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APPLICATIONS OF THE STOCHASTIC QUANTIZATION METHOD

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We enumerate the advantages of using the stochastic quantization method over the standard methods and as an example use it to quantize para-Fermi fields obeying trilinear quantum conditions. Finally chiral anomaly for spinor fields is obtained directly from the c-number Langevin equation which forms the basis of the stochastic approach for field quantization.

1. Introduction

Ever since the introduction of the stochastic quantization method by Parisi and Wu [1] in 1981, this method has been applied to a wide variety of physical systems [2] which leaves no doubt about its viability as an alternative to the standard approaches to quantum theory, namely the canonical and the path-integral quantization. This new method of field quantization is based on the equation of motion rather than the Hamiltonian and as such bypasses many of the difficulties associated with the other approaches. For example this method enables us to quantize Abelian gauge fields without gauge-fixing terms [3]. For the nonabelian gauge fields the stochastic quantization method produces the effect of Faddeev–Popov ghosts in a natural way [4]. In addition to the above remarkable features the stochastic method being based on stochastic differential equation(s) (Langevin equation(s)) facilitates the numerical simulation of correlation functions [5] similar to the Monte-Carlo method in the path-integral approach. Other than these the usefulness of the stochastic quantization method has been amply demonstrated in quantizing scalar fields and has also been extended to include fermion fields by Sakita [6]. In the framework of stochastic quantization of fermions, calculation of the chiral anomaly turns out to be much simpler as has been shown by many authors [7]. The quantization of para-Fermi fields which obey trilinear quantum conditions [7] becomes extremely involved in the standard canonical quantization. We show here that para-Fermi field quantization obtains a much simpler procedure within the context of the stochastic quantization method. Finally, to show the further usefulness of the c-number Langevin equation which forms the basis of the stochastic quantization scheme we obtain the chiral anomaly in an obvious way. In the next Section we demonstrate the use of the stochastic quantization scheme to calculate the Green function for para-Fermi fields and in the final Section we obtain the chiral anomaly based on the Langevin equations for spinor fields.

2. Stochastic quantization of para-Fermi field

The specific formalism suitable for the calculation of the two-point Green function for para-Fermi field is described below. This is then utilized to calculate the same when the order of statistics p is different from 2.

In the Euclidean field theory the propagator of a para-Fermi field [9] is given by the well-known path integral formula

$$G(x-y) = \frac{\int d\psi d\overline{\psi}\psi(x)\overline{\psi}(y)e^{-S[\psi,\overline{\psi}]}}{\int d\psi d\overline{\psi}e^{-S[\psi,\overline{\psi}]}}. \tag{2.1}$$

Here ψ and $\overline{\psi}$ are paragrassmannian variables, and x, y are 4-dimensional Euclidean co-ordinates, and $S[\psi\overline{\psi}]$ denotes the Euclidean action in bilinear form. The stochastic quantization method provides us with a simple procedure to evaluate G which is based on the Langevin equations:

$$\begin{aligned}\partial_t\psi(x,t) &= - G^{+}\frac{\delta S}{\delta\overline{\psi}(x,t)} + G^{+}\eta(x,t),\\ \partial_t\overline{\psi}(x,t) &= G^{+^{T}}\frac{\delta S}{\delta\psi(x,t)} + \overline{\eta}(x,t).\end{aligned} \tag{2.2}$$

Here $\psi(x,t)$ and $\overline{\psi}(x,t)$ are independent para-Grassmann fields satisfying:

$$\begin{aligned}[\psi(x,t),[\psi(x',t'),\psi(x'',t'')]] &= 0,\\ [\overline{\psi}(x,t),[\psi(x',t'),\psi(x'',t'')]] &= 0,\\ [\psi(x,t),[\overline{\psi}(x',t'),\psi(x'',t'')]] &= 0,\\ [\overline{\psi}(x,t),[\overline{\psi}(x',t'),\overline{\psi}(x'',t'')]] &= 0\end{aligned} \tag{2.3}$$

and G stands for a suitably chosen operator which may be chosen unity or a Dirac operator denoted by K which enters the expression for S:

$$S = \int d^4x\frac{1}{2}[\overline{\psi}(x),K\psi(x)]. \tag{2.4}$$

When the order of parastatistics p obeyed by the para-Fermi field is greater than 2, (2.4) gives the most general form for S [9]. We shall comment on the $p=2$ case at the end of this Section. The statistical properties of independent para-Grassmann white-noise sources η and $\overline{\eta}$ are summarized in (2.14) and (2.15).

Since (2.2) provides solutions of ψ and $\overline{\psi}$ as a function of η and $\overline{\eta}$, the two-point Green function is obtained by the following $\eta\overline{\eta}$-averaging procedure

$$G(x,t;x',t') = \langle\psi(x,t)\overline{\psi}(x',t')\rangle_{\eta\overline{\eta}}. \tag{2.5}$$

Here $\langle f \rangle_{\eta\bar{\eta}}$ means the average over η and $\bar{\eta}$ given by

$$\langle f(\eta\bar{\eta})\rangle_{\eta\bar{\eta}} = \frac{\int d\eta d\bar{\eta} f(\eta\bar{\eta}) \exp\left\{(-\frac{1}{2})\int d^4x dt[\eta(x,t)\bar{\eta}(x,t)]\right\}}{\int d\eta d\bar{\eta} \exp\left\{(-\frac{1}{2})\int d^4x dt[\eta(x,t)\bar{\eta}(x,t)]\right\}}. \tag{2.6}$$

Using the fact that the equivalent Fokker–Planck equation corresponding to (2.2) leads us to stationary distribution $\exp(-S[\psi,\bar{\psi}])$ in the steady state limit $t \to \infty$ we find

$$G(x,y) = \operatorname*{Lt}_{t\to\infty} G(x,t;y,t). \tag{2.7}$$

When the action for the parafield is taken to be of the form given in (2.4), it is very convenient to decompose ψ and $\bar{\psi}$ in terms of Green components

$$\psi(x) = \sum_{a=1}^{p} \psi^a(x),$$
$$\bar{\psi}(x) = \sum_{a=1}^{p} \bar{\psi}^a(x), \tag{2.8}$$

where the Green components satisfy normal anti-commutation relations for equal Green indices

$$[\psi^a(x),\psi^a(x')]_+ = [\psi^a(x),\bar{\psi}^a(x')]_+ = [\bar{\psi}^a(x),\bar{\psi}^a(x')]_+ = 0 \tag{2.9}$$

and anomalous commutation relations for unequal Green indices

$$[\psi^a(x),\psi^b(x')]_- = [\psi^a(x),\bar{\psi}^b(x')]_- = [\bar{\psi}^a(x),\bar{\psi}^b(x')] = 0. \tag{2.10}$$

In terms of Green components the action (2.4) becomes

$$S = \sum_a \int dx \bar{\psi}^a(x) K \psi^a(x). \tag{2.11}$$

The appropriate Hamiltonian for the Fokker–Planck equation

$$H_{FP} = \sum_a \int dx \frac{\delta}{\delta\psi^a(x)} G^+ \left[\frac{\delta}{\delta\bar{\psi}^a(x)} + \frac{\delta S}{\delta\bar{\psi}^a(x)}\right] - \\ - \frac{\delta}{\delta\bar{\psi}^a(x)} G^{+^T} \left[\frac{\delta}{\delta\psi^a(x)} + \frac{\delta S}{\delta\psi^a(x)}\right] \tag{2.12}$$

corresponds to the following Langevin equations

$$\partial_t \psi^a(x,t) = -G^+ \frac{\delta S}{\delta\bar{\psi}^a(x,t)} + G^+ \eta^a(x,t), \tag{2.13a}$$

$$\partial_t \bar{\psi}^a(x,t) = G^{+^T} \frac{\delta S}{\delta\psi^a(x,t)} + \bar{\eta}^a(x,t). \tag{2.13b}$$

For a straightforward proof of this assertion we refer the reader to our earlier work [10]. In (2.13a) and (2.13b) the noise sources η^a and $\overline{\eta}^a$ which act as sources for ψ^a and $\overline{\psi}^a$ satisfy the stochastic properties given by

$$\begin{aligned}\langle\eta^a(x,t)\rangle &= \langle\overline{\eta}^a(x,t)\rangle = 0,\\ \langle\eta^a(x,t)\overline{\eta}^a(x',t')\rangle &= -\langle\overline{\eta}^a(x',t')\eta^a(x,t)\rangle = 2\delta(x-x')\delta(t-t'),\\ \langle\eta^a(x,t)\eta^a(x',t')\rangle &= -\langle\eta^a(x',t')\eta^a(x,t)\rangle = 0, \quad \text{etc.}\end{aligned} \tag{2.14}$$

for equal Green indices, and

$$\begin{aligned}\langle\eta^a(x,t)\overline{\eta}^b(x',t')\rangle &= \langle\overline{\eta}^b(x',t'(\eta^a(x,t)\rangle = 0,\\ \langle\eta^a(x,t)\eta^b(x',t')\rangle &= \langle\eta^b(x',t')\eta^a(x,t)\rangle = 0, \quad \text{etc.}\end{aligned} \tag{2.15}$$

for unequal Green indices.

In order to obtain the stochastic average of $\psi(x,t)\overline{\psi}(y,t')$ we write this in terms of Green components, i. e.

$$\langle\psi(x,t)\overline{\psi}(y,t')\rangle_{\eta\overline{\eta}} = \sum_{\alpha,\beta=1}^{p} \langle\psi^\alpha(x,t)\overline{\psi}^\beta(y,t')\rangle_{\eta\overline{\eta}}$$

and evaluate this average using the solutions of ψ^a and $\overline{\psi}^a$ from (2.13a) and (2.13b)

$$\langle\psi(x,t)\overline{\psi}(y,t)\rangle_{\eta\overline{\eta}} = pK^{-1}\left(1+O(e^{-2mt})\right) \tag{2.16}$$

whose steady state limit yields the free para-Fermi field propagator for $p>2$ which is equal to p times the well-known propagator for Fermi field.

The action for the $p=2$ case which has been discussed in [9] may also be treated along similar lines. It is better, however, in this case to make a Klein transformation which reduces the action to the direct sum of two Fermionic actions with different masses m_+ and m_-. The resulting parafield propagator is the sum of two fermionic propagators corresponding to masses m_+ and m_-. Here $m_\pm = m\pm\kappa$ where m and κ are coefficients in the mass matrix which for $p=2$ case is given by $\frac{1}{2}m[\overline{\psi},\psi]_- + \frac{\kappa}{2}[\overline{\psi},\psi]_+$. Note that in the limit $\kappa\to 0$ the expression for parafield propagator reduces to twice the fermionic propagator.

3. Chiral anomaly based on Langevin equation

In the previous Section we pointed out the computational ease with which the correlation function for an Euclidean quantum field theory may be calculated by $\eta\overline{\eta}$-averaging. An even more interesting application of the stochastic differential equations (2.2) is the calculation of chiral anomaly in a direct manner which does not even require a solution of the equations of the stochastic formalism. In this way we rederive the results of our earlier work [11] in a much simpler way. Using the spinor equations of motion we arrive at the following identity

$$
\begin{aligned}
&\partial_\mu\left(\sum_a \overline{\psi}^a\gamma_5\gamma_\mu\psi^a\right) + 2m\sum_a \overline{\psi}^a\gamma_5\psi^a = \\
&= \sum_a \overline{\psi}^a\gamma_5(\gamma_\mu\overrightarrow{\partial_\mu} + m)\psi^a + \sum_a \overline{\psi}^a(-\gamma_\mu\overleftarrow{\partial_\mu} + m)\gamma_5\psi^a = \\
&= \sum_a \overline{\psi}^a\gamma_5(\gamma_\mu\overrightarrow{\partial_\mu} - ig\gamma_\mu A_\mu + m)\psi^a + \sum_a \overline{\psi}^a(-\gamma_\mu\overleftarrow{\partial_\mu} - ig\gamma_\mu A_\mu + m)\gamma_5\psi^a = \\
&= -\sum_a \overline{\psi}^a\gamma_5\partial_t\psi^a - \sum_a \partial_t\overline{\psi}^a\gamma_5\psi^a + \sum_a \overline{\psi}^a\gamma_5\eta^a + \sum_a \overline{\eta}^a\gamma_5\psi^a. \qquad (3.1)
\end{aligned}
$$

The spinor equations of motion used here are

$$
\partial_t\psi^a = -\left[\gamma_\mu(\overrightarrow{\partial_\mu} - igA_\mu) + m\right]\psi^a + \eta^a, \qquad (3.2a)
$$

$$
\partial_t\overline{\psi}^a = -\overline{\psi}^a\left[\gamma_\mu(-\overleftarrow{\partial_\mu} - igA_\mu) + m\right] + \overline{\eta}^a, \qquad (3.2b)
$$

which follow from (2.13) on setting $G = 1$ for simplicity's sake.

Thus we have finally on taking the $\eta\overline{\eta}$ average

$$
\begin{aligned}
\partial_\mu\langle\left(\sum_a \overline{\psi}^a\gamma_5\gamma_\mu\psi^a\right)\rangle + 2m\langle\sum_a \overline{\psi}^a\gamma_5\psi^a\rangle = -\,\partial_t\langle\sum_a \overline{\psi}^a\gamma_5\psi^a\rangle - \\
-\,2pTr(\delta^4(0)\gamma_5), \qquad (3.3)
\end{aligned}
$$

where the last term is obtained through Novikov's theorem. The right-hand side in (3.3) consists of the sum of two terms: the first term is t-dependent and goes to sero exponentially as t increases (see below) and the second term is clearly t-independent. In the limit $t \to \infty$ only the last term survives and the above identity reduces to the anomalous Ward identity of the equivalent quantum field theory. This is our procedure for deriving chiral anomaly from the equations of motion. In the more general case when G is taken to be different from unity the following identity is still true:

$$
\begin{aligned}
\partial_\mu\langle\sum_a \overline{\psi}^a\gamma_5\gamma_m u\psi^a\rangle + 2m\langle\sum_a \overline{\psi}^a\gamma_5\gamma_\mu\psi^a\rangle + 2m\langle\sum_a \overline{\psi}^a\gamma_5\psi^a\rangle = -\,\partial_t\langle F(\overline{\psi},\psi)\rangle - \\
-\,2pTr(\delta^4(0)\gamma_5), \\
(3.4)
\end{aligned}
$$

where $F(\psi\overline{\psi})$ is a bilinear in $\overline{\psi}(x,t)$ and $\psi(x,t)$ whose form depends on the choice of G. Now

$$
\langle F(\overline{\psi},\psi)\rangle = \int d\psi d\overline{\psi} F(\overline{\psi},\psi)P(\overline{\psi},\psi,t), \qquad (3.5)
$$

where $P(\overline{\psi}, \psi, t)$ is the Fokker–Planck distribution function which satisfies the Schrödinger type equation

$$\partial_t P(\overline{\psi}, \psi, t) = -H_{FP} P(\overline{\psi}, \psi, t). \tag{3.6}$$

Thus we have

$$\partial_t \langle F(\overline{\psi}, \psi)\rangle = -\int d\psi d\overline{\psi} F(\overline{\psi}, \psi) H_{FP} P(\overline{\psi}, \psi, t). \tag{3.7}$$

In the steady-state limit $t \to \infty$ the right-hand side of (3.7) approaches zero. Hence the left-hand side of (3.7) relaxes to zero and the anomalous conservation law follows directly. The procedure we have outlined is quite similar in spirit to the derivation of chiral anomaly given by Namiki et al [12]. In contrast to our approach based on the general results given by Novikov's theorem, the latter approach is based on the elegant Ito calculus. However, the two approaches are equivalent.

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