*Acta Physica Academiae Scientiarum Hungaricae, Tomus 45 (2), pp. 149—151 (1978)*

## *COMMUNICATIO BREVIS*

# BOUND FOR THE HARMONIC OSCILLATOR GREENS FUNCTION

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(Received 5. IX. 1978)

We derive a bound for the harmonic oscillator resolvent kernel $R(x, x', z)$ defined as

$$R(x, x'; z) = \sum_{n=0}^{\infty} \frac{u_n(x)\, u_n(x')}{n - z}, \quad z \neq 0, 1, 2, \ldots, \tag{1}$$

where $u_n(x)$ are standard harmonic oscillator eigenfunctions.

*Case I.* $\mathrm{Re}\, z < 0$

This is simple. We use the well-known integral representation [1]

$$R(x, x'; z) = \pi^{-1/2} \int_0^1 d\tau\, \tau^{-(z+1)} (1 - \tau^2)^{-1/2} L(x, x', \tau), \tag{2}$$

$$\mathrm{Re}\, z < 0,$$

where

$$L(x, x'; \tau) = \exp\left(-\frac{1}{2} \frac{1+\tau^2}{1-\tau^2} (x^2 + x'^2) + \frac{2\tau}{1-\tau^2} x x'\right). \tag{3}$$

Since $|L(x, x;, \tau)| \leq 1$ for $\mathrm{Im}\, \tau = 0$, $0 \leq \tau \leq 1$, $x$ and $x'$ real one has that

$$|R(x, x'; z)| \leq \pi^{-1/2} \int_0^1 d\tau\, \tau^{-(\mathrm{Re}\, z+1)} (1 - \tau^2)^{-1/2}, \tag{4}$$

giving the bound

$$|R(x, x; z)| \leq \frac{1}{2} \pi^{-1/2} B\left(-\frac{1}{2} \mathrm{Re}\, z, 1/2\right), \quad \mathrm{Re}\, z < 0,$$

where

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\, \Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

is the Riemann beta function.

*Case II.* $\operatorname{Re} z \geq 0,\ z \neq 0, 1, 2, \ldots$

First of all we need an integral representation of $R(x, x'; z)$ which is valid even if $\operatorname{Re} z \geq 0$. This can be obtained by distorting the path of integration in the right-hand side of Eq. (2). In fact one has that

$$R(x, x'; z) = \frac{1}{2} \frac{\pi^{-1/2}}{\sinh(i\pi z)} \int_C d\tau (-\tau)^{-(z+1)} (1 - \tau^2)^{-1/2} L(x, x'; \tau) ,$$
$$z \neq 0, 1, 2, \ldots, \quad |\arg(-\tau)| \leq \pi,$$

where the integration path $C$ in the complex $\tau$-plane can be taken to consist of a line from $\tau = 1$ to $\tau = \varrho$ ($\operatorname{Im} \varrho = 0,\ 0 < \varrho < 1$) just above the cut drawn between 0 and 1, an anti-clockwise circle of radius $\varrho$ around the origin and finally to close to contour a straight line just below the cut from $\tau = 1$ to $\tau = \varrho$.

This representation is useful only for obtaining a bound if $\varrho$ is such that $L(x, x'; \tau)$ is uniformly bounded in all its variables as long as $\tau \in C$. A straightforward computation shows that in fact

$$|L(x, x'; \tau)| \leq 1,$$

if

$$0 < \varrho < \sqrt{2} - 1$$

for all $\tau \in C$ and $x, x'$ real. From this it follows that

$$|R(x, x'; \tau)| \leq \pi^{-1/2} (1 - \varrho^2)^{-1/2} \varrho^{-\operatorname{Re} z}$$
$$\left(\pi \frac{e^{\pi |Imz|}}{|\sinh(i\pi z)|} + 2(\varrho^{-2} - 1)\right)$$

for $\operatorname{Re} z \geq 0$ and any $0 < \varrho \leq \sqrt{2} - 1$. Taking $\varrho = \sqrt{2} - 1$ we obtain our final result that

$$|R(x, x'; z)| \leq \left(2(\sqrt{2} - 1)\right)^{-1/2} \pi^{-1/2} e^{-\ln(\sqrt{2}-1) \operatorname{Re} z}$$
$$\left(\pi \frac{e^{\pi |Imz|}}{|\sinh i\pi z|} + 4 \frac{\sqrt{2} - 1}{3 - 2\sqrt{2}}\right)$$

valid for $\operatorname{Re} z \geq 0,\ z \neq 0, 1, 2, \ldots$

*Remark*

We have derived a bound for the harmonic oscillator Greens function. Such a bound can be used for estimating the rate of convergence of the Fredholm solution for a perturbed harmonic oscillator resolvent kernel. (The Fredholm solution converges provided that the perturbating potential is absolutely integrable.) In fact a simple application of Hadamard's theorem [2] for determinants gives the following estimate for the $n$th order term $I_n$ in the series expansion of the Fredholm denominator

$$| I_n | \leq g^n \frac{n^{1/2\,n}}{n!} A^n \left( \int_{-\infty}^{\infty} dx \, | V(x) | \right)^n ,$$

where $g$ is the coupling constant, $V$ is the perturbing potential and $A$ is a bound for the unperturbed harmonic oscillator Greens function.

## REFERENCES

1. See e.g. R. P. FEYNMAN and A. R. HIBBS, Quantum Mechanics and Path Integrals, McGraw-Hill. Co., New York, 1965 or Bateman Manuscript Project, McGraw-Hill Co., New York, 1953.
2. See e.g. F. G. TRICOMI, Integral Equations, Interscience Publishers Inc. New York, 1965.