Acta Physica Hungarica 63(1–2), pp. 71–87 (1988)

COULOMB PAIR CREATION II

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(Received in revised form 4 August 1987)

In a previous paper [1] we investigated the general theoretical background of electron-positron pair creation in strong external electric fields. In this paper we apply the general formalism to calculate the positron spectrum for two types of time dependent separable potential: Lorentzian time dependence and potential jump.

1. A model with separable potential

A nonlocal potential of the form

$$V = \sum_{\ell=1}^{n} \lambda_\ell \,|\beta_\ell\rangle\langle\beta_\ell|$$

is called an n-term separable potential. These types of potentials are often employed, e.g. in scattering theory because their use permits one to replace the integral equation for the scattering amplitude by a system of algebraic equations. Moreover, from the point of view of the amplitudes local potentials can be well approximated by sums of separable potentials. In what follows we do not claim realistic calculations and confine ourselves to a single-term separable potential

$$V(t) = \lambda(t)\, V = \lambda(t)|\beta\rangle\langle\beta| = (\lambda + \Delta\lambda(t))|\beta\rangle\langle\beta| = \lambda V + \Delta V \quad , \tag{1.1}$$

where $|\beta\rangle$ is an appropriately chosen normalized state in the Hilbert-space of the single-particle Dirac – Hamiltonian. In order to incorporate (1.1) into our earlier formulas they have to be rewritten for nonlocal potentials.

We have

$$M(x,x') = \langle \underline{x}|\beta\rangle\langle\beta|M(t,t')|\beta\rangle\langle\beta|\underline{x}'\rangle \quad , \tag{1.2}$$

where

$$\langle\beta|M(t,t')|\beta\rangle = \Delta\lambda(t)\,\delta(t-t') + \Delta\lambda(t)\langle\beta|K(t,t')|\beta\rangle\Delta\lambda(t')\langle\beta|K(t,t')|\beta\rangle =$$

$$= \int d^3x\, d^3x' \langle\beta|\underline{x}\rangle K(x,x')\langle\underline{x}'|\beta\rangle \;.$$

Utilizing these formulas in (4.10) – (4.11) of [1] we obtain the following integral equations for $\langle\beta|M(t,t')|\beta\rangle$:

$$\langle\beta|M(t,t')|\beta\rangle = \Delta\lambda(t)\,\delta(t-t') + \Delta\lambda(t)\int_{-\infty}^{+\infty} dt''\langle\beta|K_\lambda(t,t'')|\beta\rangle\langle\beta|M(t'',t')|\beta\rangle, \tag{1.3}$$

$$\langle\beta| M(t,t')|\beta\rangle = \Delta\lambda(t)\ \delta(t-t') + \int_{-\infty}^{+\infty} dt'' \langle\beta| M(t,t'')|\beta\rangle \langle\beta| K_\lambda\ (t'',t')|\beta\rangle \Delta\lambda(t') \quad . \qquad (1.4)$$

Let us substitute (1.2) into (4.9) of [1] and use (2.16) and (2.18) of [1] for the eigenfunctions χ:

$$A_{j\ell} = -2\pi i \langle \chi_j^{(+)out} | \beta \rangle \langle \beta | \chi_\ell^{(-)out} \rangle \tilde{M}(E_j^{(+)}, E_e^{(-)}) \quad ,$$

$$(B^+)_{j\ell} = -2\pi i \langle \chi_j^{(-)in} | \beta \rangle \langle \beta | \chi_e^{(+)in} \rangle \tilde{M}(E_j^{(-)}, E^{(+)}) \quad , \qquad (1.5)$$

$$(W_1^{+-1})_{j\ell} = \langle \chi_j^{(+)out} | \chi^{(+)in} \rangle - 2\pi i \langle \chi_j^{(+)out} | \beta \rangle \langle \beta | \chi_\ell^{(+)in} \rangle \tilde{M}(E_j^{(+)}, E^{(-)}) \quad ,$$

$$(W_4^{-1})_{j\ell} = \langle \chi_j^{(-)in} | \chi_\ell^{(-)out} \rangle + 2\pi i \langle \chi_j^{(-)in} | \beta \rangle \langle \beta | \chi_\ell^{(-)out} \rangle \tilde{M}(E_j^{(-)}, E^{(-)}) \quad ,$$

where the Fourier-transform of $\langle\beta| M(t,t')|\beta\rangle$ is defined as

$$\tilde{M}(E,E') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt\ dt'\ e^{iEt} \langle\beta| M(t,t')|\beta\rangle e^{iE't'} . \qquad (1.6)$$

Performing in (1.3), (1.4) Fourier-transformation, we obtain

$$\tilde{M}(E,E') = \Delta\tilde{\lambda}(E-E') + \int_{-\infty}^{+\infty} dE'' \Delta\tilde{\lambda}(E-E'')\ \tilde{F}_\lambda\ (E'')\ \tilde{M}(E'',E') \quad , \qquad (1.7)$$

$$\tilde{M}(E,E') = \Delta\tilde{\lambda}(E-E') + \int_{-\infty}^{+\infty} dE''\ \tilde{M}(E,E'')\ \tilde{F}_\lambda\ (E'')\ \Delta\tilde{\lambda}(E''-E') \quad , \qquad (1.8)$$

where

$$\Delta\tilde{\lambda}(E) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt\ \Delta\lambda\ (t)\ e^{iEt} ,$$

$$\tilde{F}_\lambda\ (E) = \int_{-\infty}^{+\infty} dt \langle\beta| K_\lambda\ (t)|\beta\rangle e^{iEt} \quad , \qquad (1.9)$$

since owing to the time-independence of $\lambda|\beta\rangle\langle\beta|$, $K_\lambda\ (t,t') = K_\lambda\ (t-t')$.

We see, that in the case of time dependent potential models with separable potentials are not completely solvable — the scalar integral equations (1.7), (1.8) remain to be solved.

The potential V will be assumed spherically symmetric. Then in any partial wave the Dirac equation can be reduced in a well-known manner [2] to a two-component equation for the two-component spinor $\binom{u}{v}$, in terms of which the solution of the Dirac equation ψ has the form

$$\psi_{j\ell m}(r) = \begin{pmatrix} \frac{u(r)}{r} & (i^\ell\ Y_\ell \chi)_{jm} \\ \frac{v(r)}{r} & (i^{\ell\pm1}\ Y_{\ell\pm1}\chi)_{jm} \end{pmatrix} , \qquad j = \ell \pm 1/2 \qquad (1.10)$$

being a Pauli-spinor and

$$(Y_\ell\chi)_{jm} = \sum_{\lambda\mu} \langle \ell\lambda 1/2\mu | jm \rangle Y_{\ell\lambda} \chi_\mu \ ; \qquad Y_{\ell\lambda} = Y_{\ell\lambda}\left(\frac{r}{r}\right).$$

In working with separable potentials it became customary to employ momentum representation in which (1.10) takes the form

$$\psi_{j\ell m}(p) = \begin{pmatrix} \frac{u(p)}{p} (Y_\ell \chi)_{jm} \\ \frac{v(p)}{p} (Y_{\ell \pm 1} \chi)_{jm} \end{pmatrix} \quad ; \qquad j = \ell \pm 1/2 \quad ,$$

where now $Y_{\ell\mu} = Y_{\ell\mu}\left(\frac{\underline{p}}{p}\right)$.

The Dirac-equation (2.5 – (2.6) of [1] in a given partial wave and asymptotic indices suppressed yields the two component form

$$\int_0^\infty dp' \langle p | \emptyset | p' \rangle \psi(p',t) = 0 \quad , \tag{1.11}$$

where

$$\psi(p) = \begin{pmatrix} u(p) \\ v(p) \end{pmatrix}$$

and

$$\langle p | \emptyset | p' \rangle = \langle p | \emptyset_0 | p' \rangle - \gamma_0 \langle p | V(t) | p' \rangle \quad ,$$

$$\langle p | \emptyset_0 | p' \rangle = \delta(p-p') \begin{pmatrix} i\frac{\partial}{\partial t} - m & p \\ -p & -i\frac{\partial}{\partial t} - m \end{pmatrix} \quad ; \quad \gamma_0 = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The state vector $|\beta\rangle$ in the separable potential (1.1) in the momentum representation has the components g(p), h(p), i.e.

$$\langle p | V(t) | p' \rangle = \lambda(t) \langle p | \beta \rangle \langle \beta | p' \rangle = \lambda(t) \begin{pmatrix} g(p) \\ h(p) \end{pmatrix} (g(p'), h(p')) \quad . \tag{1.12}$$

In order to decide whether a given potential produces supercritical transitions or not we must first study the adiabatic states η, i.e. solutions of (1.11) with time independent coupling λ, which belongs to the range of $\lambda(t)$. The eigenvalue equation is

$$\int dp' \langle p | H_\lambda | p' \rangle \chi(p') = E \chi(p) \quad , \tag{1.13}$$

where

$$\langle p | H_\lambda | p' \rangle = \begin{pmatrix} m & -p \\ -p & -m \end{pmatrix} \delta(p-p') + \lambda \langle p | \beta \rangle \langle \beta | p' \rangle \quad ; \quad \eta(p,t) = \chi(p)\, e^{iEt}$$

It can be formally written in the form

$$(H_0 - E) | \chi \rangle = -\lambda | \beta \rangle \langle \beta | \chi \rangle \quad , \tag{1.14}$$

where

$$\langle \beta | \chi \rangle = \int_0^\infty dp\, (gu + hv) = N \quad .$$

Let us first investigate the bound state solutions of (1.14) which will be denoted by $|\chi_B\rangle$ or $|B\rangle$. If N were equal to zero then (1.14) would reduce to the free eigenvalue equation which does not possess normalizable solutions. So, for a bound state we must have $N \neq 0$. If E is any complex number whose imaginary part does not vanish when $|\mathrm{Re}\, E| \geq m$, then the operator H_0-E has a unique inverse and the eigenvalue equation is

$$1-\lambda\,\tilde{F}(E)=0 \quad , \tag{1.15}$$

where

$$\tilde{F}(E)=\langle\beta|\frac{1}{E-H_0}|\beta\rangle=\langle\beta|\frac{\Lambda_+ + \Lambda_-}{E-H_0}|\beta\rangle=$$

$$=\int_0^\infty dp\left[\frac{\langle\beta|\Lambda_+(p)|\beta\rangle}{E-\sqrt{p^2+m^2}}+\frac{\langle\beta|\Lambda_-(p)|\beta\rangle}{E+\sqrt{p^2+m^2}}\right] \quad . \tag{1.16}$$

Here

$$\Lambda_\pm(p)=\frac{1}{\pm 2\sqrt{p^2+m^2}}\begin{pmatrix}\pm\sqrt{p^2+m^2}+m & -p\\ -p & \pm\sqrt{p^2+m^2}-m\end{pmatrix}$$

are the usual energy-projectors and

$$\langle\beta|\Lambda_\pm(p)|\beta\rangle=(g(p),h(p))\,\Lambda_\pm(p)\begin{pmatrix}g(p)\\h(p)\end{pmatrix}=(g(p),h(p))\,\Lambda^2_\pm(p)\begin{pmatrix}g(p)\\h(p)\end{pmatrix}$$

are nonnegative functions. The vanishing of either of these expressions for finite values of p would mean the absence of coupling at these momenta, a nonphysical feature, which we exclude by assuming that $\langle\beta|\Lambda_\pm(p)|\beta\rangle$ do not vanish at finite values of the argument.

As it can be seen, the energy eigenvalues are real, confined to the interval $-m<E<m$, where $\tilde{F}(E)$ is a real decreasing function, so (1.15) possesses at most a single solution.

The critical coupling constant λ_c is obviously given by the expression

$$1/\lambda_c=\tilde{F}(-m) \quad .$$

Hence, $\tilde{F}(-m)$ must be finite the condition of which is easily seen to be

$$\lim_{p\to 0}\frac{\langle\beta|\Lambda_\pm(p)|\beta\rangle}{p}=0 \quad .$$

A further natural requirement is that the bound state energy be the lower the larger is $|\lambda|$ which is fulfilled if $\lambda<0$ and $\tilde{F}(-m)<0$.

We recall now that the Dirac equation when transformed to a second order form of a Schroedinger-equation possesses at $E=-m$ a barrier which to some extent may be reflected in our model by making the coupling to the negative continuum sufficiently weak. The inequality $\tilde{F}(-m)<0$ is in conformity with this requirement.

The existence of the barrier manifests itself in the fact that at the critical charge a normalizable bound state still exists. The norm of the state B is

$$\langle B|B\rangle=\lambda^2N^2\int_0^\infty dp\left[\frac{\langle\beta|\Lambda_+(p)|\beta\rangle}{(E-\sqrt{p^2+m^2})^2}+\frac{\langle\beta|\Lambda_-(p)|\beta\rangle}{(E+\sqrt{p^2+m^2})^2}\right] \quad , \tag{1.17}$$

which is finite at $E=-m$ provided

$$\lim_{p\to 0} \langle\beta|\Lambda_+(p)|\beta\rangle < \infty \quad , \tag{1.18}$$

$$\lim_{p\to 0} \frac{\langle\beta|\Lambda_-(p)|\beta\rangle}{p^3} = 0 \quad .$$

All these requirements are met, for example, by the form factors

$$g(p) = \frac{1}{C}\,\frac{\mu^{1/2}\,p}{p^2+\mu^2} \quad ; \qquad h(p) = \frac{\alpha}{C}\,\frac{\mu^{3/2}p^2}{(p^2+\mu^2)^2} \quad , \tag{1.19}$$

$$C = \sqrt{\frac{(8+\alpha^2)\pi}{32}} \quad ; \qquad -\frac{2\mu}{m} < \alpha < 0 \quad .$$

For $|E| > m$ Eq. (1.14) has solutions normalized to delta-function, hence the domains $E < -m$, $E > m$ constitute the continuous spectrum of our Hamiltonian. The continuum eigenfunctions are

$$\chi_k^{(+)\,\substack{in\\out}}(p) = \phi_k^{(+)}(p) + \lambda G_0(p, E\pm i\varepsilon)|\beta\rangle\langle\beta|\chi_k^{(+)\,\substack{in\\out}}\rangle \quad ,$$

$$\chi_k^{(-)\,\substack{in\\out}}(p) = \phi_k^{(-)}(p) + \lambda G_0(p, E\pm i\varepsilon)|\beta\rangle\langle\beta|\chi_k^{(-)\,\substack{in\\out}}\rangle \quad , \tag{1.20}$$

where

$$E = \pm\sqrt{k^2+m^2} \quad , \qquad G_0(p,z) = \frac{\Lambda_+(p)}{z-\sqrt{p^2+m^2}} + \frac{\Lambda_-(p)}{z+\sqrt{p^2+m^2}} \quad ,$$

$$\phi_k^{(+)}(p) = \delta(k-p)\sqrt{\frac{p_0+m}{2p_0}}\begin{pmatrix} 1 \\ -\dfrac{p}{p_0+m}\end{pmatrix} \quad , \qquad \phi_k^{(-)}(p) = \delta(k-p)\sqrt{\frac{p_0+m}{2p_0}}\begin{pmatrix} \dfrac{p}{p_0+m} \\ 1\end{pmatrix} \quad .$$

Let us investigate now the bound state energy E_B as a function of the coupling constant λ. For negative values above λ_c we have E_B in the interval $(-m,m)$. When λ approaches λ_c E_B tends to $-m$. What happens to $E_B(\lambda)$ when λ becomes smaller than λ_c?

As it must be clear from our earlier considerations above equation (1.14) does not possess normalizable states, when $\lambda < \lambda_c$ and the continuum states $\chi_k^{(\pm)}$ are themselves complete. In spite of this there exists a useful extension of the notion of the bound state energy E_B below λ_c by defining it as the solution of the equation

$$1 - \lambda\tilde{F}_-(E) = 0 \quad , \tag{1.21}$$

where $\tilde{F}_-(E)$ is the analytic continuation from below of $\tilde{F}(E)$ through its cut $(-\infty, -m)$ to the Riemann-sheet R_-.

The fact that $\tilde{F}(E)$ is an analytic function of the complex variable E with cuts $(-\infty, -m)$, $(m, +\infty)$ follows from the integral representation (1.16). If in the first integral we replace the integration variable p by $x = \sqrt{p^2+m^2}$, in the second by $x = -\sqrt{p^2+m^2}$ then we obtain

$$\tilde{F}(E) = \int_{-\infty}^{-m} dx\,\frac{f_-(x)}{E-x} + \int_{m}^{+\infty} dx\,\frac{f_+(x)}{E+x} \quad ,$$

where

$$f_{\pm}(x) = \frac{\pm x}{\sqrt{x^2-m^2}} \cdot \langle\beta|\Lambda_{\pm}(p)|\beta\rangle \Big|_{p=\sqrt{x^2-m^2}} > 0$$

If we deform the integration contour into the curve C_- (see Fig. 1) then for values of E between C_- and the real axis

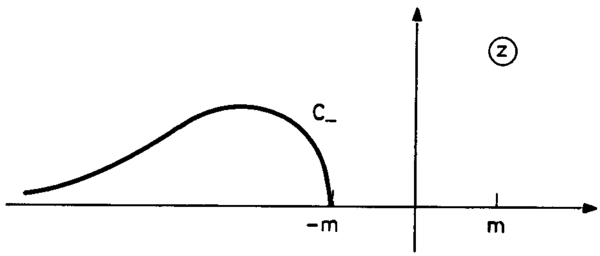

Fig. 1

$$\tilde{F}_-(E) = \int_{C_-} dz\, \frac{f_-(z)}{E-z} + \int_m^{+\infty} dx\, \frac{f_+(x)}{E+x} = \tilde{F}(E) + 2\pi i\, f_-(E)$$

is the required analytic continuation of $\tilde{F}(E)$ (we assumed that $f_-(z)$ is analytic in this domain). As it can be shown (1.21) does have a complex solution E_B even for λ smaller than but close to λ_c. It seems intuitively clear that $E_B'' = \mathrm{Im}\, E_B$ corresponds to the penetrability of the barrier at $E \cong -m$ in the Dirac equation discussed earlier. The smallness of $f_-(E)$ (see (1.18)) which is the consequence of the normalizability of the eigenstates with $E_B = -m$ leads to $E_B'' \ll |E_B|$ i.e. to small barrier penetrability. We notice that if $\tilde{F}_+(E)$ is the continuation of $\tilde{F}(E)$ through the cut $(-\infty, -m)$ from above (i.e. into R_+) then $1/\lambda = \tilde{F}_+(E)$ will be satisfied by $E = E_B^*$.

The kernel of the integral equation for $\tilde{M}$ contains the function $\tilde{F}_\lambda(E)$ defined in (1.9) through the Feynman propagator $K_\lambda(t)$. It is easy to verify that in the partial wave under consideration

$$\langle p|\, K_\lambda(t)\,|p'\rangle = -i\Big[\theta(t)\Big(\int_0^\infty dk\, e^{-i\sqrt{k^2+m^2}\,t}\,\chi_k^{(+)}(p)\,\chi_k^{(+)+}(p') + e^{-iE_Bt}\langle p|B\rangle\langle B|p'\rangle\Big) - \theta(-t)\int_0^\infty dk\, e^{i\sqrt{k^2+m^2}\,t}\,\chi_k^{(-)}(p)\,\chi_k^{(-)+}(p')\Big], \qquad (1.22)$$

where either in or out solutions can be substituted for the χ-s. For $\lambda<\lambda_c$ the term, corresponding to the bound state, is absent. The bound state is handled in (1.22) on equal footing with the particle states, so (1.22) corresponds to the unprimed description. In the primed description we move the bound state term to the antiparticle states. Substituting this equation into (1.9) we obtain

$$\tilde{F}_\lambda(E) = \int_0^\infty dp\left[\frac{\langle\beta|\chi_p^{(+)}\rangle\langle\chi_p^{(+)}|\beta\rangle}{E-\sqrt{p^2+m^2}+i\varepsilon} + \frac{\langle\beta|\chi_p^{(-)}\rangle\langle\chi_p^{(-)}|\beta\rangle}{E+\sqrt{p^2+m^2}-i\varepsilon} + \frac{N^2}{E-E_B+i\varepsilon}\right]. \qquad (1.23)$$

$\tilde{F}'_\lambda(E)$ differs from $\tilde{F}_\lambda(E)$ only in the sign of $i\varepsilon$ in the pole term. When $\lambda = 0$ the last term is absent and we observe that the eigenvalue equation (1.15) can be written as

$$1 - \lambda \tilde{F}_0(E) = 0 \quad . \tag{1.24}$$

Green functions, corresponding to different constant values λ, say λ_1 and λ_2, are connected via integral equations of the type (4.4) – (4.5) of [1] from which the connection between $\tilde{F}_{\lambda_1}(E)$ and $\tilde{F}_{\lambda_2}(E)$ is

$$F_{\lambda_2}(E) = \frac{\tilde{F}_{\lambda_1}(E)}{1 - (\lambda_2 - \lambda_1)\,\tilde{F}_{\lambda_1}(E) - i\varepsilon} \quad . \tag{1.25}$$

The choice of $-i\varepsilon$ is dictated by the pole term in (1.23). For $\lambda_1 = 0$ and $\lambda_2 = \lambda$ this gives

$$\tilde{F}_\lambda(E) = \frac{\tilde{F}_0(E)}{1 - \lambda \tilde{F}_0(E) - i\varepsilon} \quad . \tag{1.26}$$

As a consequence, for values E near E_B

$$\tilde{F}_\lambda(E) \sim \frac{-\tilde{F}_0(E_B)}{\lambda \left(\frac{d\tilde{F}_0(E)}{dE}\right)_{\tilde{E}_B} (E - E_B) + i\varepsilon} \quad . \tag{1.27}$$

From this the constant in the pole term is

$$N^2 = -\frac{\tilde{F}_0(E_B)}{\lambda \left(\frac{d\tilde{F}_0(E)}{dE}\right)_{E_B}} = -\frac{1}{\lambda^2 \left(\frac{d\tilde{F}_0(E)}{dE}\right)_{E_B}} \quad .$$

In the primed description $-i\varepsilon$ is replaced by $+i\varepsilon$.

In the subcritical case the singularity structure of $\tilde{F}_\lambda(E)$ is clearly seen from (1.23) but in the supercritical regime the pole term is absent and the influence of the pole on the R_- Riemann-sheet is hidden in the integral over the lower cut (owing to the particular $i\varepsilon$ prescription, the pole on R_+ does not influence appreciably the behaviour of $\tilde{F}_\lambda(E)$). However, the unphysical pole on R_- manifests itself explicitly in the form of $\tilde{F}_\lambda(E)$ given by (1.26). Indeed, for real values of E below $-m$ $\tilde{F}_0(E) = \tilde{F}_-(E)$ and we have

$$\tilde{F}_\lambda(E) = \frac{\tilde{F}_-(E)}{1 - \lambda \tilde{F}_-(E)} \quad .$$

So far as the imaginary part E_B'' of the solution (1.21) is small, we have for real E-s near E_B.

$$F_\lambda(E) = \frac{-\tilde{F}_-(E_B)}{\lambda \left(\frac{d\tilde{F}_-(E)}{dE}\right)_{E_B} (E - E_B)} \quad ,$$

which exhibits the pole structure due to the unphysical bound state.

2. Model calculation for Lorentzian time-dependence

There is an essentially unique choice of the function $\Delta\lambda(t)$ for which the integral equation (1.7), (1.8) can be transformed into ordinary differential equations. It is the Lorentzian form

$$\Delta\lambda\,(t) = \Delta\lambda_m \frac{T^2}{t^2 + T^2} \quad .$$

The Fourier transform of this expression

$$\Delta\tilde{\lambda}(E) = \frac{1}{2}\Delta\tilde{\lambda}_m\, T\, e^{-T|E|}$$

satisfies the Green function equation

$$\left(\frac{d^2}{dE^2} - T^2\right)\Delta\tilde{\lambda}(E) = -\Delta\lambda_m T^2\, \delta(E) \quad .$$

Applying the operator $\frac{\partial^2}{\partial E^2} - T^2$ to (1.7), $\frac{\partial^2}{\partial E'^2} - T^2$ to (1.8) we obtain the differential equations

$$\left[\frac{\partial^2}{\partial E^2} - T^2\,(1 - \Delta\lambda_m \tilde{F}_\lambda\ (E))\right] \tilde{M}(E,E') = -\Delta\lambda_m T^2\, \delta(E-E'),$$

$$\left[\frac{\partial^2}{\partial E'^2} - T^2\,(1 - \Delta\lambda_m \tilde{F}_\lambda\ (E'))\right] \tilde{M}(E,E') = -\Delta\lambda_m T^2\, \delta(E-E') \quad .$$

The necessary boundary conditions can be read off from (4.11) of [1]: $M(E,E')$ must vanish as either of its arguments becomes large in magnitude. This condition conforms with the physical meaning of $\tilde{M}(E,E')$ expressed in (1.5).

Let us put $\nu = \lambda + \Delta\lambda_m$. Then from (1.25) we can write

$$1 - \Delta\lambda_m \tilde{F}_\lambda\ (E) = \frac{\tilde{F}_\lambda\ (E)}{\tilde{F}_\nu\ (E)} \quad . \tag{2.1}$$

Therefore

$$\left(\frac{\partial^2}{\partial E^2} - T^2 \frac{\tilde{F}_\lambda\ (E)}{\tilde{F}_\nu\ (E)}\right) \tilde{M}\ (E,E') = -\Delta\lambda_m T^2\, \delta\,(E-E') \quad ,$$

$$\left(\frac{\partial^2}{\partial E'^2} - T^2 \frac{\tilde{F}_\lambda\ (E')}{\tilde{F}_\nu\ (E')}\right) \tilde{M}\ (E,E') = -\Delta\lambda_m T^2\, \delta\,(E-E') \quad .$$

These equations are satisfied by the Ansatz

$$\tilde{M}(E,E') = \theta\,(E-E')\,\mathcal{M}_1\,(E)\,\mathcal{M}_2\,(E') + \theta\,(E'-E)\,\mathcal{M}_1\,(E')\,\mathcal{M}_2\,(E) \tag{2.2}$$

provided $\mathcal{M}_j$ (E) (j = 1,2) obey the equations

$$\left(\frac{d^2}{dE^2} - T^2 \frac{\tilde{F}_\lambda\ (E)}{\tilde{F}_\nu\ (E)}\right) \mathcal{M}_j\ (E) = 0 \tag{2.3}$$

and the boundary conditions

$$\mathcal{M}_1(+\infty) = \mathcal{M}_2(-\infty) = 0 \quad , \tag{2.4}$$
$$W_{12} = \mathcal{M}_1' \mathcal{M}_2 - \mathcal{M}_1 \mathcal{M}_2' = -\Delta\lambda_m T^2 \quad .$$

We assume now that in (2.3) the ratio $\tilde{F}_\lambda(E)/\tilde{F}_\nu(E)$ can be approximated by the ratio of the pole contributions (2.27) with the relation (2.1) preserved. Due to this last relation the pole approximation incorporates the effect on $\tilde{M}$ of the moving pole as well as the contributions from the virtual processes in which one of the members of the virtual pair is in state B (in case of $\lambda < \lambda_c$ this means a superposition of those continuum states which are strongly disturbed).

We write therefore

$$\tilde{F}_\lambda(E) \to \tilde{F}_\lambda^p(E) = \frac{Z_\lambda R(E)}{E-E_B+i\varepsilon} \quad ; \qquad R(E_B) = 1, \qquad Z_\lambda = N^2 \tag{2.5}$$
$$\tilde{F}_\nu(E) \to \tilde{F}_\nu^p(E) = \frac{Z_\nu R(E)}{E-E_\nu+i\varepsilon} \quad ,$$

in which R(E) must be chosen so as to satisfy the equation

$$1 - \Delta\lambda_m \tilde{F}_\lambda^p(E) = \frac{\tilde{F}_\lambda^p(E)}{\tilde{F}_\nu^p(E)} \quad .$$

When $|\nu| < |\lambda_c|$ then Z_ν, E_ν are real, when $|\nu| > |\lambda_c|$ both are complex:

$$E_\nu = E_\nu' + iE_\nu'' \quad ; \qquad E_\nu' < -m \; , \qquad E_\nu'' > 0 \; .$$

In what follows the primed description will be primarily employed in which the $i\varepsilon$ term in (2.5) is of negative sign. It will be convenient to treat E as a complex variable. The unprimed (primed) description is obtained by approaching the real axis from above (below).

We have, therefore, in pole approximation

$$\left(\frac{d^2}{dE^2} - T^2 \frac{Z_\lambda}{Z_\nu} \frac{E-E_\nu}{E-E_B} \right) \mathcal{M}_j(E) = 0 \quad . \tag{2.6}$$

The solution, which satisfies the desired boundary conditions is

$$\begin{aligned}
\tilde{M}(E,E') = &-2\alpha\, e^{-i\pi a} \frac{\Gamma(a)}{\Gamma(2-a)} \Delta\lambda_m T^2 (E-E_B)\, e^{-\alpha(E-E_B)} \; . \\
&. \; e^{-\alpha(E'-E_B)} (E'-E_B) \; . \; \Big[\theta(E-E') \, U(a,2,2\alpha(E-E_B)) \; . \\
&. \left(e^{i\pi a} \Gamma(2-a) M(a,2,2\alpha(E'-E_B)) + U(a,2,2\alpha(E'-E_B)) \right) + \\
&+ \theta(E'-E) \, U(a,2,2\alpha(E'-E_B)) \; . \\
&. \left(-e^{i\pi a} \Gamma(2-a) M(a,2,2\alpha(E-E_B)) + U(a,2,2\alpha(E-E_B)) \right) \Big] \; .
\end{aligned} \tag{2.7}$$

M and U are the two independent confluent hypergeometric functions [3]. a and α are

$$a = 1 + \frac{\alpha}{2}(E_B - E_\nu) \quad ,$$

$$\alpha = T\sqrt{\frac{Z_\lambda}{Z_\nu}} = \frac{TN}{\sqrt{Z_\nu}} = \alpha' + i\alpha'' = |\alpha|\, e^{i\beta} \quad ; \qquad 0 \leq \beta \geq \pi/2 \quad .$$

The function U(a,b,z) is a many valued function with a cut along the negative real axis. In the primed description the semiaxis $\arg(\beta - \pi)$ which corresponds to negative real values of E must be approached from below, so the cut need not be crossed.

Now we are in a position to verify whether the condition (5.2) of [1] is satisfied or not. The last line of (1.5) gives

$$(W_4'^{-1})_{BB} = 1 + iN^2 \cdot 2\pi \cdot \tilde{M}'(E_B, E_B) \quad .$$

From this we obtain

$$(W_4'^{-1})_{BB} = e^{-2\pi\, ia}$$

and

$$\left|(W_4'^{-1})_{BB}\right| = e^{4\pi \cdot \mathrm{Im}\, a} \quad .$$

Using (2.6), a can be expressed as

$$a = 1 + \frac{1}{2} TN \sqrt{|\Delta\lambda_m|}\sqrt{E_B - E_\nu} \quad .$$

It is always true that $E_\nu < E_B$. If $E_\nu > -m$, no supercritical transition occurs, $\mathrm{Im}\, E_\nu = \mathrm{Im}\, a = 0$ and $|(W_4'^{-1})_{BB}|^2 = 1$. If, on the other hand, $E_\nu < -m$ then $\mathrm{Im}\, E_\nu > 0$ and obviously $\mathrm{Im}\, a < 0$. In particular, if $E_\nu''/(E_B - E_\nu') \ll 1$ then $\mathrm{Im}\, a = -\frac{TN}{2}\sqrt{\frac{|\Delta\lambda_m|}{E_B - E_\nu'}}\, E_\nu'' = -\frac{\gamma_L}{4\pi} T$ and $|(W_4'^{-1})_{BB}|^2 = e^{-\gamma_L T} \rightarrow 0$ as $T \rightarrow \infty$.

The criterion (5.2) of [1] is, therefore, satisfied. The quantity γ_L can be interpreted as the decay constant of the vacuum. When the unphysical pole E_ν is close to the real axis γ_L is proportional to Im E as expected. However, the proportionality factor is a nontrivial expression which reflects the effect of the moving pole.

If the electron of the pair created is bound then the spectrum of the accompanying positron is determined by the function $\tilde{M}'(E_B, -k_0)$. It is difficult to calculate the large T limit of this quantity – and of the amplitudes $\tilde{M}'(E,E')$ in general – with the aid of (2.7) since this would involve a simultaneous limit in both the argument and the parameter of the functions M and U. However, since T^{-2} plays a role similar to the role of $\hbar^2$ in a Schrödinger equation a quasiclassical treatment is available.

Let us introduce a new variable $x = E_B - E$ and a new function

$$y(x) = \frac{2\pi\, i\, N^2}{e^{-2\pi\, ia_{-1}}}\, \tilde{M}(E_B, E) \quad ,$$

which satisfies the equation

$$\eta^2\, y''(x) - c\,\frac{x-b}{x} = 0 \quad , \tag{2.8}$$

with the boundary conditions

$$y(-\infty) = 0 \ , \qquad y(0) = 1 \ ,$$

where

$$b = E_\nu - E_B = b' + ib'' \ ; \qquad c = \frac{Z_\lambda}{Z_\nu} \ ; \qquad \eta = \frac{1}{T} \ .$$

Since the imaginary part of E_ν is small we have $b'' \ll |b'|$ and in the domain $|b'| \gg x \gg 0$, x can be neglected as compared to b. We have, therefore,

$$xy'' + \frac{bc}{\eta^2} y = 0 \ .$$

The general solution of this equation can be written as a superposition of Hankel-functions:

$$y(x) = \sqrt{x} \left[A \, H_1^{(1)} \left(2 \frac{\sqrt{bc}}{\eta} \sqrt{x}\right) + B \, H_1^{(2)} \left(2 \frac{\sqrt{bc}}{\eta} \sqrt{x}\right)\right] \qquad (2.9)$$

the coefficients of which are subject to the constraint

$$y(0) = (B-A) \frac{i}{\pi\sqrt{bc}} = 1 \ . \qquad (2.10)$$

In the domain $|x| \gg \eta$ we look for the solution of (2.8) in the form

$$y = e^{\frac{1}{\eta} S} \ ,$$

$$S = S_0 + \eta \, S_1 + \eta^2 \, S_2 + \dots \ .$$

Applying the usual procedure of the quasiclassical calculations we obtain to terms linear in T^{-1}:

$$y(x) \sim K_e \exp\left\{\frac{1}{\eta} \sqrt{c} \left(\sqrt{x(x-b)} - b \ln\left(\sqrt{x-b} + \sqrt{x}\right)\right) \left(\frac{x}{x-b}\right)^{1/4}\right. \ . \qquad (2.11)$$

The coefficient K can be determined from matching the solutions (2.9) and (2.11) in the domain of overlap $|b'| \gg x \gg \eta$, where they take the form

$$y(x) \sim \sqrt[4]{\frac{x}{bc}} \sqrt{\frac{\eta}{\pi}} \left\{A e^{-i\frac{3\pi}{4}} e^{i2 \frac{\sqrt{bc}}{\eta}\sqrt{x}} + B \, e^{i\frac{3\pi}{4}} e^{-i2 \frac{\sqrt{bc}}{\eta} \sqrt{x}}\right\}$$

and

$$y(x) \sim K(-b)^{\left[\frac{1}{2\eta}(-b)\sqrt{c} - \frac{1}{4}\right]} x^{1/4} e^{i2 \frac{\sqrt{bc}}{\eta} \sqrt{x}} \ ,$$

respectively. These expressions together with (2.10) lead to

$$K = e^{-i \frac{\pi}{4}} \sqrt{\frac{\pi}{\eta}} \sqrt[4]{bc} \, (-b)^{\left[\frac{1}{4} - \frac{1}{2\eta}(-b)\sqrt{c}\right]} \ .$$

The asymptotic formula (2.11) with this value of K determines the large T limit of $\tilde{M}'(E_B,-k_0)$.

The positron spectrum was computed numerically and as it can be seen from Fig. 2 and Fig. 3. – which are typical spectra for sub- and supercritical processes – there is not any characteristic line structure which was expected on the basis of our intuitive notion. The reason for this discrepancy needs further clarification.

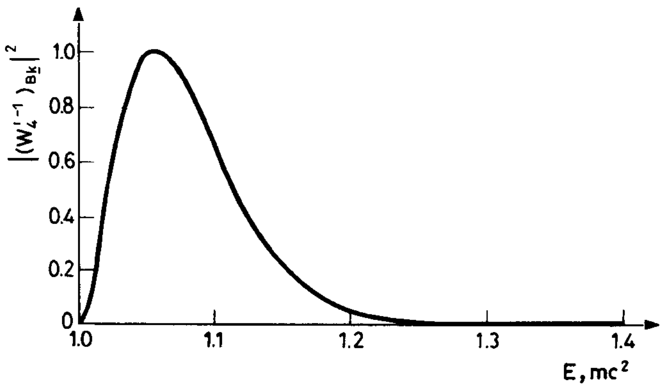

Fig. 2. The subcritical positron spectrum for Lorentzian time dependence, $E_B = -0.9$, $E_\nu = -0.987$, $T = 20$

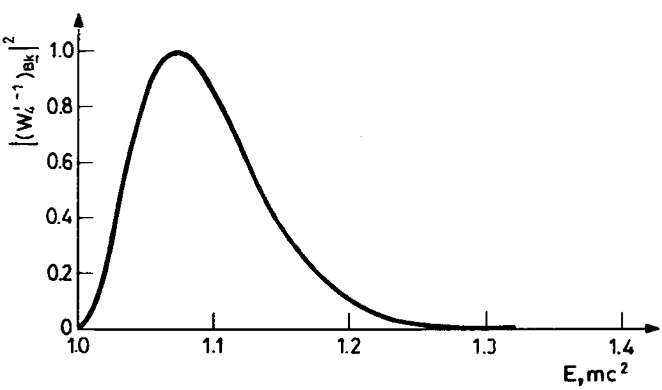

Fig. 3. The supercritical positron spectrum for Lorentzian time dependence, $E_B = -0.9$, $E_\nu = -1.05+0.0018i$, $T = 20$

3. Potential jump in pole approximation

A potential jump of duration T is described by the coupling function

$$\Delta\lambda(t) = \begin{cases} \Delta\lambda_m & \text{if} \quad -T/2 < t < T/2 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Then, according to (1.2), $\langle\beta|\, M(t,t')|\beta\rangle = 0$ unless both t and t' are within the interval $(-T/2, T/2)$. Therefore, equations (1.3), (1.4) in the primed description take on the form

$$\langle\beta|M'(t,t')|\beta\rangle = \Delta\lambda_m\, \delta(t-t') + \Delta\lambda_m \int_{-T/2}^{+T/2} dt'' \langle\beta|\, K_\lambda'(t-t'')|\beta\rangle\langle\beta|M'(t'',t')|\beta\rangle , \qquad (3.1)$$

$$\langle\beta|\,M'(t,t')\,|\beta\rangle = \Delta\lambda_m\,\delta(t-t') + \Delta\lambda_m \int_{-T/2}^{+T/2} dt''\langle\beta|\,M'(t,t'')|\beta\rangle\langle\beta|\,K'_\lambda\,(t''-t')\,|\beta\rangle\,. \qquad (3.2)$$
$$-T/2<t,t'<T/2$$

The kernel K' given in (1.28) is a highly complicated function which makes the analytic solution of (3.2) hopeless, but the pole approximation applied to the Lorentzian time dependence in the previous section leads again to a soluble problem.

At first sight one might suppose that the pole approximation consists in neglecting all contributions to $\langle\beta|\,K'_\lambda\,(t)|\beta\rangle$ except for the pole term $iN^2\theta(-t)e^{-iE_Bt}$. In such an approximation, however, equations (3.1), (3.2) would certainly be unable to account for the complexness of E_ν , the unphysical binding energy at $\nu = \lambda + \Delta\lambda_m > \lambda_c$ since the imaginary part of E_ν originates from the coupling to the continuum whose contribution to $\langle\beta|K'_\lambda(t)|\beta\rangle$ has been completely neglected. In the pole approximation suggested by the example of the preceding section we actually take into account, beside the pole term, an additional contribution also, which is just sufficient to locate E_ν at the right position. From (1.9) and (2.1) we have

$$\Delta\lambda_m\langle\beta|\,K'_\lambda\,(t)|\beta\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dE\; e^{-iEt}\,\Delta\lambda_m\,\tilde{F}'_\lambda\,(E) =$$

$$= \delta(t) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} dE\; e^{-iEt}\;\frac{\tilde{F}'_\lambda\,(E)}{\tilde{F}'_\nu\,(E)}\quad .$$

In pole approximation, according to (2.5)

$$\Delta\lambda_m\langle\beta|\,K'_\lambda\,(t)|\beta\rangle^p = \delta(t) - \frac{Z_\lambda}{Z_\nu}\;\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE\; e^{iEt}\,\frac{E-E_\nu}{E-E_B-i\varepsilon} =$$

$$= (1-\frac{Z_\lambda}{Z_\nu})\;\delta(t) - i\,\frac{Z_\lambda}{Z_\nu}\,(E_B-E_\nu\;)\;\theta\,(-t)\;e^{-iE_Bt}\quad ,$$

which, using (2.6) and the equality $Z_\lambda = N^2$, can be cast into the form

$$\langle\beta|\;K'_\lambda\,(t)\,|\beta\rangle^p = \frac{Z_\nu\;-Z_\lambda}{\Delta\lambda_m\;Z_\nu}\;\;\delta(t) + iN^2\;\theta\;(-t)\;e^{-iE_Bt}\quad .$$

Substituting this kernel into (3.1), (3.2) and multiplying by e^{iE_Bt} and e^{-iE_Bt}, we obtain

$$\frac{Z_\lambda}{Z_\nu}\;e^{iE_Bt}\,\langle\beta|\;M'(t,t')|\beta\rangle = \Delta\lambda_m\,e^{iE_Bt}\;\delta(t-t') +$$
$$+ \Delta\lambda_m\;iN^2\int_t^{T/2} dt''\;e^{iE_Bt''}\langle\beta|\;M'(t'',t')|\beta\rangle\quad , \qquad (3.3)$$

$$\frac{Z_\lambda}{Z_\nu}\langle\beta|\;M'(t,t')|\beta\rangle\;\;e^{-iE_Bt'} = \Delta\lambda_m\;e^{-iE_Bt'}\;\delta(t-t') +$$
$$+ \Delta\lambda_m\;iN^2\int_{-T/2}^{t'} dt''\,\langle\beta|\,M'(t'',t')|\beta\rangle\;\;e^{-iE_Bt''}\quad ,\qquad -T/2<t,t'<T/2\;. \qquad (3.4)$$

In the first equation in which t is a parameter we introduce the notation

$$f(t) = \int_{t}^{T/2} dt'' \, e^{iE_B t''} \langle\beta|\, M'(t'',t')|\beta\rangle \ .$$

Since

$$e^{iE_B t}\langle\beta|\, M'(t,t')|\beta\rangle = - f'(t) \tag{3.5}$$

equation (3.3) takes on the form

$$\frac{Z_\lambda}{Z_\nu} \, f'(t) + \Delta\lambda_m \, iN^2 \, f(t) = - \Delta\lambda_m \, e^{iE_B t} \, \delta(t-t') \ .$$

The solution of this differential equation, subject to the condition f(T/2) = 0 is easily found to be

$$f(t) = \frac{Z_\nu}{Z_\lambda} \, \Delta\lambda_m \, e^{i(E_B + \Delta\lambda_m Z_\nu)t'} \, e^{-i\Delta\lambda_m Z_\nu t} \, \theta(t'-t) \ .$$

Using (3.5), we obtain the solution to (3.3):

$$\langle\beta|\, M'(t,t')|\beta\rangle = \frac{Z_\nu}{Z_\lambda} \, \Delta\lambda_m \, \delta(t-t') + i \, \frac{(Z_\nu \Delta\lambda_m)^2}{Z_\lambda} \, e^{i(E_B + \Delta\lambda_m Z_\nu)(t'-t)} \theta(t'-t) \ ,$$

which satisfies (3.4), too.

Now, from (1.6) we have

$$\tilde{M}'(E,E') = \frac{Z_\nu \Delta\lambda_m}{2\pi \, Z_\lambda} \int_{-T/2}^{+T/2} dt \, e^{i(E-E')t} \, +$$

$$+ \, i \, \frac{(Z_\nu \Delta\lambda_m)^2}{2\pi \, Z_\lambda} \int_{-T/2}^{+T/2} dt dt' \, \theta(t'-t) \, e^{i(E-E_B-\Delta\lambda_m Z_\nu)t} \, e^{i(E'-E_B-\Delta\lambda_m Z_\nu)t'} \ . \tag{3.6}$$

In order to verify criterion (5.2) of [1] we calculate $\tilde{M}'(E_B, E_B)$ with the result

$$\tilde{M}'(E_B,E_B) = \frac{e^{i(E_\nu - E_B)T}}{2\pi i \, N^2}$$

from which we obtain

$$(W_4'^{-1})_{BB} = e^{i(E_\nu - E_B)T} \ ,$$

$$\left|(W_4'^{-1})_{BB}\right|^2 = e^{-2\,\mathrm{Im}\,E_\nu T} = e^{-\gamma_J T} \ .$$

which is again the expected result. A comparison with the Lorentzian time dependence shows that though the criterion (5.2) of [1] is fulfilled in both cases but the decay constants of the vacuum γ_L and γ_J are different.

Performing in (3.6) the integrations, we obtain

$$\tilde{M}'(E,E') = \frac{\Delta\lambda_m Z_\nu}{2\pi Z_\lambda}\, e^{-\frac{i}{2}(E+E')T} \left\{ \frac{\Delta\lambda_m Z_\nu\, e^{iE_\nu T}}{(E-E_\nu)(E'-E_\nu)} + \right.$$

$$\left. + \frac{1}{E-E'} \left[\frac{E-E_B}{E-E_\nu} e^{iET} - \frac{E'-E_B}{E'-E_\nu} e^{iE'T} \right] \right\} .$$

According to (1.5) the positron spectrum is essentially given by $\tilde{M}'(E_B,E)$ (the other terms are only kinematic factors):

$$\tilde{M}'(E_B,E) = \frac{1}{2\pi i N^2} \frac{E_\nu - E_B}{E_\nu - E} e^{\frac{i}{2}(E_\nu - E_B)T} \left[e^{\frac{i}{2}(E_\nu - E)T} - e^{-\frac{i}{2}(E_\nu - E)T} \right]$$

If we consider subcritical processes then E_ν is real and

$$|\tilde{M}'(E_B,E)|^2 = \frac{1}{\pi^2 N^4} (E_\nu - E_B)^2 \left(\frac{\sin\left(\frac{T(E_\nu - E)}{2}\right)}{E_\nu - E} \right)^2$$

as $E \rightarrow \infty$ this function tends to zero as E^{-2} for fixed T. As $T \rightarrow \infty$ it is a more and more rapidly oscillating function of E:

$$|\tilde{M}'(E_B,E)|^2 \xrightarrow[T \rightarrow \infty]{} \frac{1}{N^4} (E_\nu - E_B)^2 \frac{T}{2} \delta(E_\nu - E) .$$

The positron spectrum has a threshold at E = -m, so in the physically relevant region $E \leqslant -m$, thus E never coincides with E_ν .

In supercritical processes $E_\nu = E'_\nu + iE''_\nu$ is complex and

$$|\tilde{M}'(E_B,E)|^2 = \frac{1}{2\pi^2 N^4} \frac{(E'_\nu - E_B)^2 + E''^2_\nu}{(E'_\nu - E)^2 + E''^2_\nu} e^{-2E''_\nu T} (\mathrm{ch}\, 2E''_\nu T - \cos 2(E'_\nu - E)T) .$$

In the $E \rightarrow \infty$ limit it goes to zero as E^{-2} again, and as $T \rightarrow \infty$ it has a Lorentzian form

$$|\tilde{M}'(E_B,E)|^2 \xrightarrow[T \rightarrow \infty]{} \frac{1}{2\pi^2 N^4} \frac{(E'_\nu - E_B)^2 + E''^2_\nu}{(E'_\nu - E)^2 + E''^2_\nu} .$$

As it can be seen from (3.7) and (3.8) characteristic difference shows up between subcritical (Fig. 4) and supercritical (Fig. 5,6) spectra which is in agreement with our qualitative picture.

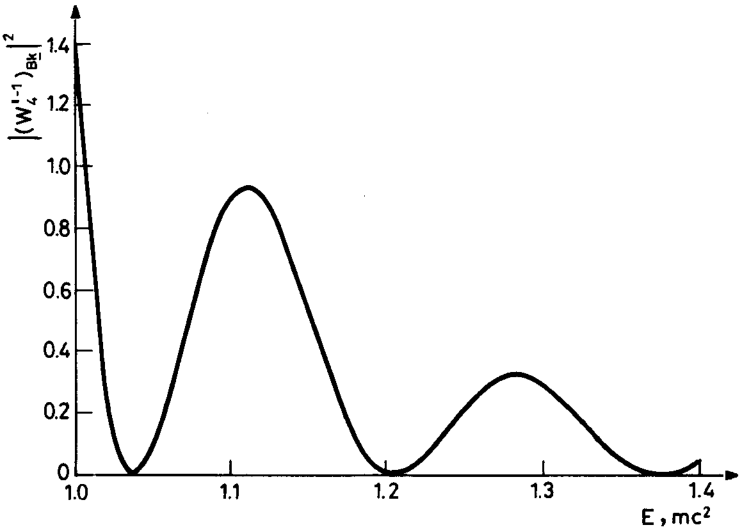

Fig. 4. The subcritical positron spectrum for potential jump, $E_B = -0.9$, $E_\nu = -0.987$, $T = 100$

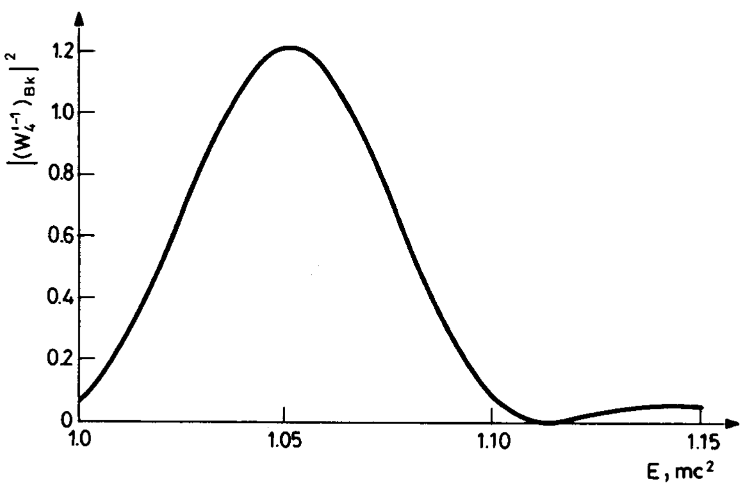

Fig. 5. The supercritical positron spectrum for potential jump, $E_B = -0.9$, $E_\nu = -1.05+0.0018i$, $T = 100$

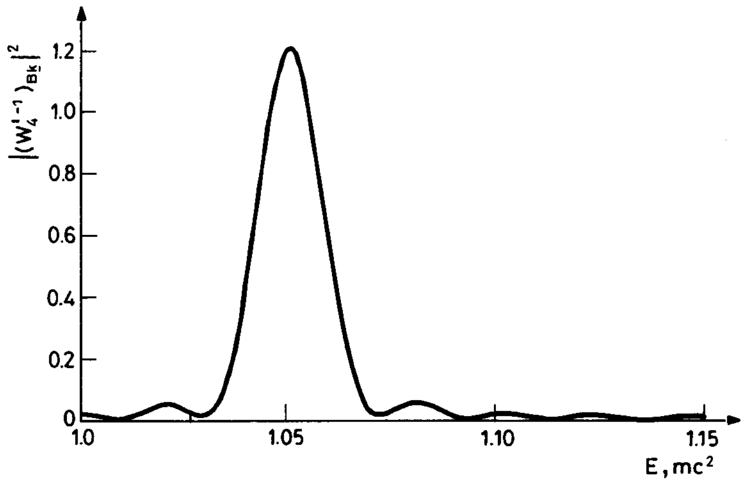

Fig. 6. The supercritical positron spectrum for potential jump, $E_B = -0.9$, $E_\nu = -1.05+0.0018i$, $T = 300$

Acknowledgement

The authors thank A. Frenkel and J. Révai for numerous fruitful discussions.

References

1. P. Hraskó, L. Földy and A. Tóth, Acta Phys. Hung., 63,
2. J.D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics Vol. I., McGraw-Hill, New York, 1964.
3. M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Washington, D.C., 1964.