_1-\Gamma_2)^2}{4}\right]^{1/2}.$$

The expression for the operator $a_1^+(t)$ can be found with subscripts 1 and 2 interchanged in (6—9).

By direct calculation one can check that in the sense of the above mentioned approximation the obtained solutions obey the commutation relations:

$$[a_i(t), a_j^+(t)]=\delta_{ij}\,,\qquad i,j=1,2\,. \tag{10}$$

If the attenuation is introduced into the equation of motion of $a_i^+(t)$ in a phenomenological way, the solutions of such a system of equations do not obey the commutation relation (10).

## 2. Statistical properties of the signals

Let us assume that at $t=0$ the density operator of the whole system is $\rho=\rho_1\rho_2\rho_T$, where $\rho_T$ is the density operator of the thermostat. It is also supposed that at $t=0$ the density operators of the idle and signal modes $\rho_1$ and $\rho_2$ are described in the $P$-representation of the coherent states [3] by weight functions $P_1^{(0)}(\beta)$ and $P_2^{(0)}(\alpha)$, respectively. The characteristic function $\chi_2(\eta, t)$ of the signal mode is given:

$$\chi_2(\eta, t)=\iint d^2\alpha\, d^2\beta\, P_2^{(0)}(\alpha)P_1^{(0)}(\beta)G(\eta, t;\alpha,\beta)\,, \tag{11}$$

where the kernfunction

$$G(\eta, t; \alpha, \beta) = tr_T\{\rho_{T2} < \alpha|_1 < \beta| e^{\eta a_2^\dagger(t)} e^{-\eta^* a_2(t)} |\beta>_1 |\alpha>_2\} \tag{12}$$

is the characteristic function of the signal mode if both the signal and idle modes were in pure coherent states $|\alpha>_2$ and $|\beta>_1$ at $t=0$. Using (6—8) and retaining only the secular terms one can evaluate $G(\eta, t; \alpha, \beta)$

$$G(\eta, t; \alpha, \beta) = \exp\{-|\eta|^2 \mu(t) +$$
$$+ \eta[\alpha^* c(t) + \beta^* s(t)] - \eta^*[\alpha c^*(t) + \beta s^*(t)]\}, \tag{13}$$

where $s(t)$, $c(t)$ can be obtained from $S(t)$ and $C(t)$ by the replacement of operators by $C$-numbers:

$$\Gamma_{1,2} \to \gamma_{1,2} = tr_T(\rho_T \Gamma_{1,2}); \quad \Delta\omega_{1,2} \to tr_T(\rho_T \Delta\omega_{1,2});$$

$$\mu(t) = 2\int_0^t dt' \{\gamma_2 n_2(T)|c(t')|^2 + \gamma_1 n_1(T)|s(t')|^2\}. \tag{14}$$

Here $n_{1,2}(T)$ are the occupation numbers of the idle and signal modes in thermodynamical equilibrium. Let us suppose that at $t=0$ each of the considered modes is in a state that corresponds to the superposition of coherent signal with equilibrium thermal noise:

$$P_{1,2}^{(0)}(\alpha) = \frac{1}{\pi n_{1,2}(T)} \exp\left\{-\frac{|\alpha - z_{1,2}|^2}{n_{1,2}(T)}\right\}. \tag{15}$$

Substituting (13) and (15) into (11) one finds the characteristic function of the signal mode

$$\chi_2(\eta, t) = \exp\{-|\eta|^2 N(t) + \eta R^*(t) - \eta^* R(t)\}, \tag{16}$$

and the corresponding weight function

$$P_2(\alpha, t) = \frac{1}{\pi N(t)} \exp\left\{-\frac{|\alpha - R(t)|^2}{N(t)}\right\}, \tag{17}$$

where

$$R(t) = z_2 c^*(t) + z_1 s^*(t) \tag{18}$$

and

$$N(t) = n_2(T)|c(t)|^2 + n_1(T)|s(t)|^2 + \mu(t). \tag{19}$$

One can see that Eqs (16, 17) describe a superposition of a coherent state having a parametrically transformed amplitude $R(t)$ with a Gaussian noise with variance $N(t)$.

## 3. Discussion

As it can be seen from Eqs (6—9) there are two different dynamic conditions of the parametric converter depending on the relation between $|g|$ and $|\gamma_1-\gamma_2|/2$. Let us assume that there is a pure coherent signal $z_1$ in mode 1 at $t=0$. If $|g|>|\gamma_1-\gamma_2|/2$ the signal will periodically transform from mode 1 to mode 2 and vice versa with a damping amplitude. In this case the damping constant is equal to $(\gamma_1+\gamma_2)/2$. If, on the contrary, $|g|<|\gamma_1-\gamma_2|/2$, there is also some conversion but the converted signal will attenuate without oscillation. In this case at $g\to 0$ Eqs (16—19) transform into a form describing non-interacting modes.

To understand the cause of the secondly mentioned dynamic duty of the parametric converter let us consider two oscillators with weak and strong dampings, respectively. Let us assume that at $t=0$ only the oscillator with weak damping is excited and a weak parametric coupling begins at that moment. The energy transfer from the first oscillator to the second one will be slower than the potential velocity of energy loss by the second oscillator. Therefore there will be only a one way energy conversion between these oscillators.

For $|g|>|\gamma_1-\gamma_2|/2$, using the characteristic function, let us evaluate the average number of quanta in mode 2 if at $t=0$ there were signals in both modes. From Eq. (17) we have:

$$\langle a_2^+(t)\, a_2(t)\rangle = |R(t)|^2 + N(t)\,, \tag{20}$$

The first term $|\mathrm{R(t)}|^2$ in Eq. (20) describes the kinetics of the signal intensity, the second term $N(t)$ shows the level of noise in mode 2.

The expressions $N(t)$ in Eq. (19) can be divided into two parts:

$$N_i(t) = N(t) - \mu(t) \quad \text{and} \quad \mu(t)\,. \tag{21}$$

The quantity

$$N_i(t) = n_2(T)\exp\left[-(\gamma_1+\gamma_2)t\right]\left(\cos rt + \frac{\gamma_1-\gamma_2}{2r}\sin rt\right)^2 +$$

$$+ n_1(T)\frac{|g|^2}{r^2}\exp\left[-(\gamma_1+\gamma_2)\mathrm{t}\right]\sin^2 rt \tag{22}$$

originates from the initial levels of noise of the signal and idle modes. It should be noticed that in this expression the temperature $T$ can be different from the crystal temperature (e.g. in case of ballistic phonons [4]). In this case if $z_{1,2}=0$ Eq. (19) describes parametric conversion of input signals with Gaussian statistics.

The term

$$\mu(t) = N_a + N_t(t)\,, \tag{23}$$

where

$$N_a = (|g|^2 + \gamma_1\gamma_2)^{-1}\left[n_2\gamma_1\gamma_2 + \frac{\gamma_1 n_1 + \gamma_2 n_2}{\gamma_1+\gamma_2}|g|^2\right] \tag{24}$$

and

$$N_t(t) = -e^{-(\gamma_1+\gamma_2)t}\left\{\frac{|g|^2(n_2\gamma_2+n_1\gamma_1)}{r^2(\gamma_1+\gamma_2)} + \right.$$

$$+\frac{\cos 2rt}{2(|g|^2+\gamma_1\gamma_2)}\left[\frac{|g|^2(\gamma_1+\gamma_2)(n_2\gamma_2-n_1\gamma_1)}{2r^2} + n_2\gamma_2(\gamma_1-\gamma_2)\left(1-\frac{\gamma_1^2-\gamma_2^2}{4r^2}\right)\right]+$$

$$\left.+\frac{\sin 2rt}{2(|g|^2+\gamma_1\gamma_2)}\left[\frac{|g|^2(n_1\gamma_1-n_2\gamma_2)}{r} + n_2\gamma_2(\gamma_1-\gamma_2)\left(1-\frac{\gamma_1-\gamma_2}{2r}\right)\right]\right\} \tag{25}$$

describe Gaussian noise in mode 2 connected with the persistent influence of the thermostat on modes 1 and 2 in the interval of time from 0 to $t$. The quantity $N_t(t)\to 0$ in the limit $t\to\infty$ and corresponds to rather complicated transient conditions of noise level. The term $N_a$ describes the asymptotic level of noise in mode 2 in the limit $t\to\infty$.

From Eq. (24) in the limit $g\to 0$, as it can be expected, we have $N_a\to n_2(T)$. On the contrary, if $|g|\gg\gamma_1\gamma_2$, then

$$N_a \approx \frac{\gamma_1 n_1(T)+\gamma_2 n_2(T)}{\gamma_1+\gamma_2}. \tag{26}$$

In the last case (i.e. $|g|\gg\gamma_1\gamma_2$) the number of quanta of noise in mode 1 has the same limit. This limit will be between the equilibrium numbers of quanta in modes 1 and 2; $n_1(T)$ and $n_2(T)$. In case of acoustooptical conversion the asymptotic level of noise in the optical idle mode will be higher than that which corresponds to Planck's formula and the asymptotic level of noise in the acoustical signal mode will be correspondingly lower than in thermal equilibrium.

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