Acta Physica Hungarica 70 (4), pp. 289–297 (1991)

ON THE BONNOR COUNTERPARTS OF THE TOMIMATSU–SATO SOLUTIONS*

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(Received 30 April 1991)

In the literature a belief is spreading that the static electrovac counterparts of the Tomimatsu–Sato solutions are known. However, as we show, the counterpart metrics have been obtained by means of a wrong method, and do not describe electrovac fields. In this paper we give the true static electrovac counterparts.

1. Introduction

In this paper we construct the Bonnor counterparts of the Tomimatsu–Sato solutions. The Bonnor transformation connects a stationary vacuum solution of the Einstein equation with a static electrovac one [1], [2], [3]; the number of free parameters is the same in both solutions, except for a trivial duality rotation in the electromagnetic field (always possible in the sourcefree Einstein–Maxwell problem). The Tomimatsu–Sato series is a 3-parameter family of axisymmetric stationary vacuum solutions; the free parameters are the mass, angular momentum, and a number influencing the oblateness. These solutions are asymptotically flat; in the central regions singularities and acausalities are reported. The first member of the family is the Kerr solution, which is the unique black hole solution with *regular* horizon [4]. Therefore, the TS solutions seem to be the most important stationary axisymmetric vacua available in the present state of art, and so their Bonnor counterparts may be important, too. Unfortunately, just in the time when the TS solution had been found some confusion occurred in the literature about the form of the Bonnor transformation, so now different articles mention different line elements as Bonnor counterparts of the Tomimatsu–Sato metrics. Here we are going to calculate the true Bonnor counterparts in a methodical way, and analyze their physical properties.

Section 2 gives the field equation of the stationary axisymmetric electrovac problem. Section 3 lists the TS solutions. The essence of controversies about the Bonnor transformation is discussed in Sections 4 and 5. In Section 6 we perform the transformation and give the counterpart metrics. Section 7 investigates the asymptotic behaviour of the new solutions, and Section 8 contains some observations about the possible sources of the metrics.

*Dedicated to Prof. R. Gáspár on his 70th birthday.

2. Stationary axisymmetric electrovac metrics

Stationary axisymmetric geometries can be written into the canonical form

$$ds^2 = f(dt + \omega d\Psi)^2 - f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\Psi^2], \tag{2.1}$$

where no quantity depends on the symmetry coordinates (t, Ψ). The Maxwell field can be represented by a complex potential Φ, which is some combination of the time and space components of the vector potential A; for the details see [5]. Then, the Einstein–Maxwell equations can be rewritten in terms of γ, Φ, f and φ, where φ is defined via

$$\text{grad}\,\varphi = -(f^2/\rho)(\text{nxrot}\,\vec{\omega}), \tag{2.2}$$

where n is the azimuthal unit vector.

The general form of the field equations can be found in [5]. Here we are interested only in two special cases.

Case 1: Stationary axisymmetric vacuum. Then the equations get the form

$$\begin{aligned} f\Delta f &= (\text{grad}\, f)^2 - (\text{grad}\,\varphi)^2, \\ f\delta\varphi &= 2(\text{grad}\, f \bullet \text{grad}\,\varphi), \\ \gamma_{,\rho} &= (\rho/4f^2)(f_{,\rho}^2 + \varphi_{,\rho}^2 - f_{,z}^2 - \varphi_{,z}^2), \\ \gamma_{,z} &= (\rho/2f^2)(f_{,\rho} f_{,z} + \varphi_{,\rho}\varphi_{,z}). \end{aligned} \tag{2.3}$$

Case 2: Static axisymmetric electrovac. A theorem exists for this case that for any such solution

$$\Phi = |\Phi| e^{iC}, C = \text{const}. \tag{2.4}$$

[6]. Then by a trivial duality rotation first Φ can be made real, and henceforth this will be the convention. After this the equations read as

$$\begin{aligned} f\Delta f &= (\text{grad}\, f)^2 + 2f(\text{grad}\,\Phi)^2, \\ f\Delta\Phi &= (\text{grad}\, f \bullet \text{grad}\,\Phi), \\ \gamma_{,\rho} &= (\rho/4f^2)(f_{,\rho}^2 - 4f\Phi_{,\rho}^2 - f_{,z}^2 + 4f\Phi_{,z}^2), \\ \gamma_{,z} &= (\rho/2f^2)(f_{,\rho} f_{,z} - 4f\Phi_{,\rho}\Phi_{,z}). \end{aligned} \tag{2.5}$$

Having got a solution, the inverse of the original duality rotation, i.e.

$$\Phi \to \Phi e^{iC} \tag{2.6}$$

still remains as a freedom (and this is always a freedom of any Einstein–Maxwell problem without sources).

In both cases the background metric is flat cylindrical. The first two equations always separate and then there remains a quadrature for γ.

There is a tempting structural similarity between the two sets of equation, suggesting transformations between their solutions. The first such transformation was found by Bonnor [1]. In the present language the transformation formally has the form [2], [3]

$$f(\text{static. el. vac}) = f^2(\text{station. vac}), \qquad (2.7)$$
$$\Phi(\text{static. el. vac}) = i\varphi(\text{station. vac}).$$

The symmetry can be verified by a simple substitution. However, by construction, both φ and Φ must be real, otherwise at some places complex conjugates would occur. So Eq. (2.7) cannot be taken in face value. The relation holds between complex analytic extensions of these field quantities. The method was demonstrated in [2], reproducing the Bonnor solution [7], which is the static electrovac counterpart of the Kerr metric.

There is at least a good chance that Transformation (2.7) keeps the asymptotic flatness. Namely, the first transformation keeps Schwarzschild asymptotics. Then, for φ the asymptotic form is $J \cos\theta/r^2$, which would lead to a dipole field, although the complex extension may complicate the matter. As for the Bonnor solution, from outside the solution is a charged mass dipole. The internal region contains both singularities and acausal domains [7] suggesting a rigid support between the opposite charges.

3. The Tomimatsu–Sato series

Until 1973 there was no other candidate than Kerr as the starting point of the transformation (henceforth called Bonnorification). In that year, however, Tomimatsu and Sato found an infinite series with good asymptotics, whose first member is the Kerr metric [8]. For later use here we briefly recapitulate the fundamentals of the TS series.

Consider the stationary axisymmetric vacuum problem, and introduce the complex Ernst variable ξ:

$$f + i\varphi \equiv (\xi - 1)/(\xi + 1). \qquad (3.1)$$

Furthermore, introduce the spheroidal coordinates x, y as

$$\rho \equiv (x^2 - 1)^{1/2}(1 - y^2)^{1/2}/c, \qquad (3.2)$$
$$z \equiv xy/c,$$

where c is an appropriate dimensional constant. Then take an Ansatz

$$\xi = (\alpha_1 + iqy\alpha_2)/(\beta_1 + iqy\beta_2), \qquad (3.3)$$

where q is a constant and the small Greek symbols are polynomials with the structure

$$\alpha_1 = \alpha_1(p \equiv (1-q^2)^{1/2}, q^2; x, y^2; \delta), \text{ etc.} \tag{3.4}$$

(δ being a serial number labelling the solution).

Now it turns out that the first two of Eqs (2.3) do have such solutions. The equations give two constraints on the four polynomials; clearly a common factor cancels in ξ, and then the remaining freedom leaves us with a series of one degree of freedom labelled by δ. The constraint equations are rather complicated; [8] gave the first four members of the series of which here we recapitulate only the first two. (Remember that δ gives the Kerr solution.)

δ	α_1	α_2	β_1	β_2
1	px	-1	$+1$	0
2	$p^2x^4+q^2y^4-1$	$-2px(x^2-y^2)$	$2px(x^2-1)$	$-2(1-y^2)$

As for the meaning of δ observe that

$$\begin{aligned} &\lim \alpha_1 = p^\delta x^{\delta^2}, \\ &x \to \infty \\ &\lim \beta_1 = p^{\delta-1} x^{\delta^2-1}, \\ &x \to \infty. \end{aligned} \tag{3.5}$$

Then, having a whole series of promising stationary solutions, it would have been natural to Bonnorify them.

4. Towards the Bonnorified TS solutions

This process, however, was misled by an accident, although some papers propagate the belief that the Bonnorified TS solutions had been found. (Instead of the full review here we mention only [9]. It discusses a Bonnor–Misra–Pandey–Srivastava–Tripathi–Wang family of polarized charge solutions, saying that "the K-TS family of solutions and the B-MPST-W family of solutions are mathematically identical, though of course physically quite different".) What happened, had been initiated by a sign error.

In 1973, Misra et al claimed to find a transformation between solutions of the stationary vacuum and static electrovac problems [10]. It is necessary to follow some steps of the authors in their own notation for clarity.

They started from the static electrovac problem with

$$g_{oo} \equiv e^{2u} \tag{4.1}$$

and the electromagnetic field is obtained from a *real* potential C. (That is, in our notation, in Eq. (2.5) $f \equiv e^{2u}$ and $\Phi \equiv C$.) Then they introduced a complex quantity E:

$$E \equiv e^u + iC, \tag{4.2}$$

they perform a transformation to X as

$$E \equiv (X-1)/(X+1) \tag{4.3}$$

and then X satisfies the vacuum Weyl equations. Then based on this observation, they started back from a solution where X was one of the spheroidal coordinates. Hence a static dipole solution was obtained. In their Note added in proof they mentioned a similarity to the Bonnor solution. Later in an Erratum [11] they corrected a sign error in the energy-momentum tensor. The Erratum states that the error can be corrected by writing iC instead of C and, in the specific example, ie instead of e.

In the meantime Wang reformulated their statement in the language of the Ernst equation, and applied the symmetry on the TS series to get electrovac solutions [12]. Unfortunately, she still started from the false energy-momentum tensor. The resulting solutions turned out to be some dipoles. Later Ward rediscovered the sign error [13], and gave prescriptions to correct them by multiplying the electromagnetic potential, charge and dipole parameters by i. (For a review, see [14].) Still, 4 years later [9] still gave the counterpart solutions from the original transformation. Now let us stop here with four remarks.

1) It may quite be true that in special cases sign changes in the squared charge and dipole momentum suffice to get the solutions of the correct Einstein–Maxwell system from those of the false ones, but this is not necessarily so in general, for solutions of many parameters. The TS solutions contain a lot of parameters (mainly number constants), and Misra et al's Erratum did not deal with the TS series.

2) The suggested corrections include imaginary values for parameters real by construction.

3) There is an uncertainty in the literature about the static electrovac counterparts of the TS solutions.

4) Even that is doubtful, if the Bonnor counterparts have already been manufactured at all. To my knowledge nobody directly calculated them. Wang's solutions are the false counterparts, and [9] regards the Wang line elements as the Bonnor counterparts.

5. On the connection of the Bonnor and Misra–Pandey–Srivastava–Tripathi transformations

[11] (the final Note) suggests a close connection between the two generating methods and it seems that [9] identifies them with each other. This is indeed so, except for the sign error.

Namely, consider Eqs (4.1-3). In the Ernst language they would mean simply

$$\begin{aligned} f &\to f^2, \\ \varphi &\to \Phi. \end{aligned} \tag{5.1}$$

(Observe that e^u stands for f, C for φ and X for ξ.) By performing the suggested correction $C \to iC$ we arrive to (2.7), which is Bonnorification. Of course, Φ must not be imaginary, but that problem was discussed in Section 2. So by correcting the field equations the claimed symmetry of [11] and [12] is just Bonnorification. Therefore the Wang series is the *false* Bonnor counterpart of the TS series.

Now, if one needs the true Bonnor counterparts, two possibilities exists. Either one may try to perform all the sign changes on the Wang line elements, containing a lot of constants in the polynomials α_i, β_i, or one can calculate the correct formulae by the correct method *ab initio*. The second way seems to be safer.

6. The transformation

We want to perform Transformation (2.7) on the TS solutions. Then first we need the original f and φ. For them Eqs (3.1-4) give

$$\begin{aligned} f &= [\alpha_1^2 - \beta_1^2 + q^2y^2(\alpha_2^2 - \beta_2^2)]/[(\alpha_1+\beta_1)^2 + q^2y^2(\alpha_2+\beta_2)^2], \\ \varphi &= -2qy(\alpha_1\beta_2 - \alpha_2\beta_1)/[(\alpha_1+\beta_1)^2 + q^2y^2(\alpha_2+\beta_2)^2], \end{aligned} \tag{6.1}$$

Now we need a complex extension of φ in order to choose the imaginary special value for the transformation. The possibilities are restricted, because the coordinates must remain real and changes in the polynomials α_i, β_i would generally complexify f as well. However, this does not happen if

$$q \to iq, \tag{6.2}$$

with all the *number* constants in α, β unchanged. (This, as we shall see, is *not* at all a general rule, but is accidentally true for the whole TS series because of the special structure of the polynomials.) Note that by construction

$$p \to p^* = (1+q^2)^{1/2}. \tag{6.3}$$

Then all polynomials change into their starred counterparts according to the scheme

$$\alpha_1^* \equiv \alpha_1(p^*, -q^2; x, y^2; \delta), \text{etc.} \tag{6.4}$$

Now we are in the position to perform the transformation. According to (2.7) we get

$$\begin{aligned} f_{(\text{el. vac})} &= U^2/W^2, \\ \Phi_{(\text{el. vac})} &= -2qy(\alpha_1^*\beta_2^* - \alpha_2^*\beta_1^*)/W, \\ U &\equiv \alpha_1^{*2} - \beta_1^{*2} - q^2y^2(\alpha_2^{*2} - \beta_2^{*2}), \\ W &\equiv (\alpha_1^* + \beta_1^*)^2 - q^2y^2(\alpha_2^* + \beta_2^*)^2. \end{aligned} \tag{6.5}$$

The remaining two of Eqs (2.5) give the new γ. Since the coordinates are unchanged, and the polynomials depend on them in the old way except for the parameter *values* instead of p and q^2, Wang's result [12] is valid, except that the polynomials must be starred. I.e.

$$\begin{aligned}\gamma_{(\text{el. vac})} &= \alpha_1^{*2} - \beta_1^{*2} - q^2 y^2 (\alpha_2^{*2} - \beta_2^{*2})/V,\\ V &\equiv p^{*2\delta}(x^2 - y^2)^{\delta^2}.\end{aligned} \tag{6.6}$$

Now we can reconstruct the metric tensor in the usual way [2], [15], [16]. The result is as follows:

$$\begin{aligned}ds^2 = (U^2/W^2)\,dt^2 - c^{-2}\{&U^2W^2p^{*-8\delta}(x^2 -\\ &- y^2)^{1-4\delta^2}[(x^2-1)^{-1}\,dx^2 + (1-y^2)^{-1}\,dy^2] +\\ &+ (W^2/U^2)(x^2-1)(1-y^2)\,d\Psi^2\},\end{aligned} \tag{6.7}$$

where U and W are given in Eq. (6.5), p^* in (6.3), and c is the dimensional constant introduced together with the spheroidal coordinates. The potential Φ is given in Eq. (6.5); there it is real and then is equal to the only nonzero component A_o (pure electrostatic field). As told, still a special duality rotation (2.6) is possible to transform some part of the electric field into a magnetic one of the same form.

Of course, our line element differs from that of [12] due to the starring of the polynomials. It also differs from the formula given in [9]. Namely, there the *original* vacuum γ appears in the line element, with a prefactor 4. Now, the prefactor is correct as shown directly by the second two of Eqs (2.5), but γ must feel the effects of starring the polynomials α, β as well. At the same time, it turns out that in this special case the solutions might, indeed, have been obtained in the indirect way suggested in [14] in the following steps. First perform a transformation with the *false* energy-momentum tensor, resulting in the formulae of [12]; then change q into imaginary, and take this into account in p, *and change no other parameters;* finally change the now imaginary Φ back to real. However, this possibility to get the solution in two ways seems to be the consequence of the special structure of the solution. Namely, observe that in Eq. (6.1) there is only *one* dimensional parameter in the *two* quantities f and φ; in addition f is an even function of this parameter q while φ is its odd function. This gave the possibility of making φ imaginary and keeping f real by simply writing iq instead of q. For more complicated stationary vacua such thumb rules would not work and even now it was safer to go through the steps of the direct way.

7. The behaviour of the solutions

Now, we are going to investigate the behaviour and possible physical meaning of the Bonnor counterparts of the Tomimatsu–Sato solutions. In this we concentrate on the asymptotic ($r \to \infty$) region. For the central one [7] and [13] report singularities and acausal regions when $\delta = 1$. No doubt, such anomalies arise from the similar ones of the original TS solutions; e.g. Eq. (2.7) clearly shows that *coordinate* singularities survive the transformation.

In the asymptotic region Eq. (3.2) gives

$$x \to \mathrm{cr}, \quad y = \cos\theta. \tag{7.1}$$

Therefore, the asymptotic region lies at $x \to \infty$. There, neglecting the $\sigma(x^{-2})$ terms,

$$ds^2 \approx (1 - 2\delta/(\sqrt{1+q^2}cr))\, dt^2 - (1 + 2\delta/(\sqrt{1+q^2}cr)) \cdot \{dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\Psi^2)\}. \tag{7.2}$$

So an asymptotically Schwarzschild line element has been obtained with the mass parameter

$$m = \delta/(c\sqrt{1+q^2}) \tag{7.3}$$

and now one sees the physical meaning of the dimensional constant c.

For the asymptotics of the electromagnetic potential Φ one gets

$$\Phi = -e^{iQ} 2m^2 q r^{-2} \cos\theta (1 + m/r) + \sigma(r^{-4}). \tag{7.4}$$

(The leading term is the same as obtained by Wang [12].)

We must confess that the asymptotic formulae (7.2–4) have been deduced only from the first four TS solutions, and no effort was made to prove their general validity *for all* δ. However, as seen, in the Schwarzschild coordinates $\{t, r, \theta, \Psi\}$ the obtained asymptotic formulae do not even contain the serial parameter δ.

8. Conclusions

For the physical meaning of the Bonnorified Tomimatsu–Sato solutions we can deduce the following facts. For $Q = 0$ (no duality rotation) one sees a dipole *electric* field. Since in Φ the r^{-3} term comes only from the Schwarzschild metric, the quadrupole momentum is 0. This, together with the complete axial symmetry, implies a globally neutral system of collinear electric charges. In Minkowski space such a system could not be stable; here singularities in the central region may correspond to rigid supports for the point sources. So one can conclude that $m^2 q$ is the net dipole momentum eR. The duality rotation may produce some magnetic dipole momentum as well. While its source may be a circular current, this is alien from a *static* solution, so a system of magnetic monopoles is more convenient.

For the mass, Eq. (7.3) gives a limiting value for vacuum ($q = 0$). Therefore it would be tempting to regard m as a result of the point masses + field energy. However, with increasing electric field m is *decreasing*. So it is more prudent to restrain ourselves from contemplating on the origin of the mass, maybe partly hidden in the singular region.

So, the Bonnor counterparts of the Tomimatsu–Sato solutions describe the asymptotically flat external metric of some globally neutral collinear structures of point electric charges with rigid supports.

Acknowledgements

I would like to thank Dr. Z. Perjés for suggesting this problem and for helpful personal communications, and also Dr. Á. Sebestyén for illuminating remarks.

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